TOKYO J. MATH.
VOL. 26, NO. 1, 2003

On Average Curvatures of Convex Curves in Surfaces

Jin LU* and Minoru TANAKA**

Fudan University and Tokai University

(Communicated by R. Miyaoka)

Abstract. In this paper, an upper bound of the average curvature of a convex curve in a simply connected surface is obtained.

1. Introduction

In [1] M. Bridgeman defined the average curvature of a curve, and gave an upper bound of the average curvature of a convex curve embedded in the hyperbolic plane H^2. He also proved that the average curvature of a bi-infinite convex curve in H^2 is bounded above by one. It is natural to ask such a question: What is the upper bound of the average curvature of a convex curve embedded in a surface? In this paper we establish this upper bound.

In this paper a *surface* means a 2-dimensional complete Riemannian manifold. A *convex curve* in a surface is defined as following:

DEFINITION 1. Let M be a surface. A Jordan arc (i.e., a curve diffeomorphic to a closed interval) α in M is called convex curve if any minimal geodesic joining two points of α intersects α only at those two points and if for any point p on α, there is no cut point of p on α.

The *average curvature* of a curve α in the surface M is defined as following:

DEFINITION 2 ([1]). If α is a finite length curve in the surface M, the average curvature $K(\alpha)$ of α is defined by

$$K(\alpha) = \frac{\displaystyle\int_\alpha k_g ds}{\displaystyle\int_\alpha |\alpha^{'}| ds} = \frac{\text{Total curvature along } \alpha}{\text{Length of } \alpha} ,$$

where k_g is the geodesic curvature of α, and s is the arc-length along α.

Received April 25, 2002
2000 Mathematics Subject Classification. Primary 53C45, 58A05; Secondary 52A10, 52A40
* Supported in part by National Natural Science Foundation of China (No. 10271029).
** Supported in part by a Grant-in-Aid for Scientific Research (No. 11640094) from Japan Society for the Promotion of Science.

If α is an infinite length curve in the surface M, then the average curvature $K(\alpha)$ is defined by

$$K(\alpha) = \limsup_{L\to\infty}\{K(\bar{\alpha}) \mid \bar{\alpha} \text{ is a subarc of } \alpha \text{ of length } L\}\,.$$

Let α be a finite length curve in the surface M, define

$$\delta(\alpha) := \sup\{d(p,q) \mid p, q \in \alpha\}\,,$$

where $d(p,q)$ denotes the distance between p and q in M.

Denote by $m_k(t)$ the solution of the following differential equation

$$m''(t) + km(t) = 0\,, \quad m(0) = 0\,, \quad m'(0) = 1\,,$$

where k is a constant.

The main result of this paper is:

THEOREM 1. *Let M be a simply connected surface whose Gaussian curvature G satisfies*

$$k_1 \le G \le k_2\,, \quad \text{and} \quad k_1 \le 0\,,$$

where k_1 and k_2 are constants. Set

$$d = \begin{cases} +\infty\,, & \text{if } k_2 \le 0\,, \\ \dfrac{\pi}{\sqrt{k_2}}\,, & \text{if } k_2 > 0\,. \end{cases}$$

If α is a convex curve of length L in M, and satisfies $\delta(\alpha) < d$, then the average curvature $K(\alpha)$ of α satisfies

$$K(\alpha) \le \frac{2\pi}{L} - \frac{k_1}{2L} f\left(2L\frac{m_{k_1}(\delta(\alpha))}{m_{k_2}(\delta(\alpha))}\right)\,,$$

where $f(t) := 2\pi \int_0^{m_{k_1}^{-1}(\frac{t}{2\pi})} m_{k_1}(\rho)d\rho$, which is a monotonically increasing function on $(0, +\infty)$.

Some interesting corollaries of this theorem will be discussed in Section 3 of this paper.

2. Notations and lemmas

Let M be a simply connected surface whose Gaussian curvature G satisfies $k_1 \le G \le k_2$, $k_1 \le 0$, and $\alpha : [0, L] \to M$ be a convex curve parameterized by arclength in M, whose endpoints are x, y, and $x = \alpha(0)$. Join x and y by a unit-speed geodesic γ such that $\gamma(0) = x$. Let T_xM be the tangent space of M at x, (ρ, θ) be the polar coordinate of T_xM, and the metric in T_xM be taken as $ds^2 = d\rho^2 + \rho^2 d\theta^2$.

Denote by Ω the closed domain bounded by α and γ such that the minimal geodesic joining x to the midpoint $\alpha(\frac{1}{2}L)$ of α lies in Ω. By the convexity of α, such a Ω can be defined.

LEMMA 1. *For any distinct points p, q on the convex curve α, the minimal geodesic joining p and q lies in Ω.*

PROOF. For each $s \in [0, L]$, let D_s be the set of all parameter values $t \in [0, L]$ such that the minimal geodesic joining $\alpha(t)$ to $\alpha(s)$ lies in Ω. It is easy to see that D_s is closed for each s because Ω is closed. Since $\alpha(s)$ has no cut point on α, D_s is relatively open in $[0, L]$. Thus D_s is empty or $[0, L]$ for each $s \in [0, L]$. Since $\frac{1}{2}L \in D_0$, $D_0 = [0, L]$. This implies $0 \in D_s$. Therefore $D_s = [0, L]$ for each $s \in [0, L]$. Hence for any distinct points p, q on the convex curve α, the minimal geodesic joining p and q lies in Ω. □

Denote by θ_0 the interior angle formed by α and γ at x, by θ_1 the interior angle formed by α and γ at y. By the convexity of curve α, we can express α and Ω in the following way: When we take the orthonormal basis $\{e_1, e_2\}$ of T_xM to be suitable,

$$\alpha : \quad \exp_x(\rho(\theta)\cos\theta e_1 + \rho(\theta)\sin\theta e_2)\,, \quad 0 \le \theta \le \theta_0\,,$$

and

$$\Omega = \{\exp_x(\rho\cos\theta e_1 + \rho\sin\theta e_2) \mid 0 \le \rho \le \rho(\theta)\,, \quad 0 \le \theta \le \theta_0\}\,,$$

where $\rho(\theta)$, $0 \le \theta \le \theta_0$, is a function of θ satisfying $\rho(\theta_0) = 0$. It is easy to see that $\rho(\theta) \le \delta(\alpha) \le L$, $0 \le \theta \le \theta_0$.

LEMMA 2. *The two angles θ_0, θ_1 do not exceed π.*

PROOF. Assume that at least one of two angles θ_0, θ_1 is greater than π. Since there are no cut points of x and y on α and M is complete, we can extend the geodesic γ to infinity at both two directions. Denote the extended geodesic by $\bar{\gamma}$. If, for example, the angle θ_0 at x is greater than π, since the M is a simply connected complete manifold of dimension two, there exist two points P_1 and P_2 on α such that they lies in the different sides of $\bar{\gamma}$. Since α is a curve passing though P_1 and P_2, it must intersects $\bar{\gamma}$ at least a point P_3 which is different from x and y. Let β be the part connecting three points x, y and P_3 of geodesic $\bar{\gamma}$, and it can be parameterized by arclength. Obviously, β lies in Ω since the angle θ_0 at x is greater than π. Notice that Lemma 1 implies that there are no cut points of x and y on Ω, hence β is a minimal geodesic, and it intersects α at least three distinct points x, y and P_3, this contradicts that the curve α is convex. □

We denote by $Area(\cdot)$ the area of a domain, and by $Length(\cdot)$ the length of a curve.

Let $M(k_1)$ be a simply connected surface of constant curvature k_1. Take a fixed point O in $M(k_1)$, and an orthonormal basis $\{\bar{e}_1, \bar{e}_2\}$ of $T_OM(k_1)$. Let α_1 be the curve in $M(k_1)$ such that

$$\alpha_1 : \quad \exp_O(\rho(\theta)\cos\theta\bar{e}_1 + \rho(\theta)\sin\theta\bar{e}_2)\,, \quad 0 \le \theta \le \theta_0\,,$$

where θ_0 and $\rho(\theta)$ was defined above. We call α_1 the curve associated with α. Furthermore, set

$$\Omega_1 = \{\exp_O(\rho\cos\theta\bar{e}_1 + \rho\sin\theta\bar{e}_2) \mid 0 \le \rho \le \rho(\theta),\ 0 \le \theta \le \theta_0\}$$

and

$$\gamma_1 = \exp_O(\rho\bar{e}_1)\,, \quad 0 \le \rho \le \rho(0)\,,$$

so γ_1 is the unit-speed geodesic connecting the endpoints of α_1, and Ω_1 is the closed domain bounded by α_1 and γ_1.

LEMMA 3. $Area(\Omega) \le Area(\Omega_1)$.

PROOF. Denote by $\frac{\partial}{\partial\theta}$ the vector field in $T_x M$ which orthogonal to radical direction (we use this notation all over this paper). Since the Gaussian curvature G of M satisfies $G \ge k_1$, by the Rauch comparison theorem we have

$$\left| d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right| \le m_{k_1}(\rho)\,.$$

Since

$$Area(\Omega) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} \left| d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right| d\rho\,,$$

and

$$Area(\Omega_1) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} m_{k_1}(\rho) d\rho\,,$$

we get the conclusion. □

LEMMA 4.

$$L \ge \frac{m_{k_2}(\delta(\alpha))}{m_{k_1}(\delta(\alpha))} L(\alpha_1)\,,$$

where $L(\alpha_1)$ denotes the length of curve α_1 in $M(k_1)$ associated with α.

PROOF. Let $W(t) = m'_{k_2}(t)m_{k_1}(t) - m'_{k_1}(t)m_{k_2}(t)$, then $W(0) = 0$. From

$$\left(\frac{m_{k_2}}{m_{k_1}}\right)'(t) = \frac{W(t)}{m_{k_1}^2(t)}\,,$$

and

$$\begin{aligned} W'(t) &= m''_{k_2}(t)m_{k_1}(t) - m''_{k_1}(t)m_{k_2}(t) \\ &= (k_1 - k_2)m_{k_1}(t)m_{k_2}(t) < 0, \quad 0 < t < d\,, \end{aligned}$$

we can see that $W(t) < 0$ $(0 < t < d)$. Therefore, the function $\frac{m_{k_2}(t)}{m_{k_1}(t)}$ is monotonically decreasing when $0 \le t < d$. Obviously,

$$\lim_{t\to 0} \frac{m_{k_2}(t)}{m_{k_1}(t)} = 1\,.$$

By the Rauch comparison theorem, we have $\left|d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right| \geq m_{k_2}(\rho)$. Hence from $\rho(\theta) \leq \delta(\alpha) < d$ $(0 \leq \theta \leq \theta_0)$ we have

$$\begin{aligned}
L &= \int_0^{\theta_0} \sqrt{\rho'(\theta)^2 + \left|d\exp_x \frac{\partial}{\partial\theta}(\rho,\theta)\right|^2}\, d\theta \\
&\geq \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + m_{k_2}^2(\rho(\theta))}\, d\theta \\
&= \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + \frac{m_{k_2}^2(\rho(\theta))}{m_{k_1}^2(\rho(\theta))} m_{k_1}^2(\rho(\theta))}\, d\theta \\
&\geq \frac{m_{k_2}(\delta(\alpha))}{m_{k_1}(\delta(\alpha))} \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + m_{k_1}^2(\rho(\theta))}\, d\theta \\
&= \frac{m_{k_2}(\delta(\alpha))}{m_{k_1}(\delta(\alpha))} L(\alpha_1)\,.
\end{aligned}$$

LEMMA 5. *Let $M(k)$ be a simply connected surface of constant curvature k $(k \leq 0)$, and C be a circle with circumference L in $M(k)$. Then the area $A(L)$ of the domain bounded by C is*

$$A(L) = f(L),$$

where $f(t) := 2\pi \int_0^{m_k^{-1}(\frac{t}{2\pi})} m_k(\rho)d\rho$, which is a monotonically increasing function on $(0, \infty)$.

PROOF. Assume that the radius of C is r. It is well known that

$$L = 2\pi m_k(r).$$

Notice that $m_k(t)$ is a strictly monotonic function, we have $r = m_k^{-1}\left(\frac{L}{2\pi}\right)$. Hence, the area

$$A(L) = 2\pi \int_0^r m_k(\rho)d\rho = 2\pi \int_0^{m_k^{-1}(\frac{L}{2\pi})} m_k(\rho)d\rho\,.$$

□

LEMMA 6. $LK(\alpha) \leq 2\pi - k_1 Area(\Omega)$.

PROOF. By the Gauss-Bonnet theorem we have

$$\int_\Omega G dV + \int_{\alpha\cup\gamma} k_g ds + \pi - \theta_0 + \pi - \theta_1 = 2\pi\chi(\Omega)\,,$$

where k_g denotes the geodesic curvature.

Since γ is a geodesic, $k_g = 0$ on γ. Obviously $\chi(\Omega) = 1$, hence

$$\int_\Omega G dV + \int_\alpha k_g ds = \theta_0 + \theta_1\,.$$

By the assumption $G \geq k_1$ we have

$$k_1 Area(\Omega) \leq \theta_0 + \theta_1 - \int_\alpha k_g ds\,.$$

Since $\int_\alpha k_g ds = LK(\alpha)$, from Lemma 2 we have

$$LK(\alpha) \leq 2\pi - k_1 Area(\Omega)\,.$$

□

LEMMA 7. *$2Area(\Omega_1) \leq f(2L(\alpha_1))$, where $f(t)$ is the function defined in Lemma* 5, *and $L(\alpha_1)$ denotes the length of the curve α_1 in $M(k_1)$ associated with α.*

PROOF. In $M(k_1)$, take the curve

$$\bar{\alpha}_1 : \quad \exp_O(\rho(|\theta|)\cos\theta\bar{e}_1 + \rho(|\theta|)\sin\theta\bar{e}_2)\,, \quad -\theta_0 \leq \theta \leq 0\,,$$

and set

$$\bar{\Omega}_1 = \{\exp_O(\rho\cos\theta\bar{e}_1 + \rho\sin\theta\bar{e}_2)\,|\,0 \leq \rho \leq \rho(|\theta|),\ -\theta_0 \leq \theta \leq 0\}\,.$$

Graphically, curve $\bar{\alpha}_1$ and domain $\bar{\Omega}_1$ are the reflection of α_1 and Ω_1 about γ_1 in $M(k_1)$ respectively.

Notice that $\left|d\exp_O \frac{\partial}{\partial\theta}(\rho,\theta)\right| = m_{k_1}(\rho)$ in $M(k_1)$, where $\frac{\partial}{\partial\theta}$ also denotes the vector field in $T_O M(k_1)$ which orthogonal to radical direction, we have

$$Area(\Omega_1) = Area(\bar{\Omega}_1) = \int_0^{\theta_0} d\theta \int_0^{\rho(\theta)} m_{k_1}(\rho)d\rho\,,$$

and

$$Length(\alpha_1) = Length(\bar{\alpha}_1) = \int_0^{\theta_0} \sqrt{\rho'^2(\theta) + m_{k_1}^2(\rho(\theta))}\,d\theta\,.$$

From isoperimetric inequality, $Area(\Omega_1 \cup \bar{\Omega}_1)$ is less than or equal to the area of the disk of circumference $Length(\alpha_1) + Length(\bar{\alpha}_1) = 2L(\alpha_1)$, from Lemma 5 we have

$$2Area(\Omega_1) = Area(\Omega_1 \cup \bar{\Omega}_1) \leq f(2L(\alpha_1))\,.$$

□

3. Proof of the Theorem and discussions

PROOF OF THEOREM 1. From Lemma 6 we have

$$K(\alpha) \leq \frac{2\pi}{L} - \frac{k_1}{L} Area(\Omega)\,,$$

and from Lemma 4 we have

$$L(\alpha_1) \leq L\frac{m_{k_1}(\delta(\alpha))}{m_{k_2}(\delta(\alpha))}\,.$$

Since $k_1 \le 0$ and $f(t)$ is a monotonically increasing function, from Lemma 7 we have

$$K(\alpha) \le \frac{2\pi}{L} - \frac{k_1}{2L} f(2L(\alpha_1)) \le \frac{2\pi}{L} - \frac{k_1}{2L} f\left(2L \frac{m_{k_1}(\delta(\alpha))}{m_{k_2}(\delta(\alpha))}\right).$$

□

REMARK. It is well known that the solution of differential equation

$$m''(t) + km(t) = 0\,, \quad m(0) = 0\,, \quad m'(0) = 1\,,$$

is:

$$m_k(t) = \begin{cases} \frac{1}{\sqrt{-k}} \sinh(\sqrt{-k}t), & \text{if } k < 0\,, \\ t\,, & \text{if } k = 0\,, \\ \frac{1}{\sqrt{k}} \sin(\sqrt{k}t)\,, & \text{if } k > 0\,. \end{cases}$$

Hence the estimate of upper bound of the average curvature in Theorem 1 can be expressed explicitly according to the different cases of k_1.

When M is the hyperbolic plane H^2, taking $k_1 = k_2 = -1$ in Theorem 1, we have

$$f(t) = 2\pi\left(\sqrt{1 + \left(\frac{t}{2\pi}\right)^2} - 1\right).$$

Hence we deduce that

COROLLARY 1. *If α is a convex curve of length L in the hyperbolic plane H^2, then*

$$K(\alpha) \le \sqrt{1 + \left(\frac{\pi}{L}\right)^2} + \frac{\pi}{L}\,.$$

Furthermore, if α is a convex curve of infinite length, then $K(\alpha) \le 1$.

This corollary is Theorem 2 and Corollary 1 in [1].

When M is the Euclidean plane R^2, taking $k_1 = k_2 = 0$ in Theorem 1 we have

COROLLARY 2. *The curvature k of convex curve α in the Euclidean plane R^2 satisfies*

$$\int_\alpha k ds \le 2\pi\,.$$

Hence the average curvature of a convex curve of infinite length in R^2 that is defined just replacing k_g by k in Definition 2 is bounded above by zero.

If $M(k)$ is a surface of positive constant Gaussian curvature k, for any convex curve α in $M(k)$, it is easy to see that $\delta(\alpha) < \pi/\sqrt{k}$ naturally holds, taking $k_1 = 0$ in Theorem 1 we have

COROLLARY 3. *Let $M(k)$ be a surface of constant Gaussian curvature k $(k > 0)$. If α is a convex curve in $M(k)$, then the geodesic curvature of α satisfies*

$$\int_{\alpha} k_g ds \leq 2\pi .$$

References

[1] M. BRIDGEMAN, Average curvature of convex curves in H^2, Proc. Amer. Math. Soc. **126** (1998), 221–224.

[2] I. CHAVEL, *Riemannian Geometry: A modern introduction*, Cambridge University Press (1993).

[3] W. KLINGENBERG, *A course in differential geometry*, Springer (1978).

[4] W. KLINGENBERG, D. GROMOLL and W. MEYER, *Riemannsche geometrie in Grossen*, Springer (1968).

[5] P. PETERSEN, *Riemannian geometry*, Springer (1998).

[6] K. SHIOHAMA, The role of total curvature on complete noncompact Riemannian 2-manifolds, Illinois J. of Math. **28** (1984), 597–620.

[7] M. TANAKA, On the cut loci of a von Mangoldt's surface of revolution, J. Math. Soc. Japan **44** (1992), 631–641.

Present Address:
JIN LU
DEPARTMENT OF MATHEMATICS, FUDAN UNIVERSITY,
SHANGHAI 200433, CHINA.
e-mail: jinluk@online.sh.cn

MINORU TANAKA
DEPARTMENT OF MATHEMATICS, SCHOOL OF SCIENCE,
TOKAI UNIVERSITY, HIRATSUKA 259–1292 JAPAN.
e-mail: m-tanaka@sm.u-tokai.ac.jp