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Subordination of Semidynamical Systems

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Abstract. We develop the fundamental properties of multiplicative functional $\mathcal{M}$ defined on a semidynamical system $(X, \mathcal{B}, \Phi, w)$. We give a characterization of semigroups which are subordinate to the deterministic semigroup **H** and we show that they are generated by a multiplicative functional. We study the case when a multiplicative semigroup defined on a measurable space $(X, \mathcal{B})$ is deterministic.

1. Introduction

The semidynamical system arise from a dynamical interpretation of functional differential equations with time lag and evolution type partial differential equations (i.e the heat diffusion equation). In this case a solution $\Phi(t, x)$ with initial condition x is defined on $[0, \rho(x)[$ only and jumps into a "coffin" state "w" afterwards.

Also, a multiplicative functional arise when we "kill" a semidynamical system if it enters in a domain D.

So, starting from a semidynamical system $(X, \mathcal{B}, \Phi, w)$ we give the definition of a terminal time as a first time some physical event occurs and for a subset U of X, we give the notion of a first hitting time and first entry time and we show particularly that any measurable hitting time is a terminal time.

Next, we give some characterization of deterministic semigroups. We show essentially that if $Q_t = M_t H_t$ where $\mathcal{M}$ is a right continuous multiplicative functional and **H** is deterministic, then $(Q_t)_{t\geq 0}$ is deterministic if and only if $\mathcal{M} = 1_{[0,T[}$ where T is a terminal time and therefore $(Q_t)_{t\geq 0}$ is subordinate to **H** (Theorem 2).

Conversely, we prove that any semigroup which is subordinate to **H** is generated by a multiplicative functional $\mathcal{M}$ (Theorem 3).

Notice that any deterministic semigroup is multiplicative. However, if we consider a Lusin space (cf. [7]) we will show that under some restrictions any multiplicative semigroup is deterministic (Theorem 6). Note that this Theorem gives a generalization to the results given in [11], [12] and [3]. Indeed, in [11] and [12], the author considered a topological space

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$(X, \mathcal{T})$ which is locally compact and having a countable base. In [3], the authors showed that a multiplicative potential cone is associated to a deterministic semigroup. In our case, we show that every positive constant is excessive and that **H** is the unique semigroup associated and also any multiplicative semigroup is a right continuous deterministic semigroup.

2. Preliminary

In this section, we will introduce some definitions which will be useful in the remainder of this paper (For more details see [3], [5], [10] and [15]).

DEFINITION 1. Let $(X, \mathcal{B})$ be a separable measurable space with a distinguished point ω. A measurable map $\Phi : \mathbf{R}_+ \times X \to X$ is called a semidynamical system with cofinal point ω if the following conditions are fulfilled:

(S_1) For any x in X, there exists an element $\rho(x)$ in $[0, \infty]$ such that $\Phi(t, x) \neq \omega$ for all $t \in [0, \rho(x))$ and $\Phi(t, x) = \omega$ for all $t \geq \rho(x)$,

(S_2) For any $s, t \in \mathbf{R}_+$ and any $x \in X$ we have

$$\Phi(s, \Phi(t, x)) = \Phi(s + t, x)\,,$$

(S_3) $\Phi(0, x) = x$ for all $x \in X$,

(S_4) If $\Phi(t, x) = \Phi(t, y)$ for all $t > 0$, then $x = y$.

Next, we will denote by $X_0 = X \setminus \{w\}$. For any $x \in X_0$ we denote by Γ_x the trajectory of x, i.e:

$$\Gamma_x = \{\Phi(t, x);\, t \in [0, \rho(x))\}$$

and we define the function Φ_x on $[0, \rho(x))$ by $\Phi_x(t) = \Phi(t, x)$. So for any $x, y \in X_0$ we put

$$x \underset{\Phi}{\leq} y \Leftrightarrow y \in \Gamma_x\,.$$

A maximal trajectory is a totally ordered subset Γ of $X \setminus \{\omega\}$ with respect to the above order, such that there is no $x_0 \in X_0 \setminus \Gamma$ which is minorant of Γ and such for any $x \in \Gamma$, we have $\Gamma_x \subset \Gamma$.

In what follows, we shall suppose that $(X, \mathcal{B}, \Phi, \omega)$ is a transient semidynamical system (cf. [3]). It is proved that the map Φ_x is a measurable isomorphism between $[0, \rho(x))$ and Γ_x endowed with trace measurable structures.

In the next, let us denote by $\mathcal{B}_0 = \{U \in \mathcal{B};\, U \subset X_0\}$. Let Λ be the Lebesgue measure associated with the semidynamical system $(X, \mathcal{B}, \Phi, \omega)$ given by $\Lambda(A) = \lambda(\Phi_x^{-1}(A))$ for any $x \in X_0$, $A \in \mathcal{B}_0$ and $A \subset \Gamma_x$, where λ is the Lebesgue measure on $\mathbf{R}$ (cf. [4]). We recall (cf. [1]) that in the same way Λ can be defined on the σ-algebra $\mathcal{B}_0(\Lambda)$ which is the set of all subsets A of X_0 such that $A \cap M \in \mathcal{B}_0$ for any countable union M of trajectories of X_0.

One can show that the resolvent family $\mathbf{V} = (V_\alpha)_{\alpha \geq 0}$ may be considered on the measurable space $(X_0, \mathcal{B}_0(\Lambda))$ and we denote by $\mathcal{F}(X_0, \Lambda)$ the set of all positive $\mathcal{B}_0(\Lambda)$ measurable functions on X_0. For every $f \in \mathcal{F}(X_0, \Lambda)$, we have

$$V_\alpha f(x) = \int_0^\infty e^{-\alpha t} f(\Phi(t,x))dt.$$

In the sequel, we define the inherent topology $\mathcal{T}_\Phi^0$ as being the set of all subsets D of X_0 satisfying the following condition:

$$(\forall x \in X_0, \forall t_0 \in [0, \rho(x)[\text{ such that } \Phi(t_0, x) \in D) \Rightarrow$$
$$(\exists \varepsilon > 0, \text{ such that } \forall t \in]t_0 - \varepsilon, t_0 + \varepsilon[\cap [0, \rho(x)[, \Phi(t,x) \in D)$$

(see [3], [10]).

Let us denote by $\mathbf{H} = (H_t)_{t \in \mathbf{R}_+}$ the deterministic semigroup introduced in [11] and [13] and defined by

$$\varepsilon_x H_t = \begin{cases} \varepsilon_{\Phi(t,x)} & \text{if } t < \rho(x), \\ 0 & \text{if } t \geq \rho(x), \end{cases}$$

for every $(t,x) \in \mathbf{R}_+ \times X_0$.

Then, we get the following results (see [2]).

THEOREM 1. *The map $t \to \Phi(t,x)$ is right continuous with respect to the inherent topology $\mathcal{T}_\Phi^0$.*

PROOF. Let $f : X_0 \to \mathbf{R}^+$ be a bounded $\mathcal{B}_0(\Lambda)$-measurable function. For every $x \in X_0$, we have

$$\begin{aligned} V_0 f(\Phi(t,x)) &= \int_0^\infty f(\Phi(s, \Phi(t,x)))ds \\ &= \int_t^\infty f(\Phi(s,x))ds. \end{aligned}$$

Therefore $t \to (V_0 f)\Phi(t,x)$ is right continuous, i.e $t \to \Phi(t,x)$ is right continuous with respect to $\mathcal{T}_\Phi^0$. □

COROLLARY 1. *The deterministic semigroup is right continuous.*

NOTATION. Throughout this paper we will denote by ε_x the Dirac measure concentrated in x. For every subset U of X, we denote $U^C = X \setminus U$ and

$$1_U = \begin{cases} 1 & \text{if } x \in U, \\ 0 & \text{if not}. \end{cases}$$

3. Multiplicative functionals

DEFINITION 2 (see Definition (6.1) of Ch.I from [6]). A mapping $T : X \to [0, +\infty]$ is called a stopping time if it is $\mathcal{B}_0(\Lambda)$ measurable.

DEFINITION 3 (see (10.1) of Ch.I from [6]). Let A be a subset of X. For each $x \in X$, we define the first entry time of A by

$$D_A(x) = \inf\{t \geq 0 : \Phi(t, x) \in A\}$$

and the first hitting time of A by

$$T_A(x) = \inf\{t > 0 : \Phi(t, x) \in A\}$$

where in both cases the infinimum of the empty set is understood to be $+\infty$.

DEFINITION 4 (see (2.18) of Ch.II from [6]). A stopping time T is a terminal time if for each $t \geq 0$,

$$T = t + T(\Phi(t, .)) \quad \text{on } \{T > t\}\,.$$

PROPOSITION 1 (see (2.18) of Ch.II from [6]). *Any measurable hitting time is a terminal time.*

PROOF. Let $(\alpha_n)_n$ be a sequence which decreases to 0 and such that $\Phi(T_A(\Phi(t, x)) + \alpha_n, \Phi(t, x)) \in A$. Since

$$\Phi(T_A(\Phi(t, x)) + \alpha_n, \Phi(t, x)) = \Phi(T_A(\Phi(t, x)) + \alpha_n + t, x)$$

we get

$$T_A(x) \leq t + \alpha_n + T_A(\Phi(t, x))\,.$$

By letting $n \to \infty$ we get

$$T_A(x) \leq T_A(\Phi(t, x)) + t\,. \tag{3.1}$$

On the other hand, let t be such that $T_A(x) > t$, then there exists $(\beta_n)_n$ which decreases to 0 such that $\Phi(T_A(x) + \beta_n, x) \in A$. Since

$$\Phi(T_A(x) - t + t + \beta_n, x) = \Phi(T_A(x) - t + \beta_n, \Phi(t, x))$$

we get that

$$T_A(\Phi(t, x)) \leq T_A(x) - t + \beta_n$$

for every $n \in \mathbf{N}$, which yields that

$$T_A(\Phi(t, x)) \leq T_A(x) - t. \tag{3.2}$$

The proof then is achieved by using (3.1) and (3.2). □

DEFINITION 5 (see Definition (1.1) of Ch.III from [6]). A family $\mathcal{M} = \{M_t; 0 \leq t < \infty\}$ of measurable functions on $(X, \mathcal{B})$ is called a multiplicative functional provided:

(1) for every $t \geq 0$, M_t is $\mathcal{B}_0(\Lambda)$ measurable,

(2) For each $x \in X$, for every $t, s \geq 0$,

$$M_{t+s}(x) = M_t(x).M_s(\Phi(t, x))\,,$$

(3) $0 \leq M_t(x) \leq 1$ for all t and x.
We say that $\mathcal{M}$ is right continuous (or continuous) provided $t \to M_t(x)$ is right continuous (or continuous) for every $x \in X$.

REMARK 1. We remark that $M_0(x) = M_0^2(x)$ i.e that $M_0(x) = 0$ or $M_0(x) = 1$. Thus, we say that an element x of X is permanent if $M_0(x) = 1$. Moreover the conditions (2) and (3) gives us that

$$M_t(x) \leq M_0(x) \;\; \forall t \geq 0\,.$$

Hence, we will assume that every $x \in X$ is permanent.

The following examples are issued from (1.2), (1.3), (1.4) and (1.5) of Chapter III in [6].

EXAMPLE 1. For each $\alpha \geq 0$, define $M_t = e^{-\alpha t}$. Then $\{M_t; t \geq 0\}$ is a continuous multiplicative functional.

EXAMPLE 2. Let T be a terminal time and define

$$M_t(x) = 1_{[0,T(x)[}(t)\,.$$

Then $\mathcal{M} = \{M_t; 0 \leq t < \infty\}$ is a right continuous multiplicative functional. In fact, if $s < T(\Phi(t,x))$ and $t < T(x)$, then

$$M_t(x)M_s(\Phi(t,x)) = 1\,.$$

Since

$$s < T(\Phi(t,x)) = T(x) - t \text{ on } \{T > t\}$$

we get $t + s < T(x)$ and therefore

$$M_{t+s}(x) = 1_{[0,T(x)[}(t+s) = 1 = M_t(x)M_s(\Phi(t,x))\,.$$

Now, if $s \geq T((\Phi(t,x))$ or $t \geq T(x)$, then $M_t(x)M_s(\Phi(t,x)) = 0$.
In the first case , we have

$$s \geq T((\Phi(t,x)) = T(x) - t \quad \text{if } \; T(x) > t$$

which yields that $t + s \geq T(x)$ and therefore

$$M_{t+s}(x) = 1_{[0,T(x)[}(t+s) = 0\,.$$

In the second case, $t \geq T(x) \Rightarrow t + s \geq T(x)$ which implies that $M_{t+s}(x) = 0$.

EXAMPLE 3. For every $f \in \mathcal{F}(X_0, \Lambda)$, define

$$M_t(x) = \exp\left(- \int_0^t f(\Phi(s,x))ds \right).$$

It is obvious that $\mathcal{M}$ is a continuous multiplicative functional when f is bounded.

EXAMPLE 4. For every $f \in \mathcal{F}(X_0, \Lambda)$, define

$$T = \inf\left\{t : \int_0^t f(\Phi(s, x))ds = \infty\right\}$$

Then T is a terminal time and

$$M_t(x) = 1_{[0,T(x)[}(t) \exp\left(- \int_0^t f(\Phi(s, x))ds\right)$$

defines a right continuous multiplicative functional.

4. Subordinate semigroups

In this section, we will deal with the properties of multiplicative semigroups.

4.1 Deterministic semigroup and subordination

DEFINITION 6. Let $(X, \mathcal{B})$ be a measurable space and let $\mathbf{P} = (P_t)_{t\geq 0}$ be a family of operators such that

(1) For all $A \in \mathcal{B}$, the mapping $(t, x) \to P_t 1_A(x)$ is measurable,
(2) For all $x \in X$, $t \geq 0$, the mapping $A \to P_t 1_A(x)$ is a measure on $\mathcal{B}$,
(3) $P_{t+s} = P_t P_s$,
(4) $P_0 = Id$.

Then $(P_t)_{t\geq 0}$ is called a semigroup.

DEFINITION 7. Let $\mathbf{P} = (P_t)_{t\geq 0}$ be a semigroup. We say that $\mathbf{P}$ is multiplicative if for every measurable functions f and g on $(x, \mathcal{B})$, we have

$$P_t(f.g) = P_t f.P_t g\ .$$

DEFINITION 8. Let $(X, \mathcal{B})$ be a measurable space. We say that a semigroup $\mathbf{P}$ is deterministic if there exists a semidynamical system Φ such that $\mathbf{P}$ is the deterministic semigroup associated.

REMARK 2. Let $(Q_t)_{t\geq 0}$ be a deterministic semigroup defined on a measurable space $(X, \mathcal{B})$. Then, $(Q_t)_{t\geq 0}$ is multiplicative.

REMARK 3. If $\mathcal{M}$ is a multiplicative functional defined on a semidynamical system $(X, \mathcal{B}, \Phi, w)$, we define for each $t \geq 0$ an operator Q_t on $\mathcal{F}(X_0, \Lambda)$ by

$$Q_t f(x) = M_t(x).H_t f(x)$$

where $(H_t)_{t\geq 0}$ is the deterministic semigroup. It is clear that $(Q_t)_{t\geq 0}$ is a linear map from $\mathcal{F}(X_0, \Lambda)$ to $\mathcal{F}(X_0, \Lambda)$ such that $Q_t \leq H_t$. Moreover, we have

$$\begin{aligned} Q_{t+s}f(x) &= M_{t+s}(x).f(\Phi(t+s,x)) \\ &= M_t(x).M_s(\Phi(t,x)).f(\Phi(s,(\Phi(t,x)))) \\ &= M_t(x).(Q_s(f)(\Phi(t,x))) \\ &= Q_s Q_t f(x) \end{aligned}$$

and so $\{Q_t; t \geq 0\}$ is a semigroup called the semigroup generated by $\mathcal{M}$.

THEOREM 2. *Let $(X, \mathcal{B}, \Phi, w)$ be a semidynamical system and* **H** *be the deterministic semigroup associated. Let $(Q_t)_{t\geq 0}$ be a semigroup such that $Q_t = M_t H_t$ where $(M_t)_{t\geq 0}$ is a right continuous multiplicative functional. Then $(Q_t)_{t\geq 0}$ is deterministic if and only if there exists a terminal time T such that $M_t(x) = 1_{[0,T(x)[}(t)$.*

PROOF. Suppose that $(Q_t)_{t\geq 0}$ is deterministic, then by Remark 2 $(Q_t)_{t\geq 0}$ is multiplicative and therefore for $f = g = 1$ we get

$$Q_t(f.g)(x) = Q_t(f)(x).Q_t(g)(x)$$

for all $x \in X$, i.e $M_t(x) = M_t(x)^2$. Thus, for each $x \in X$ there exists $A(x)$ such that $M_t(x) = 1_{A(x)}(t)$.

On the other hand, since $(M_t)_{t\geq 0}$ is multiplicative, then for each $t, s \geq 0$ we have

$$\begin{aligned} 1_{A(x)}(t+s) &= M_{t+s}(x) \\ &= M_t(x).M_s(\Phi(t,x)) \\ &= 1_{A(x)}(t).1_{A(\Phi(t,x))}(s)\,. \end{aligned}$$

Hence

$$t+s \in A(x) \Leftrightarrow t \in A(x) \quad \text{and} \quad s \in A(\Phi(t,x))\,. \tag{4.1}$$

Note that for $t = 0$ we have

$$Q_0 1(x) = M_0(x) P_0 1(x) = M_0(x) = 1 = 1_{A(x)}(0)$$

by Remark 1 which yields that $0 \in A(x)$.

Next, we shall prove that $A(x)$ is an interval. Indeed, let $t \in A(x)$ such that $t > 0$ and let $0 < t' < t$. Then, there exists $s > 0$ such that $t = t' + s$. By (4.1) we get that $t' \in A(x)$.

Set $T(x) = \sup\{t \geq 0 : t \in A(x)\}$. If $T(x) < \infty$, we shall prove that $T(x) \notin A(x)$. So assume that $T(x) \in A(x)$, then for all $\varepsilon > 0$, we have

$$\begin{aligned} 0 &= M_{T(x)+\varepsilon}(x) \\ &= 1_{A(x)}(T(x)).M_\varepsilon(\Phi(T(x),x)) \\ &= M_\varepsilon(\Phi(T(x),x))\,. \end{aligned}$$

By letting $\varepsilon \to 0$, we get

$$0 = M_0(\Phi(T(x), x)) = 1_{A(\Phi(T(x),x))}(0) = 1$$

which is impossible. Hence

$$A(x) = [0, T(x)[\,.$$

Hence, we define a mapping on X, by setting

$$T(x) = \sup\{t \geq 0 : t \in A(x)\}\,.$$

Next, we claim that $T : X \to \overline{\mathbf{R}_+}$ is a stopping time. In fact, for each $\alpha \geq 0$, we have

$$\begin{aligned}\{T > \alpha\} &= \{x : \alpha \in [0, T(x)[\} \\ &= \{x : M_\alpha(x) \neq 0\}\end{aligned}$$

which is measurable.

Next, we claim that T is a terminal time. Indeed, let $x \in X$ and let $t \geq 0$ such that $T(x) > t$ (i.e $t \in A(x)$).

For every $s \in A(\Phi(t, x))$ we have by (4.1) that $t + s \in A(x)$ and therefore $t + s < T(x)$.

Hence, by taking the supremum over all $s \in A(\Phi(t, x))$, we get

$$t + T(\Phi(t, x)) \leq T(x)\,. \tag{4.2}$$

Conversely, for every $s : t < s < T(x)$, there exists $s' > 0$ such that $s = t + s' < T(x)$. Since $t + s' \in A(x)$, again by (4.1) we get that $s' \in A(\Phi(t, x))$. Consequently, we get

$$s \leq t + T(\Phi(t, x))$$

which yields

$$T(x) \leq t + T(\Phi(t, x))\,. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain

$$T(x) = t + T(\Phi(t, x)) \quad \text{on } \{T > t\}\,.$$

□

DEFINITION 9. Let $\mathbf{H}$ be the deterministic semigroup of $(X, \mathcal{B}, \Phi, w)$. A semigroup $(Q_t)_{t\geq 0}$ is subordinate to $\mathbf{H}$ if

$$Q_t f \leq H_t f$$

for each $t \geq 0$ and $f \in \mathcal{F}(X_0, \Lambda)$.

Next, we will prove the following result called the Theorem of Meyer (see [6] and [14]).

THEOREM 3. *If* $(Q_t)_{t\geq 0}$ *is subordinate to* $\mathbf{H}$, *then there exists a multiplicative function* $\mathcal{M}$ *such that* $Q_t = M_t.H_t$.

PROOF. Since $\varepsilon_x Q_t \leq \varepsilon_x H_t$, then by Radon Nikodym Theorem there exists a function $M_t(x)$ such that $0 \leq M_t(x) \leq 1$ and

$$\varepsilon_x Q_t = M_t(x)\varepsilon_x H_t = M_t(x)\varepsilon_{\Phi(t,x)}\,.$$

By setting $f = 1$, we get that

$$M_t(x) = Q_t 1(x)$$

and therefore for every $t \geq 0$ M_t is measurable. On the other hand, for every $s, t \geq 0$ we have

$$\begin{aligned} M_{t+s}(x) f(\Phi(t+s,x)) &= Q_{t+s} f(x) \\ &= Q_t(Q_s f)(x) \\ &= M_t(x)(Q_s f)(\Phi(t,x)) \\ &= M_t(x) M_s(\Phi(t,x)) f(\Phi(t+s,x))\,. \end{aligned}$$

Hence

$$M_{t+s}(x) = M_t(x).M_s(\Phi(t,x))\,. \qquad \square$$

COROLLARY 2. *If $(Q_t)_{t\geq 0}$ is subordinate to $(H_t)_{t\geq 0}$ and $(Q_t)_{t\geq 0}$ is deterministic, then there exists a terminal time T such that $Q_t = 1_{[0,T[} H_t$.*

DEFINITION 10. Let $(X, \mathcal{B}, \Phi, w)$ be a semidynamical system. A semidynamical system Φ' is said to be subordinated to Φ if there exists a terminal time T such that

$$\Phi'(t,x) = \begin{cases} \Phi(t,x) & \text{if } t < T(x)\,, \\ w & \text{if } t \geq T(x)\,. \end{cases}$$

PROPOSITION 2. *Let $(Q_t)_{t\geq 0}$ be a deterministic semigroup associated to a semidynamical system Φ' and such that $(Q_t)_{t\geq 0}$ is subordinate to $(H_t)_{t\geq 0}$. Then, Φ' is subordinate to Φ.*

PROOF. By Corollary 2, there exists a terminal time T such that $Q_t = 1_{[0,T[} H_t$. Hence

$$\varepsilon_x Q_t = \begin{cases} \varepsilon_x H_t = \varepsilon_{\Phi(t,x)} & \text{if } t < T(x)\,, \\ 0 & \text{if } t \geq T(x)\,. \end{cases}$$

It is obvious that Φ' is subordinate to Φ. $\square$

EXAMPLE 5. Let T be a terminal time and set $A = \{T = 0\}$. Next, we claim that $T \leq T_A$. Indeed, let $s \geq 0$ such that $\Phi(s,x) \in A$. Since

$$T(x) = s + T(\Phi(s,x)) = s \quad \text{on } \{T > s\}$$

we conclude that $T(x) \leq s$ which yields that $T(x) \leq T_A(x)$. By setting

$$\Phi'(t,x) = \begin{cases} \Phi(t,x) & \text{if } t < T(x)\,, \\ w & \text{if } t \geq T(x)\,. \end{cases}$$

We get that Φ' is subordinate to Φ.

4.2 Characterization of multiplicative semigroups. In this section, we will show that under mild restrictions any multiplicative semigroup is deterministic with respect to some semidynamical system. So let $(X, \mathcal{B})$ be a measurable space and let $\mathbf{P} = (P_t)_{t \geq 0}$ be a multiplicative semigroup defined on $(X, \mathcal{B})$.

THEOREM 4. *There exists a stopping T time such that for each $x \in X$,*
(1) $T(x) > 0$,
(2) $P_t 1(x) = 1_{[0,T(x)[}(t), \forall t \geq 0$.

PROOF. Since $\mathbf{P}$ is multiplicative, then $P_t 1(x)$ is 1 or 0. Set

$$A(x) = \{t : P_t 1(x) = 1\} .$$

Then

$$P_t 1(x) = 1_{A(x)}(t) .$$

Next, we shall prove that $A(x)$ is an interval. Indeed, by (4) in Definition 6 we have that $0 \in A(x)$ and we will prove that for each $t \in A(x)$ we have $[0, t] \subset A(x)$. Let $t \in A(x)$ and $s < t$, then $t = s + s'$ for some $s' > 0$. Using the fact that $t \in A(x)$, we get

$$1 = P_t 1(x) = P_s(P_{s'} 1)(x) \leq P_s 1(x)$$

which gives us that $P_s 1(x) = 1$ and therefore $s \in A(x)$.

Set $T(x) = \sup A(x)$. Suppose that $T(x) < \infty$ and that $T(x) \in A(x)$. We choose a sequence (ε_n) which decreases to 0. Using the fact that $T(x) = \sup A(x)$, we get that

$$P_{T(x)}(P_{\varepsilon_n})1(x) = P_{T(x)+\varepsilon_n} 1(x) = 0 .$$

On the other hand, for every $y \in X$ we have $P_{\varepsilon_n} 1(y) = 1_{[0,T(y))}(\varepsilon_n)$ which converges to 1 as $n \to \infty$. Since $P_{\varepsilon_n} 1$ is increasing, then $P_{T(x)}(P_{\varepsilon_n})1(x)$ converges to $P_{T(x)} 1(x) = 1$ which is impossible.

Finally, we shall prove that T is measurable. So, let $\alpha \in \mathbf{R}$, since the map $x \to P_\alpha 1(x)$ is measurable, we get that the set $\{x : P_\alpha 1(x) = 1\}$ is measurable. On the other hand,

$$\{x : P_\alpha 1(x) = 1\} = \{x : \alpha \in [0, T(x)[\} = \{T(x) > \alpha\} .$$

Thus, T is a stopping time. □

REMARK 4. For every measurable function f on X, we have that

$$P_t f(x) = 1_{[0,T(x)[}(t) P_t f(x)$$

and consequently, we have $P_t f(x) = 0$ if $t \geq T(x)$.

Next, we introduce the following notation which will be needed later. Let us denote by $\mathcal{E}_\mathbf{P}$ the set of excessive functions of $\mathbf{P}$ (cf. [8]) and by

$$V1(x) = \int_0^\infty P_t 1(x) dt = T(x)$$

the potential of $\mathbf{P}$.

PROPOSITION 3. *The function* 1 *is excessive and therefore every nonnegative constant is excessive.*

PROOF. Since, by Theorem 4, $T(x) > 0$ and $P_t 1(x) = 1_{[0,T(x)[}(t)$, then $\sup_{t\geq 0} P_t 1(x) = 1$. □

Next, we denote by

$$x \underset{\mathcal{E}_{\mathbf{P}}}{\leq} y$$

if $s(y) \leq s(x)$ for all $s \in \mathcal{E}_{\mathbf{P}}$ and by

$$\Gamma_x^{\mathcal{E}_{\mathbf{P}}} = \{y : x \underset{\mathcal{E}_{\mathbf{P}}}{\leq} y\}.$$

Note that if $\mathcal{E}_{\mathbf{P}}$ separates the elements of X, then "$\underset{\mathcal{E}_{\mathbf{P}}}{\leq}$" is an order on X.

THEOREM 5. *Suppose that* $V1 < \infty$ *and that* $\mathcal{E}_{\mathbf{P}}$ *is minstable and that it separates the elements of* X. *Then,* $\mathcal{E}_{\mathbf{P}}$ *is equal to the set of all positive decreasing functions with respect to* "$\underset{\mathcal{E}_{\mathbf{P}}}{\leq}$" *and lower semicontinuous with respect to the fine topology which is the coarsest topology on* X *for which all the excessive functions are continuous.*

PROOF. Let f, g tow excessive functions, then

$$\sup_{t\geq 0} P_t(f.g) = \sup_{t\geq 0} P_t(f).P_t(g) = \sup_{t\geq 0} P_t(f).\sup_{t\geq 0} P_t(g) = f.g$$

and therefore $f.g$ is excessive. On the other hand, by Proposition 3 all positive constants are excessive. By using Theorem 16 in [3], we get that $\mathcal{E}_{\mathbf{W}}$ is equal to the set of all positive decreasing functions with respect to "$\underset{\mathcal{E}_{\mathbf{P}}}{\leq}$" and lower semicontinuous with respect to the fine topology where $\mathbf{W}$ is the resolvent associated to $\mathbf{P}$. On the other hand, by [8], we get that $\mathcal{E}_{\mathbf{W}} = \mathcal{E}_{\mathbf{P}}$. □

THEOREM 6. *Suppose that* $(X, \mathcal{B})$ *is a Lusin space (cf.* [7]), $V1 < \infty$ *and that* $\mathcal{E}_{\mathbf{P}}$ *is minstable and that it separates the elements of* X. *Moreover, assume that for each* $x \in X$, *there exists* $\alpha_x < \beta_x$ *such that* T *is an isomorphism from* $\Gamma_x^{\mathcal{E}_{\mathbf{P}}}$ *to* $]\alpha_x, \beta_x]$. *Then, the semigroup* $\mathbf{P}$ *is a right continuous deterministic semigroup and* T *is a terminal time with respect to* $\mathbf{P}$.

PROOF. Let $x \in X$ and $t \in [0, T(x)[$. Since $\mathbf{P}$ is multiplicative, then for each $A \in \mathcal{B}$ we have $P_t(1_A) \in \{0, 1\}$. By Hunt's approximation Theorem (see [8]) we get that

$$x \underset{\mathcal{E}_{\mathbf{P}}}{\leq} y \Leftrightarrow Vf(y) \leq Vf(x)$$

for every positive bounded measurable function on X. On the other hand, since V is proper, then there exists $(B_n)_n \subset \mathcal{B}^{\mathbf{N}}$ such that

$$\mathcal{C} = \{B_n : n \in \mathbf{N}\}$$

is a ring of sets satisfying the following properties

(1) For each $n \in \mathbf{N}$, $V1_{B_n}$ is bounded,

(2) $X = \bigcup_{n\in\mathbf{N}} B_n$,

(3) The σ-algebra generated by $\mathcal{C}$ is equal to $\mathcal{B}$.

Hence

$$\Gamma_x^{\mathcal{E}_\mathbf{P}} = \bigcap_{n\in\mathbf{N}} \{V(1_{B_n}) \le V(1_{B_n})(x)\}$$

is measurable. Next, we claim that $1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}$ is excessive. Indeed, Let $y \underset{\mathcal{E}_\mathbf{P}}{\le} z$, then $z \in \Gamma_x^{\mathcal{E}_\mathbf{P}}$ when $y \in \Gamma_x^{\mathcal{E}_\mathbf{P}}$ and hence

$$1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}(z) = 1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}(y)\,.$$

But if $y \notin \Gamma_x^{\mathcal{E}_\mathbf{P}}$, we get

$$1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}(y) = 1 \ge 1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}(z)$$

which yields that $1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}$ is decreasing with respect to $\underset{\mathcal{E}_\mathbf{P}}{\le}$.

Now, let α be a real, then

$$\{1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C} > \alpha\} = \begin{cases} X & \text{if } \alpha < 0\,, \\ (\Gamma_x^{\mathcal{E}_\mathbf{P}})^C & \text{if } \alpha \in [0, 1[\,, \\ \emptyset & \text{if } \alpha \ge 1\,, \end{cases}$$

which gives us that $1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}$ is lower semicontinuous with respect to the fine topology and therefore is excessive by Theorem 5.

Next, we claim that $\varepsilon_x P_t$ is concentrated in $(\Gamma_x^{\mathcal{E}_\mathbf{P}})$. Indeed, since $1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}$ is excessive, we get that

$$\varepsilon_x P_t(1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}) \le 1_{(\Gamma_x^{\mathcal{E}_\mathbf{P}})^C}(x) = 0\,.$$

In the following, let

$$T : \Gamma_x^{\mathcal{E}_\mathbf{P}} \to]\alpha_x, \beta_x]$$

be an isomorphism. Let us denote by $\mathcal{T}_x$ the topology defined on $\Gamma_x^{\mathcal{E}_\mathbf{P}}$ and generated by the collection of subsets $V \subset \Gamma_x^{\mathcal{E}_\mathbf{P}}$ such that $\forall y \in V$ there exists $\varepsilon > 0$ such that $]T(y) -$

$\varepsilon, T(y) + \varepsilon[\subset]\alpha_x, \beta_x]$ if $y \neq x$ and $]T(y) - \varepsilon, T(y)] \subset]\alpha_x, \beta_x]$ if $y = x$. It follows that T is an homeomorphism from $(\Gamma_x^{\mathcal{E}_\mathbf{P}}, \mathcal{T}_x)$ to $]\alpha_x, \beta_x]$ and by Lusin Theorem (cf. [7]), T is a measurable isomorphism. Let us denote by

$$\mathrm{Supp}(\varepsilon_x P_t) = \{y \in \Gamma_x^{\mathcal{E}_\mathbf{P}} : \forall V \in \mathcal{T}_x \ \varepsilon_x \ P_t(V) > 0\}.$$

In the sequel, let us assume that there exist $y, z \in \mathrm{Supp}(\varepsilon_x P_t)$ such that $y \neq z$. We can choose $y \in U \in \mathcal{T}_x$ and $z \in W \in \mathcal{T}_x$ such that $U \cap W = \emptyset$. Since

$$0 = \varepsilon_x P_t(1_U 1_W) = \varepsilon_x P_t(1_U)\varepsilon_x P_t(1_W)$$

we get that $\varepsilon_x P_t(1_U) = 0$ or $\varepsilon_x P_t(1_W) = 0$ which is impossible. Thus there exists a unique element y of $\Gamma_x^{\mathcal{E}_\mathbf{P}}$ which will be denoted by $\Phi_0(t, x)$ such that $\varepsilon_x P_t = \varepsilon_{\Phi_0(t,x)}$.

In what follows, Let $X_w = X \cup \{w\}$ where w is an element not in X and $\mathcal{B}_w$ be the σ-algebra on X_w generated by $\mathcal{B}$ and $\{w\}$. Note that $\{w\} \in \mathcal{B}_w$. We define Φ on X_w by

$$\Phi(t, x) = \begin{cases} \Phi_0(t, x) & \text{if } \ x \in X\,, \quad t \in [0, T(x)[\,, \\ w & \text{if } \ x \in X\,, \quad t \geq T(x)\,, \\ w & \text{if } \ x = w\,, \quad t \geq 0\,. \end{cases}$$

Next, we claim that $(X_w, \mathcal{B}_w, \Phi, w)$ is a semidynamical system. In fact, let $s, t \geq 0$ such that $t + s < T(x)$. Then

$$\varepsilon_{\Phi(t+s,x)} = \varepsilon_x P_{t+s} = \varepsilon_x P_t P_s = \varepsilon_{\Phi(t,x)} P_s = \varepsilon_{\Phi(s,\Phi(t,x))}$$

which yields that $\Phi(t + s, x) = \Phi(s, \Phi(t, x))$.

For $t = 0$, we have $\varepsilon_x = \varepsilon_{\Phi(0,x)}$ which gives us that $\Phi(0, x) = x$.

Now, consider $x, y \in X$ such that $\Phi(t, x) = \Phi(t, y)$ for every $t > 0$. Thus, we get that

$$1_{[0,T(x)[}(t) = 1_{[0,T(y)[}(t)$$

for every $t > 0$. Hence $T(x) = T(y)$ and therefore $x = y$.

Next, we shall prove that Φ is measurable and T is a terminal time. In fact, let $t < T(x)$. Since

$$\varepsilon_{\Phi(t,x)} P_s 1 = \varepsilon_{\Phi(t+s,x)} 1 = \varepsilon_x P_{(t+s)} 1\,,$$

we get that

$$1_{[0,T(\Phi(t,x))[}(s) = 1_{[0,T(x)[}(t + s)$$

which gives us that

$$s < T(\Phi(t, x)) \Leftrightarrow s + t < T(x)$$

and hence

$$t + T(\Phi(t, x)) = T(x)\,. \tag{4.4}$$

Now, using the fact that T is a measurable isomorphism, we get that

$$\Phi(t, x) = T^{-1}(T(x) - t)$$

on the set $\{t < T(x)\}$. $(X_w, \mathcal{B}_w)$ being a Lusin space, we get then that Φ is measurable and hence is a semidynamical system. Moreover, by (4.4), we obtain that T is a terminal time.

Finally, since $V1 < \infty$, then $(X_w, \mathcal{B}_w, \Phi, w)$ is a transient semidynamical system (cf. [3] and [9]) and by Corollary 1 we get that **P** is right continuous. □

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