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The Modified Jacobi-Perron Algorithm over $\mathbf{F}_q(X)^d$

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Introduction

The Jacobi-Perron type algorithms are some kinds of multi-dimensional continued fraction algorithms, which have been studied by many authors in the case of real numbers. The following map S is associated to the Jacobi-Perron algorithm:

$$S(x_1, x_2, \ldots, x_d) = \left(\frac{x_2}{x_1} - \left[\frac{x_2}{x_1}\right], \ldots, \frac{x_d}{x_1} - \left[\frac{x_d}{x_1}\right], \frac{1}{x_1} - \left[\frac{1}{x_1}\right]\right)$$

for $(x_1, x_2, \ldots, x_d) \in [0, 1]^d$. Then the ergodic properties of S give some metric results of the Jacobi-Perron algorithm. We refer to [12] for the real number case. In this paper, we consider a modified version of this algorithm, which is called Brun's algorithm, over formal power series.

Let $\mathbf{F}_q$ be a finite fields with q elements and we consider the following:

$\mathbf{F}_q[X] = \{a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 , a_i \in \mathbf{F}_q, \ 0 \le i \le n\}$
: the set of polynomials of $\mathbf{F}_q$-coefficients,

$\mathbf{F}_q(X) = \{\frac{P}{Q} : \ P, Q \in \mathbf{F}_q[X], \ Q \neq 0\}$
: the set of rational functions,

$\mathbf{F}_q((X^{-1})) = \{a_n X^n + a_{n-1} X^{n-1} + \cdots , \ a_i \in \mathbf{F}_q, \ i \le n, \ a_n \neq 0, \ n \in \mathbf{Z}\}$
: the set of formal Laurent power series of $\mathbf{F}_q$-coefficients.

We regard $\mathbf{F}_q[X]$, $\mathbf{F}_q(X)$, and $\mathbf{F}_q((X^{-1}))$ as the set of integers, of rational numbers, and of real numbers, respectively. Then we consider the set of formal Laurent power series of negative degree as the unit interval and can define the map "S" in the same way. We call the algorithm together with S the Jacobi-Perron algorithm over $\mathbf{F}_q(X)^d$. This algorithm has been studied by Paysant-Leroux and Dubois [9], [10], Feng and Wang [2] and Inoue [4]. Indeed, they showed the convergence of the expansion ([2], [9] and [10]), some simple metric properties ([9] and [10]) and the exponential convergence ([4]). In this paper, we modify this

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algorithm in the following way. For $(f_1, f_2, \dots, f_d)$, first we choose j so that $\deg f_j \geq \deg f_i$ for $i \neq j$, then transform the vectors to

$$\left(\frac{f_1}{f_j} - \left[\frac{f_1}{f_j}\right], \dots, \frac{f_{j-1}}{f_j} - \left[\frac{f_{j-1}}{f_j}\right], \frac{1}{f_j} - \left[\frac{1}{f_j}\right], \frac{f_{j+1}}{f_j} - \left[\frac{f_{j+1}}{f_j}\right], \dots, \frac{f_d}{f_j} - \left[\frac{f_d}{f_j}\right]\right).$$

By iterations of this map, we obtain the n-th simultaneous convergence $(\frac{B_1^{(n)}}{B_0^{(n)}}, \dots, \frac{B_d^{(n)}}{B_0^{(n)}})$, $B_i \in \mathbf{F}_q[X]$, $0 \leq i \leq d$, $n \geq 1$ of $(f_1, \dots, f_d)$. Because we choose f_j as one of the maximum degree among $(f_1, \dots, f_d)$, we may choose the same j all the time. This may destroy the exponential convergence of $\frac{B_i^{(\nu)}}{B_0^{(\nu)}}$, $i \neq j$ for such a $(f_1, \dots, f_d)$. However, we will see that this never occur for "most" of $(f_1, \dots, f_d)$. In §1, we give some definitions and basic properties including the convergence of $(\frac{B_1^{(n)}}{B_0^{(n)}}, \dots, \frac{B_d^{(n)}}{B_0^{(n)}})$. In §2, we give an upper estimate of the convergent rate. First we estimate the convergent rate under an assumption on the choice of "j" in the above and show that the convergent rate is exponentially fast a.e. where a.e. means a.e. with respect to the d fold product of the Haar measure. Next we prove that there exists an absolute constant $c_d > 0$ and a constant $C = C(f_1, \dots, f_d) > 0$ such that

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C}{|B_0^{(\nu)}|^{1+c_d}}, \quad 1 \leq i \leq d$$

for a.e. $(f_1, \dots, f_d)$. In Theorem 5, we also give an upper estimate of c_d. To get this result, we prove that this algorithm preserves the d fold product of the Haar measure and the induced expansion coefficients provide an independent and identically distributed sequence of random variables. Such an estimate of convergent rate was studied by [5] (and [3]) for a modified version of two-dimensional Jacobi-Perron algorithm introduced by Podsypanin [11] in the case of real numbers. Then [12] gives an estimate for the Jacobi-Perron algorithm. We also refer [7], [8] and [1] for related topics. In §3, we count the number of d-dimensional vectors $(\frac{B_1}{B_0}, \dots, \frac{B_d}{B_0})$ such that $B_0, B_1, \dots, B_d$ have no non-trivial common factor by using this algorithm (see [6] for the one-dimensional case).

1. Definitions and basic properties

In this section, we define a map T which is arisen from the modified Jacobi-Perron algorithm (MJPA).

We denote by 0 and 1 the additive unity and the multiplicative unity of $\mathbf{F}_q$, respectively. Note that we identify $a_0 X^0 \in \mathbf{F}_q[X]$ with $a_0 \in \mathbf{F}_q$. For $f = a_n X^n + a_{n-1} X^{n-1} + \cdots \in$

$\mathbf{F}_q((X^{-1}))$, we put

$$\deg f = \begin{cases} n & \text{if } a_n \neq 0, \\ -\infty & \text{if } f \equiv 0. \end{cases}$$

We define the absolute value of f by

$$|f| = q^{\deg f}.$$

Also we put

$$[f] = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0 \quad \text{for } f \in \mathbf{F}_q((X^{-1})).$$

We define

$$\mathbf{L} = \{ f = a_{-1} X^{-1} + a_{-2} X^{-2} + \cdots, \quad a_i \in \mathbf{F}_q \quad \text{for } i \leq -1 \},$$

which is a compact Abelian group with the addition and the metric $d(f, g) = |f - g|$. Now, for $1 \leq j \leq d$, we put

$$\mathbf{L}_j^d = \left\{ \begin{matrix} (f_1, \ldots, f_d) : & \deg f_j > \deg f_i & \text{for } 1 \leq i < j, \\ & \deg f_j \geq \deg f_i & \text{for } j < i \leq d \end{matrix} \right\},$$

then

$$\mathbf{L}^d = \mathbf{L}_1^d \cup \cdots \cup \mathbf{L}_d^d.$$

Note that $(0, \ldots, 0) \in \mathbf{L}_1^d$. We denote by m_d the normalized Haar measure on $\mathbf{L}^d$. For $(f_1, \ldots, f_d) \in \mathbf{L}_j^d$, we define

$$a_i = a_i(f_1, \ldots, f_d) = \begin{cases} \left[\dfrac{f_i}{f_j} \right] & \text{for } 1 \leq i \leq d, \\[2ex] \left[\dfrac{1}{f_j} \right] & \text{for } i = d + 1. \end{cases}$$

if $(f_1, \ldots, f_d) \neq (0, \ldots, 0)$ and

$$a_i = 0 \quad \text{for } 1 \leq i \leq d$$

if $(f_1, \ldots, f_d) = (0, \ldots, 0)$. From the above, we see

$$a_i = \begin{cases} 0 & \text{for } 1 \leq i \leq j - 1, \\ a_i \in \mathbf{F}_q & \text{for } j \leq i \leq d, \\ a_i \in \mathbf{F}_q[X],\ \deg a_i \geq 1 & \text{for } i = d + 1. \end{cases} \tag{1}$$

Now we define the map $T : \mathbf{L}^d \to \mathbf{L}^d$ by

$$
\begin{aligned}
T(f_1, \ldots, f_d) &= \left(\frac{f_1}{f_j}, \ldots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - \left[\frac{1}{f_j}\right], \frac{f_{j+1}}{f_j} - \left[\frac{f_{j+1}}{f_j}\right], \ldots, \frac{f_d}{f_j} - \left[\frac{f_d}{f_j}\right]\right) \\
&= \left(\frac{f_1}{f_j}, \ldots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - a_{d+1}, \frac{f_{j+1}}{f_j} - a_{j+1}, \ldots, \frac{f_d}{f_j} - a_d\right)
\end{aligned}
$$

for $(f_1, \ldots, f_d) \in \mathbf{L}_j^d$, $(f_1, \ldots, f_d) \neq (0, 0, \ldots, 0)$ and

$$T(0, \ldots, 0) = (0, \ldots, 0).$$

We put

$$(f_1^{(\nu)}, \ldots, f_d^{(\nu)}) = T^\nu(f_1, \ldots, f_d) \quad \text{for } \nu \geq 1$$

and

$$a_i^{(\nu)} = a_i(f_1^{(\nu-1)}, \ldots, f_d^{(\nu-1)}) \quad \text{for } 1 \leq i \leq d+1,$$

that is,

$$
\begin{aligned}
T^\nu(f_1, \ldots, f_d) &= T(f_1^{(\nu-1)}, \ldots, f_d^{(\nu-1)}) \\
&= \left(\frac{f_1^{(\nu-1)}}{f_j^{(\nu-1)}}, \ldots, \frac{f_{j-1}^{(\nu-1)}}{f_j^{(\nu-1)}}, \frac{1}{f_j^{(\nu-1)}} - \left[\frac{1}{f_j^{(\nu-1)}}\right], \frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}} - \left[\frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}}\right], \ldots, \frac{f_d^{(\nu-1)}}{f_j^{(\nu-1)}} - \left[\frac{f_d^{(\nu-1)}}{f_j^{(\nu-1)}}\right]\right) \\
&= \left(\frac{f_1^{(\nu-1)}}{f_j^{(\nu-1)}}, \ldots, \frac{f_{j-1}^{(\nu-1)}}{f_j^{(\nu-1)}}, \frac{1}{f_j^{(\nu-1)}} - a_{d+1}^{(\nu)}, \frac{f_{j+1}^{(\nu-1)}}{f_j^{(\nu-1)}} - a_{j+1}^{(\nu)}, \ldots, \frac{f_d^{(\nu-1)}}{f_j^{(\nu-1)}} - a_d^{(\nu)}\right)
\end{aligned}
$$

for $(f_1^{(\nu-1)}, \ldots, f_d^{(\nu-1)}) \in \mathbf{L}_j^d$. Also we put $\kappa(\nu) := j$ such that

$$\deg f_j^{(\nu-1)} > \deg f_i^{(\nu-1)} \quad \text{for } 1 \leq i < j,$$

$$\deg f_j^{(\nu-1)} \geq \deg f_i^{(\nu-1)} \quad \text{for } j < i \leq d.$$

We define a $(d+1) \times (d+1)$ matrix $M = (m_{i_1 i_2})$, $m_{i_1 i_2} \in \mathbf{F}_q[X]$, associated to $(f_1, \ldots, f_d) \in \mathbf{L}_j^d$, $(f_1, \ldots, f_d) \neq (0, \ldots, 0)$ in the following way;

(i) $1 \leq i_2 \leq d,\ i_2 \neq j$

$$m_{i_1 i_2} = \delta_{i_1 i_2} \quad \text{for } 1 \leq i_1 \leq d+1, \tag{2}$$

(ii) $i_2 = j$

$$m_{i_1 j} = \begin{cases} 1 & \text{for } i_1 = d+1 \\ 0 & \text{for } 1 \leq i_1 \leq d, \end{cases} \tag{3}$$

(iii) $i_2 = d+1,\ 1 \le i_1 \le d+1$

$$m_{i_1 i_2} = a_{i_1} ,$$

that is,

$$M = M(f_1, \dots, f_d)$$
$$= \left(\begin{array}{ccc|ccccc} 1 & & 0 & & & & & \\ & \ddots & & & & 0 & & \\ 0 & & 1 & & & & & \\ \hline & & & 0 & \cdots & \cdots & 0 & 1 \\ & & & \vdots & 1 & & 0 & a_{j+1} \\ & 0 & & \vdots & & \ddots & & \vdots \\ & & & 0 & & & 1 & a_d \\ & & & 1 & 0 & & & a_{d+1} \end{array} \right) . \qquad (4)$$

For $(f_1, \dots, f_d) = (0, \dots, 0)$, we define M the $(d+1) \times (d+1)$ unit matrix I_{d+1}. We put

$$M^{(0)} = I_{d+1} ,$$

$$M^{(\nu)} = M(f_1^{(\nu-1)}, \dots, f_d^{(\nu-1)}) \quad \text{for } \nu \ge 1 ,$$

where $(f_1^{(0)}, \dots, f_d^{(0)}) = (f_1, \dots, f_d)$. Since we consider the columns of the matrix $M^{(1)} \cdots M^{(\nu)}$, we denote

$$M^{(1)} \cdots M^{(\nu)} = \begin{pmatrix} \beta_{1\,1}^{(\nu)} & \cdots & \cdots & \beta_{1\,d}^{(\nu)} & B_1^{(\nu)} \\ \vdots & & & \vdots & \vdots \\ \beta_{\kappa(\nu)\,1}^{(\nu)} & \cdots & \cdots & \beta_{\kappa(\nu)\,d}^{(\nu)} & B_j^{(\nu)} \\ \vdots & & & \vdots & \vdots \\ \beta_{d\,1}^{(\nu)} & \cdots & \cdots & \beta_{d\,d}^{(\nu)} & B_d^{(\nu)} \\ \beta_{0\,1}^{(\nu)} & \cdots & \cdots & \beta_{0\,d}^{(\nu)} & B_0^{(\nu)} \end{pmatrix}$$

and

$$M^{(0)} = \begin{pmatrix} B_1^{(-d)} & \cdots & B_1^{(-1)} & B_1^{(0)} \\ \vdots & & \vdots & \vdots \\ B_d^{(-d)} & \cdots & B_d^{(-1)} & B_d^{(0)} \\ B_0^{(-d)} & \cdots & B_0^{(-1)} & B_0^{(0)} \end{pmatrix} .$$

By the definition of $B_0^{(\nu)}$, it is easy to see that $\deg B_0^{(\nu)} = \sum_{i=1}^{\nu} \deg a_{d+1}^{(i)}$ which we use often. $B_0^{(\nu)}$ will be the denominator of the ν-th convergent and $B_i^{(\nu)}$, $1 \le i \le d$, will be the numerator. Evidently,

$$
\begin{aligned}
&M^{(1)} \cdots M^{(\nu)} \\
&= \begin{pmatrix}
\beta_{1\,1}^{(\nu-1)} & \cdots & \beta_{1\,d}^{(\nu-1)} & B_1^{(\nu-1)} \\
\vdots & & \vdots & \vdots \\
\beta_{\kappa(\nu)\,1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,d}^{(\nu-1)} & B_{\kappa(\nu)}^{(\nu-1)} \\
\vdots & & \vdots & \vdots \\
\beta_{d\,1}^{(\nu-1)} & \cdots & \beta_{d\,d}^{(\nu-1)} & B_d^{(\nu-1)} \\
\beta_{0\,1}^{(\nu-1)} & \cdots & \beta_{0\,d}^{(\nu-1)} & B_0^{(\nu-1)}
\end{pmatrix}
\left(\begin{array}{ccc|ccccc}
1 & & 0 & & & & & \\
 & \ddots & & & & \mathbf{0} & & \\
0 & & 1 & & & & & \\
\hline
 & & & 0 & \cdots & \cdots & 0 & 1 \\
 & & & \vdots & 1 & & 0 & a_{\kappa(\nu)+1}^{(\nu)} \\
 & \mathbf{0} & & \vdots & & \ddots & & \vdots \\
 & & & 0 & & & 1 & a_d^{(\nu)} \\
 & & & 1 & 0 & & & a_{d+1}^{(\nu)}
\end{array}\right)
\end{aligned} \tag{5}
$$

$$
= \begin{pmatrix}
\beta_{1\,1}^{(\nu-1)} & \cdots & \beta_{1\,\kappa(\nu)-1}^{(\nu-1)} & B_1^{(\nu-1)} & \beta_{1\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{1\,d}^{(\nu-1)} & \beta_{1\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{1\,k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_1^{(\nu-1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\beta_{\kappa(\nu)\,1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,\kappa(\nu)-1}^{(\nu-1)} & B_{\kappa(\nu)}^{(\nu-1)} & \beta_{\kappa(\nu)\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{\kappa(\nu)\,d}^{(\nu-1)} & \beta_{\kappa(\nu)\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{\kappa(\nu)\,k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_{\kappa(\nu)}^{(\nu-1)} \\
\vdots & & \vdots & \vdots & \vdots & & \vdots & \vdots \\
\beta_{d\,1}^{(\nu-1)} & \cdots & \beta_{d\,\kappa(\nu)-1}^{(\nu-1)} & B_d^{(\nu-1)} & \beta_{d\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{d\,d}^{(\nu-1)} & \beta_{d\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{d\,k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_d^{(\nu-1)} \\
\beta_{0\,1}^{(\nu-1)} & \cdots & \beta_{0\,\kappa(\nu)-1}^{(\nu-1)} & B_0^{(\nu-1)} & \beta_{0\,\kappa(\nu)+1}^{(\nu-1)} & \cdots & \beta_{0\,d}^{(\nu-1)} & \beta_{0\,\kappa(\nu)}^{(\nu-1)} + \sum\limits_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{0\,k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_0^{(\nu-1)}
\end{pmatrix}
$$

Since $\det(M^{(1)} \cdots M^{(\nu)}) = \pm 1$, which follows from (4), we see that $B_0^{(\nu)}, \ldots, B_{d-1}^{(\nu)}$ and $B_d^{(\nu)}$ have no non-trivial common factor. By a simple calculation, for $(f_1^{(\nu-1)}, \ldots, f_d^{(\nu-1)}) \in \mathbf{L}_{\kappa(\nu)}^d$, we see that

(i) $i_2 \neq \kappa(\nu)\,,\ d+1$

$$\beta_{i_1\,i_2}^{(\nu)} = \beta_{i_1\,i_2}^{(\nu-1)} \quad \text{for } 1 \le i_1 \le d\,, \tag{6}$$

(ii) $i_2 = \kappa(\nu)$

$$\beta_{i_1\,i_2}^{(\nu)} = B_{i_1}^{(\nu-1)} \quad \text{for } 0 \le i_1 \le d\,, \tag{7}$$

(iii) $i_2 = d+1$

$$\beta_{i_1\,i_2}^{(\nu)} = B_{i_1}^{(\nu)} = \beta_{i_1\,\kappa(\nu)}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{i_1\,k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_{i_1}^{(\nu-1)} \quad \text{for } 0 \le i_1 \le d\,. \tag{8}$$

The above (i) and (ii) mean $\begin{pmatrix} \beta_{1\,i}^{(\nu')} \\ \vdots \\ \beta_{d\,i}^{(\nu')} \\ \beta_{0\,i}^{(\nu')} \end{pmatrix}$ is one of $\begin{pmatrix} B_{1\,i}^{(\nu')} \\ \vdots \\ B_{d\,i}^{(\nu')} \\ B_{0\,i}^{(\nu')} \end{pmatrix}$, $-d \leq \nu' \leq \nu - 1$. From (5), we find that $B_i^{(\nu)}$ increases as ν increases and

$$\deg B_{i_1}^{(\nu)} > \deg \beta_{i_1\,\kappa(\nu)}^{(\nu)} > \deg \beta_{i_1\,i_2}^{(\nu)}$$

if $i_2 \neq \kappa(\nu), d+1$ for $0 \leq i_1 \leq d$. We put

$$M^{(1)} \cdots M^{(\nu)} \begin{pmatrix} f_1^{(\nu)} \\ \vdots \\ f_d^{(\nu)} \\ 1 \end{pmatrix} = \begin{pmatrix} \beta_{1\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{1\,d}^{(\nu)} f_d^{(\nu)} + B_1^{(\nu)} \\ \vdots \\ \beta_{d\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{d\,d}^{(\nu)} f_d^{(\nu)} + B_d^{(\nu)} \\ \beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)} \end{pmatrix}$$

and see the following theorem.

THEOREM 1. *For any* $(f_1, \ldots, f_d) \in \mathbf{L}^d$, *we have*

$$f_i = \frac{\beta_{i\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{i\,d}^{(\nu)} f_d^{(\nu)} + B_i^{(\nu)}}{\beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)}} \quad \textit{for } 1 \leq i \leq d\,,$$

whenever $T^{\nu'}(f_1, \ldots, f_d) \neq (0, \ldots, 0)$ *for any* $0 \leq \nu' < \nu$.

PROOF. From the definition, for $(f_1, \ldots, f_d) \in \mathbf{L}_j^d$,

$$\begin{aligned} T(f_1, \ldots, f_d) &= (f_1^{(1)}, \ldots, f_d^{(1)}) \\ &= \left(\frac{f_1}{f_j}, \ldots, \frac{f_{j-1}}{f_j}, \frac{1}{f_j} - a_{d+1}^{(1)}, \frac{f_{j+1}}{f_j} - a_{j+1}^{(1)}, \ldots, \frac{f_d}{f_j} - a_d^{(1)}\right). \end{aligned}$$

Then

$$f_i = \begin{cases} \dfrac{1 \cdot f_i^{(1)}}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } 1 \leq i < j\,, \\ \dfrac{1}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } i = j\,, \\ \dfrac{1 \cdot f_i^{(1)} + a_i^{(1)}}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } j < i \leq d\,. \end{cases} \tag{9}$$

On the other hand, for $(f_1, \dots, f_d) \in \mathbf{L}_j^d$,

$$\frac{\beta_{i\,1}^{(1)} f_1^{(1)} + \cdots + \beta_{i\,d}^{(1)} f_d^{(1)} + B_i^{(1)}}{\beta_{0\,1}^{(1)} f_1^{(1)} + \cdots + \beta_{0\,d}^{(1)} f_d^{(1)} + B_0^{(1)}} = \begin{cases} \dfrac{1 \cdot f_i^{(1)}}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } 1 \le i < j\,, \\ \dfrac{1}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } i = j\,, \\ \dfrac{1 \cdot f_i^{(1)} + a_i^{(1)}}{1 \cdot f_j^{(1)} + a_{d+1}^{(1)}} & \text{for } j < i \le d\,. \end{cases} \tag{10}$$

From (9) and (10), the assertion of the theorem holds for $\nu = 1$. Now we assume that the assertion of the theorem holds by ν, and we will show that the assertion holds for $\nu + 1$. Note that $\kappa(\nu + 1)$ is chosen by $(f_1^{(\nu)}, \dots, f_d^{(\nu)}) \in \mathbf{L}_{\kappa(\nu+1)}^d$.

$$\begin{aligned} &\frac{\beta_{i\,1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{i\,d}^{(\nu+1)} f_d^{(\nu+1)} + B_i^{(\nu+1)}}{\beta_{0\,1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{0\,d}^{(\nu+1)} f_d^{(\nu+1)} + B_0^{(\nu+1)}} \\ &= \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{i\,k}^{(\nu+1)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{i\,\kappa(\nu+1)}^{(\nu+1)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - a_{d+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{i\,k}^{(\nu+1)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - a_k^{(\nu+1)} \right) + B_i^{(\nu+1)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0\,k}^{(\nu+1)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{0\,\kappa(\nu+1)}^{(\nu+1)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - a_{d+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{0\,k}^{(\nu+1)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - a_k^{(\nu+1)} \right) + B_0^{(\nu+1)}} \\ &= \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{i\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_i^{(\nu)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - a_{d+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{i\,k}^{(\nu)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - a_k^{(\nu+1)} \right) + B_i^{(\nu+1)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_0^{(\nu)} \left(\frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} - a_{d+1}^{(\nu+1)} \right) + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{0\,k}^{(\nu)} \left(\frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} - a_k^{(\nu+1)} \right) + B_0^{(\nu+1)}}\,. \end{aligned}$$

From (8),

$$\begin{aligned} &\frac{\beta_{i\,1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{i\,d}^{(\nu+1)} f_d^{(\nu+1)} + B_i^{(\nu+1)}}{\beta_{0\,1}^{(\nu+1)} f_1^{(\nu+1)} + \cdots + \beta_{0\,d}^{(\nu+1)} f_d^{(\nu+1)} + B_0^{(\nu+1)}} \\ &\quad = \frac{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{i\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_i^{(\nu)} \frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{i\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{i\,\kappa(\nu+1)}^{(\nu)}}{\displaystyle\sum_{k=1}^{\kappa(\nu+1)-1} \beta_{0\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + B_0^{(\nu)} \frac{1}{f_{\kappa(\nu+1)}^{(\nu)}} + \sum_{k=\kappa(\nu+1)+1}^{d} \beta_{0\,k}^{(\nu)} \frac{f_k^{(\nu)}}{f_{\kappa(\nu+1)}^{(\nu)}} + \beta_{0\,\kappa(\nu+1)}^{(\nu)}} \\ &\quad = \frac{\beta_{i\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{i\,d}^{(\nu)} f_d^{(\nu)} + B_i^{(\nu)}}{\beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)}} = f_i\,. \end{aligned}$$

Thus the assertion holds for $\nu + 1$ and the proof is complete. □

Now we call $\frac{B_i^{(\nu)}}{B_0^{(\nu)}}$ the ν-th convergent of the MJPA and $M^{(1)}, \dots, M^{(\nu)}$ the expansion by this algorithm when $T^\nu(f_1, \dots, f_d) \neq (0, \dots, 0)$ for $\nu \geq 0$. Moreover the expansion by the MJPA is said to be finite or infinite if $T^\nu(f_1, \dots, f_d) = 0$ for some $\nu \geq 0$ or $T^\nu(f_1, \dots, f_d) \neq 0$ for any $\nu \geq 0$, respectively. In the sequel, we show some lemmas about the expansion.

LEMMA 1. *For $(f_1, \dots, f_d) \in \mathbf{L}^d$ with $f_i \in \mathbf{F}_q(X)$, $1 \leq i \leq d$, the expansion by the MJPA is finite.*

PROOF. As $f_i \in \mathbf{F}_q(X)$ $(1 \leq i \leq d)$, we can write $(f_1, \dots, f_d) = (\frac{P_1}{Q}, \dots, \frac{P_d}{Q})$ where $P_1, \dots, P_d$ and Q are in $\mathbf{F}_q[X]$ and have no non-trivial common factor. Then it is clear that

$$T(f_1, \dots, f_d) = \left(\frac{P_1'}{P_j}, \dots, \frac{P_d'}{P_j}\right) \tag{11}$$

for some $P_1', \dots, P_d' \in \mathbf{F}_q[X]$ if $(f_1, \dots, f_d) \in \mathbf{L}_j^d$. If we put $T^\nu(f_1, \dots, f_d) = (\frac{P_1^{(\nu)}}{Q^{(\nu)}}, \dots, \frac{P_d^{(\nu)}}{Q^{(\nu)}})$, $P_1^{(\nu)}, \dots, P_d^{(\nu)}$ and $Q^{(\nu)}$ have no non-trivial common factor, then (11) implies

$$\deg Q^{(\nu)} < \deg Q^{(\nu-1)} \quad \text{for } \nu \geq 1 .$$

Consequently, for some $\nu_0 \geq 1$, $P_1^{(\nu_0)} = \cdots = P_d^{(\nu_0)} = 0$ since $\deg P_i^{(\nu)} < \deg Q^{(\nu)}$ and $Q^{(\nu)}$ is a polynomial for any $\nu \geq 1$. □

LEMMA 2. *For $(f_1, \dots, f_d) \in \mathbf{L}^d$, if $f_i \notin \mathbf{F}_q(X)$ for some i $(1 \leq i \leq d)$, the expansion by the MJPA is infinite.*

PROOF. If the expansion of f_i is finite, which means $T^\nu(f_1, \dots, f_d) = (0, \dots, 0)$ for some $\nu \geq 0$, then from Theorem 1, we see

$$f_i = \frac{B_i^{(\nu)}}{B_0^{(\nu)}} = \frac{\beta_{ij}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{ik}^{(\nu-1)} + a_{d+1}^{(\nu)} B_i^{(\nu-1)}}{\beta_{0j}^{(\nu-1)} + \sum_{k=\kappa(\nu)+1}^{d} a_k^{(\nu)} \beta_{0k}^{(\nu-1)} + a_{d+1}^{(\nu)} B_0^{(\nu-1)}} .$$

But this shows that f_i is a rational function, which contradicts with the assertion $f_i \notin \mathbf{F}_q(X)$. □

LEMMA 3. *For any sequence $M^{(1)}, \dots, M^{(\nu+1)}, \dots$ of the form* (5),

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \leq \frac{1}{q}$$

holds for any $\nu \geq 1$.

PROOF. Note that $\kappa(\nu) = \min_{1\le i\le d+1}\{i : m_{i\,d+1}^{(\nu)} \neq 0\}$ where $m_{i\,d+1}^{(\nu)}$ is the $(i, d+1)$ component of $M^{(\nu)}$. Then if $1 \le \kappa(1) < \kappa(2)$,

$$\left|\frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}}\right| = \left|\frac{a_i^{(2)}}{a_{d+1}^{(1)}a_{d+1}^{(2)}}\right| \quad \text{for } 1 \le i \le d\,.$$

Since $\deg a_{d+1}^{(\nu)} \ge 1$ and $\deg a_{d+1}^{(\nu)} > \deg a_i^{(\nu)}$, $1 \le i \le d$, for $\nu \ge 1$, we have

$$|B_0^{(1)}|\left|\frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}}\right| \le \frac{1}{q}\,. \tag{12}$$

We also see if $\kappa(1) = \kappa(2)$,

$$\left|\frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}}\right| = \begin{cases} \left|\dfrac{a_i^{(2)}}{(1 + a_{d+1}^{(1)}a_{d+1}^{(2)})a_{d+1}^{(1)}}\right| & \text{for } 1 \le i \le \kappa(1)\,, \\[2ex] \left|\dfrac{a_{d+1}^{(1)}a_i^{(2)} - a_i^{(1)}}{(1 + a_{d+1}^{(1)}a_{d+1}^{(2)})a_{d+1}^{(1)}}\right| & \text{for } \kappa(1) < i \le d \end{cases}$$

and if $\kappa(2) < \kappa(1) \le d$,

$$\left|\frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}}\right| = \begin{cases} \left|\dfrac{a_i^{(2)}}{a_{\kappa(1)}^{(2)} + a_{d+1}^{(1)}a_{d+1}^{(2)}}\right| & \text{for } 1 \le i \le \kappa(1)\,, \\[2ex] \left|\dfrac{a_{d+1}^{(1)}a_i^{(2)} - a_i^{(1)}a_{\kappa(1)}^{(2)}}{(a_{\kappa(1)}^{(2)} + a_{d+1}^{(1)}a_{d+1}^{(2)})a_{d+1}^{(1)}}\right| & \text{for } \kappa(1) < i \le d\,. \end{cases}$$

Then similarly, we have

$$|B_0^{(1)}|\left|\frac{B_i^{(2)}}{B_0^{(2)}} - \frac{B_i^{(1)}}{B_0^{(1)}}\right| \le \frac{1}{q}\,.$$

Now we suppose the assertion of Lemma 3 holds by $\nu - 1$. For $\nu \ge 2$,

$$\left|\frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right| = \left|\frac{\beta_{i\,\kappa(\nu+1)}^{(\nu)} + \displaystyle\sum_{k=\kappa(\nu+1)+1}^{d} a_k^{(\nu+1)}\beta_{i\,k}^{(\nu)} + a_{d+1}^{(\nu+1)}B_i^{(\nu)}}{\beta_{0\,\kappa(\nu+1)}^{(\nu)} + \displaystyle\sum_{k=\kappa(\nu+1)+1}^{d} a_k^{(\nu+1)}\beta_{0\,k}^{(\nu)} + a_{d+1}^{(\nu+1)}B_0^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}}\right|$$

$$= \left| \frac{\beta^{(\nu)}_{i\,\kappa(\nu+1)} B^{(\nu)}_0 - \beta^{(\nu)}_{0\,\kappa(\nu+1)} B^{(\nu)}_i + \sum_{k=\kappa(\nu+1)+1}^{d} a^{(\nu+1)}_k (\beta^{(\nu)}_{i\,k} B^{(\nu)}_0 - \beta^{(\nu)}_{0\,k} B^{(\nu)}_i)}{\left(\beta^{(\nu)}_{0\,\kappa(\nu+1)} + \sum_{k=\kappa(\nu+1)+1}^{d} a^{(\nu+1)}_k \beta^{(\nu)}_{0\,k} + a^{(\nu+1)}_{d+1} B^{(\nu)}_0\right) B^{(\nu)}_0} \right|$$

$$= \frac{\left| \sum_{k=\kappa(\nu+1)}^{d} a^{(\nu+1)}_k (\beta^{(\nu)}_{i\,k} B^{(\nu)}_0 - \beta^{(\nu)}_{0\,k} B^{(\nu)}_i) \right|}{\left| a^{(\nu+1)}_{d+1} B^{(\nu)2}_0 \right|}$$

$$= \frac{1}{|a^{(\nu+1)}_{d+1} B^{(\nu)}_0|} \left| \sum_{k=\kappa(\nu+1)}^{d} \beta^{(\nu)}_{0\,k} \left(\frac{\beta^{(\nu)}_{i\,k}}{\beta^{(\nu)}_{0\,k}} - \frac{B^{(\nu)}_i}{B^{(\nu)}_0} \right) \right| ,$$

here we use the facts that $\deg a^{(\nu+1)}_k \beta^{(\nu)}_{0\,k} < \deg a^{(\nu)}_{d+1} B^{(\nu)}_0$ for the third equality. By (6) and (7), we can replace $\beta^{(\nu)}_{i\,k}$ by $B^{(l_k)}_i$ for some l_k, $-d \le l_k \le \nu - 1$, but $B^{(l_k)}_i = 0$ for $i \neq 0$ and $l_k < 0$,

$$\begin{aligned} |B^{(\nu)}_0| \left| \frac{B^{(\nu+1)}_i}{B^{(\nu+1)}_0} - \frac{B^{(\nu)}_i}{B^{(\nu)}_0} \right| &= \frac{1}{|a^{(\nu+1)}_{d+1}|} \left| \sum_{l_k=1}^{d} B^{(l_k)}_0 \left(\frac{B^{(l_k)}_i}{B^{(l_k)}_0} - \frac{B^{(\nu)}_i}{B^{(\nu)}_0} \right) \right| \\ &= \frac{1}{|a^{(\nu+1)}_{d+1}|} \left| \sum_{l_k=1}^{d} B^{(l_k)}_0 \sum_{l=1}^{\nu-1} \left(\frac{B^{(l)}_i}{B^{(l)}_0} - \frac{B^{(l+1)}_i}{B^{(l+1)}_0} \right) \right| \\ &= \frac{1}{|a^{(\nu+1)}_{d+1}|} \max_{1\le k\le \nu-1} \max_{k\le l\le \nu-1} |B^{(k)}_0| \left| \frac{B^{(l)}_i}{B^{(l)}_0} - \frac{B^{(l+1)}_i}{B^{(l+1)}_0} \right| . \end{aligned}$$

Then, from the assumption of the induction,

$$|B^{(\nu)}_0| \left| \frac{B^{(\nu+1)}_i}{B^{(\nu+1)}_0} - \frac{B^{(\nu)}_i}{B^{(\nu)}_0} \right| \le \frac{1}{q^2} \le \frac{1}{q} . \qquad \square$$

THEOREM 2. (i) *If $T^{\nu}(f_1, \dots, f_d) \not\equiv 0$ for any $\nu \ge 1$,*

$$\lim_{\nu\to\infty} \frac{B^{(\nu)}_i}{B^{(\nu)}_0} = f_i \quad \textit{for } 1 \le i \le d ,$$

on the other hand, if $T^{\nu-1}(f_1, \dots, f_d) \not\equiv 0$ and $T^{\nu}(f_1, \dots, f_d) \equiv 0$, then

$$\frac{B^{(\nu)}_i}{B^{(\nu)}_0} = f_i \quad \textit{for } 1 \le i \le d .$$

(ii) *For a given sequence of arrays* $\{a_i^{(\nu)} : 1 \le i \le d+1\,,\ \nu \ge 1\}$;

$$\begin{gathered} a_{d+1}^{(\nu)} \in \mathbf{F}_q[X], \quad \deg a_{d+1}^{(\nu)} \ge 1\,, \\ a_i^{(\nu)} = 0 \quad \text{for } 1 \le i < j(\nu), \quad a_i^{(\nu)} \in \mathbf{F}_q \quad \text{for } j(\nu) \le i \le d \end{gathered} \tag{13}$$

with a sequence $j(1), j(2), \ldots$ $(1 \le j(\nu) \le d$ *for* $\nu \ge 1)$, *there exists* $(f_1, \ldots, f_d) \in \mathbf{L}^d$ *such that* $\kappa(\nu) = j(\nu)$.

PROOF. (i) We see

$$\begin{aligned} \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| &= \left| \frac{\beta_{i\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{i\,d}^{(\nu)} f_d^{(\nu)} + B_i^{(\nu)}}{\beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \\ &= \left| \frac{\displaystyle\sum_{k=1}^{d} (\beta_{i\,k}^{(\nu)} B_0^{(\nu)} - \beta_{0\,k}^{(\nu)} B_i^{(\nu)}) f_k^{(\nu)}}{(\beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)}) B_0^{(\nu)}} \right| \\ &= \left| \frac{\displaystyle\sum_{k=1}^{d} \left(\frac{\beta_{i\,k}^{(\nu)}}{\beta_{0\,k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \beta_{0\,k}^{(\nu)} f_k^{(\nu)}}{(\beta_{0\,1}^{(\nu)} f_1^{(\nu)} + \cdots + \beta_{0\,d}^{(\nu)} f_d^{(\nu)} + B_0^{(\nu)})} \right| . \end{aligned}$$

For each k, $1 \le k \le d$, there exists some l_k, $-d \le l_k < \nu$, such that

$$\beta_{i\,k}^{(\nu)} = B_i^{(l_k)}\,.$$

Then, we have

$$\begin{aligned} & \left| \sum_{k=1}^{d} \beta_{0\,k}^{(\nu)} \left(\frac{\beta_{i\,k}^{(\nu)}}{\beta_{0\,k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| \\ &\quad = \left| \sum_{k=1}^{d} B_0^{(l_k)} \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| \\ &\quad = \sum_{k=1}^{d} |B_0^{(l_k)}| \left| \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(l_k+1)}}{B_0^{(l_k+1)}} \right) + \cdots + \left(\frac{B_i^{(\nu-1)}}{B_0^{(\nu-1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| \\ &\quad \le \max_{1 \le l \le \nu-1} |B_0^{(l)}| \left| \frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}} \right|, \quad \text{since } B_i^{(l_k)} = 0 \quad \text{for} \quad i \ne l_k,\ -d \le l_k < 0 \\ &\quad \le \frac{1}{q}\,. \end{aligned}$$

Since $\deg B_0^{(\nu)} = \sum_{k=1}^{\nu} \deg a_{d+1}^{(k)} \geq \nu$ for any $\varepsilon > 0$, there exists $\nu_0 \geq 1$ such that

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \leq \frac{1}{q} \frac{1}{|B_0^{(\nu)}|} < \varepsilon \quad \text{for any} \quad \nu \geq \nu_0 .$$

This implies

$$\lim_{\nu \to \infty} \frac{B_i^{(\nu)}}{B_0^{(\nu)}} = f_i \quad \text{for } 1 \leq i \leq d .$$

(ii) Now we suppose that such a sequence of arrays $\{a_i^{(\nu)}\}$ satisfying (13) is given. Since

$$\left| \frac{B_i^{(\nu)}}{B_0^{(\nu)}} - \frac{B_i^{(\nu+l)}}{B_0^{(\nu+l)}} \right| \leq \max_{\nu \leq k \leq \nu+l-1} \left| \frac{B_i^{(k)}}{B_0^{(k)}} - \frac{B_i^{(k+1)}}{B_0^{(k+1)}} \right| ,$$

it is easy to see, from Lemma 3, that $(\frac{B_i^{(\nu)}}{B_0^{(\nu)}})$ is a Cauchy sequence for $1 \leq i \leq d$. Then we have the existence of the limit of $(\frac{B_i^{(\nu)}}{B_0^{(\nu)}})$, because $\mathbf{L}$ is complete. □

2. The rate of convergence

In this section, we shall give a stronger estimate of the convergence than that of Lemma 3 under an assumption on $\{\kappa(\nu),\ \nu \geq 1\}$.

THEOREM 3. *Suppose* $\{a_i^{(\nu)} : 1 \leq \kappa(\nu) \leq i \leq d,\ \nu \geq 1\}$ *is the expansion of* $(f_1, \ldots, f_d) \in \mathbf{L}^d$. *If* $\#\{\nu : \kappa(\nu) = i\} = \infty$,

$$\lim_{\nu \to \infty} |B_i^{(\nu)} - f_i B_0^{(\nu)}| = 0 \quad \textit{for any } i, \quad 1 \leq i \leq d .$$

Here the condition $\#\{\nu : \kappa(\nu) = i\} = \infty$ holds for a.e. We prove it later. Before we prove Theorem 3, we give a definition and some lemmas which are necessary for the proof.

DEFINITION 1. For any $(f_1^{(\nu-1)}, \ldots, f_d^{(\nu-1)}) \in \mathbf{L}_{\kappa(\nu)}^d$, we put

$$u(\nu) := \min_{1 \leq k \leq d} \{l_k : \beta_{i\,k}^{(\nu)} = B_i^{(l_k)},\ \text{for any} \quad 0 \leq i \leq d\} .$$

Also we put, for $s \geq 2$,

$$n_{s,\,i} := \{\nu : \min_{\tau_{s-1} \leq \nu} \kappa(\nu) = i\}$$

for $s \geq 2$, $1 \leq i \leq d$ and

$$\tau_s := \max_{1 \leq i \leq d} n_{s,\,i} + 1$$

with $\tau_1 = 0$.

LEMMA 4. *Suppose* $\#\{\nu : \kappa(\nu) = i\} = \infty$ *for any* i, $1 \le i \le d$. *Then*

$$\tau_{s-1} \le u(\nu) < \tau_s \quad \textit{for} \ \ \tau_s \le \nu < \tau_{s+1} .$$

PROOF. From the definition of τ_s,

$$0 = \tau_1 < u(\nu) \quad \text{for} \quad \nu \ge \tau_2 .$$

Note that $u(\nu)$ is non-increasing. If $\nu = \tau_3 - 1$, then $u(\nu) < \tau_2$ also by the definition of τ_s. So

$$\tau_1 \le u(\nu) < \tau_2$$

holds for $\tau_2 \le \nu < \tau_3$. In general, the assumption of the lemma implies $\tau_s < \infty$ for any $s \ge 2$ and we have $\tau_s \le u(\nu)$ for $\nu \ge \tau_{s+1}$. Also we have $u(\nu) = \tau_s$ if $\nu = \tau_{s+1} - 1$. □

LEMMA 5. *For any sequence* $M^{(1)}, \dots, M^{(\nu+1)}, \dots$ *of the form* (1) *we have*

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^s} \quad \textit{for} \ \ \nu \ge \tau_s .$$

PROOF. From (12), it is clear that

$$|B_0^{(0)}| \left| \frac{B_i^{(1)}}{B_0^{(1)}} - \frac{B_i^{(0)}}{B_0^{(0)}} \right| = \left| \frac{B_i^{(1)}}{B_0^{(1)}} \right| = \frac{1}{q} . \tag{14}$$

For $\nu \ge 1$,

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| = \frac{1}{|a_{d+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{d} \beta_{0\,k}^{(\nu)} \left(\frac{\beta_{i\,k}^{(\nu)}}{\beta_{0\,k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| .$$

We can replace $\beta_{i\,k}^{(\nu)}$ to $B_i^{(l_k)}$ for some l_k, $-d \le l_k \le \nu - 1$, but $B_i^{(l_k)} = 0$ for $i \ne l_k$, $-d \le i < 0$, then

$$\begin{aligned} |B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| &= \frac{1}{|a_{d+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{d} B_0^{(l_k)} \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) \right| \\ &= \frac{1}{|a_{d+1}^{(\nu+1)}|} \left| \sum_{k=\kappa(\nu)}^{d} B_0^{(l_k)} \sum_{l=l_k}^{\nu-1} \left(\frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}} \right) \right| \\ &\le \frac{1}{|a_{d+1}^{(\nu+1)}|} \max_{u(\nu) \le l \le \nu-1} |B_0^{(l)}| \left| \frac{B_i^{(l)}}{B_0^{(l)}} - \frac{B_i^{(l+1)}}{B_0^{(l+1)}} \right| . \end{aligned}$$

By (14) and Lemma 3, for $\nu \ge \tau_2$,

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^2} .$$

By the induction, for $\nu \ge \tau_s$, we have

$$|B_0^{(\nu)}| \left| \frac{B_i^{(\nu+1)}}{B_0^{(\nu+1)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \le \frac{1}{q^s} . \qquad \square$$

PROOF OF THEOREM 3. For some $l_k,\ 1 \le k \le d$, we have

$$\begin{aligned} |B_i^{(\nu)} - f_i B_0^{(\nu)}| &= |B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \\ &= |B_0^{(\nu)}| \frac{1}{|B_0^{(\nu)}|} \left| \sum_{k=1}^{d} \left(\frac{\beta_{i\,k}^{(\nu)}}{\beta_{0\,k}^{(\nu)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) f_k^{(\nu)} \beta_{0\,k}^{(\nu)} \right| \\ &\le \left| \max_{1 \le k \le d} \left(\frac{B_i^{(l_k)}}{B_0^{(l_k)}} - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right) f_k^{(\nu)} \beta_{0\,k}^{(\nu)} \right| \\ &< \max_{1 \le k \le d} \max_{l_k \le t \le \nu-1} |B_0^{(t)}| \left| \frac{B_i^{(t)}}{B_0^{(t)}} - \frac{B_i^{(t+1)}}{B_0^{(t+1)}} \right| . \end{aligned}$$

By Lemma 5, for $t \ge \tau_s$,

$$|B_0^{(t)}| \left| \frac{B_i^{(t)}}{B_0^{(t)}} - \frac{B_i^{(t+1)}}{B_0^{(t+1)}} \right| < \frac{1}{q^s} .$$

Then,

$$\lim_{\nu \to \infty} |B_i^{(\nu)} - f_i B_0^{(\nu)}| = 0 . \qquad \square$$

We show that S is Haar measure preserving.

For a fixed $\nu \ge 1$, we denote by $\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle$ the cylinder set induced from $(\mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)})$, that is, we put

$$\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle = \left\{ (f_1, \dots, f_d) : \begin{pmatrix} a_1^{(1)} \\ \vdots \\ a_{d+1}^{(1)} \end{pmatrix} = \mathbf{a}^{(1)}, \dots, \begin{pmatrix} a_1^{(\nu)} \\ \vdots \\ a_{d+1}^{(\nu)} \end{pmatrix} = \mathbf{a}^{(\nu)} \right\} .$$

THEOREM 4. (i) *For any Borel set $B \subset \mathbf{L}^d$,*

$$m_d(T^{-1}B) = m_d(B),$$

that is, m_d, the normalized Haar measure on $\mathbf{L}^d$, is an invariant probability measure for S.

(ii) $\left\{ \begin{pmatrix} a_1^{(\nu)} \\ \vdots \\ a_{d+1}^{(\nu)} \end{pmatrix} : \nu \geq 1 \right\}$ *is an independent and identically distributed sequence of random variables with respect to m_d.*

PROOF. (i) It is enough to show that

$$m_d(T^{-1}\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) = m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle)$$

for every cylinder set $\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle$. Let

$$\mathbf{a} = \begin{pmatrix} a_1 \\ \vdots \\ a_{d+1} \end{pmatrix}$$

with

$$a_i = \begin{cases} 0 & \text{for } 1 \leq i < j, \\ 1 & \text{for } i = j, \\ a_i \in \mathbf{F}_q & \text{for } j < i \leq d, \\ a_i \in \mathbf{F}_q[X],\ \deg a_i \geq 1 & \text{for } i = d+1. \end{cases} \tag{15}$$

Then we see that

$$T^{-1}\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle = \bigcup_{\mathbf{a}} \langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle,$$

where $\mathbf{a}$ takes all such vectors with $1 \leq j \leq d$. If we fix $\mathbf{a}$, then $S|_{\langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle}$ is 1-1 and onto $\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle$. For any $f \in \mathbf{L}$ of $\deg f = -n$, we consider

$$S_f(g) = \frac{g}{f} - \left[\frac{g}{f} \right] \quad \text{for } g \in \mathbf{L}.$$

The composition $m_1 \circ S_f$ of the normalized Haar measure on $\mathbf{L}$ and S_f is defined by

$$(m_1 \circ S_f)(A) = m_1(S_f A)$$

for a Borel subset A of $\mathbf{L}$. Then it is easy to see that

$$\frac{dm_1 \circ S_f}{dm_1}(g) = q^n \quad a.e.$$

holds. Also we consider

$$V(f) = \frac{1}{f} - \left[\frac{1}{f}\right]$$

and have

$$\frac{dm_1 \circ V}{dm_1}(f) = q^{2n} \quad a.e.$$

This means that the Radon-Nikodym derivatives of S_f and V are constants (a.e.) if $\deg f = -n$. This shows

$$\frac{dm_d \circ S}{dm_d}(f_1, \dots, f_d) = q^{2n} \cdot q^{n(d-1)} \quad a.e.$$

on $\langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle$. Hence we have

$$q^{2n} q^{n(d-1)} m_d(\langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) = m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle)$$

when $\deg a_{d+1} = n \geq 1$. Moreover, the number of $\mathbf{a}$ with (7) is $q^{d-j} q^n (q-1)$. Therefore,

$$\begin{aligned} m_d(T^{-1}\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) &= m_d\left(\bigcup_{\mathbf{a}} \langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle\right) \\ &= \sum_{\mathbf{a}} m_d(\langle \mathbf{a}, \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) \\ &= \sum_{j=1}^{d} \sum_{n=1}^{\infty} (q-1) q^n q^{d-j} \frac{m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle)}{q^{2n} q^{n(d-1)}} \\ &= \sum_{j=1}^{d} \sum_{n=1}^{\infty} \frac{q-1}{q^{(n-1)d+j}} m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) \\ &= m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) . \end{aligned}$$

(ii) A similar calculation shows that

$$m_d(\langle \mathbf{a}^{(1)}, \dots, \mathbf{a}^{(\nu)} \rangle) = m_d(\langle \mathbf{a}^{(1)} \rangle) \cdots m_d(\langle \mathbf{a}^{(\nu)} \rangle) .$$

This means that the coefficients of the MJPA induce an independent and identically distributed sequence of $(d+1)$-dimensional $\mathbf{F}_q[X]$-valued random variables. □

From Theorem 4, we have the following.

PROPOSITION 1. *For a.e.* $(f_1, \dots, f_d) \in \mathbf{L}^d$,

(i)

$$\lim_{\nu \to \infty} \frac{\#\{\eta : 1 \leq \eta \leq \nu, \ \kappa(\eta) = j\}}{\nu} = \frac{(q-1)q^{d-j}}{q^d - 1} \quad \textit{for } 1 \leq j \leq d ,$$

(ii)

$$\lim_{\nu\to\infty} \frac{\#\{\eta:\ 1 \le \eta \le \nu,\ \deg a_{d+1}^{(\eta)} = n\}}{\nu} = \frac{q^d - 1}{q^{dn}}\,,$$

(iii)

$$\lim_{\nu\to\infty} \frac{\#\{\eta : 1 \le \eta \le \nu,\ \kappa(\eta) = j,\ \deg a_{d+1}^{(\eta)} = n\}}{\nu} = \frac{q-1}{q^{(n-1)d+j}} \quad \textit{for}\ 1 \le j \le d\,.$$

PROOF. It is easy to see that

$$m_1(\{f:\ \deg f = -n\}) = \frac{q-1}{q^n}\,,$$

$$m_1(\{f:\ \deg f < -n\}) = \frac{1}{q^n}\,,$$

and

$$m_1(\{f:\ \deg f \le -n\} = \frac{1}{q^{n-1}}\,.$$

So,

$$\begin{aligned} &m_d(\{(f_1, \dots, f_d) : (f_1, \dots, f_d) \in \mathbf{L}_j^d\,,\ \deg f_j = -n\}) \\ &= \frac{q-1}{q^n}\left(\frac{1}{q^n}\right)^{j-1}\left(\frac{1}{q^{n-1}}\right)^{d-j} \\ &= \frac{q-1}{q^{(n-1)d+j}}\,. \end{aligned} \tag{16}$$

Then the strong law of large numbers shows (iii). Also (i) and (ii) are easily shown by

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{q-1}{q^{(n-1)d+j}} &= \frac{q-1}{q^j} \sum_{n=1}^{\infty} \left(\frac{1}{q^d}\right)^{n-1} \\ &= \frac{(q-1)q^{d-j}}{q^d - 1} \end{aligned}$$

and

$$\sum_{j=1}^{d} \frac{q-1}{q^{(n-1)d+j}} = \frac{q-1}{q^{(n-1)d}} \sum_{j=1}^{d} \frac{1}{q^j}$$

$$= \frac{q^d - 1}{q^{dn}} . \qquad \square$$

PROPOSITION 2. *For a.e.* $(f_1, \dots, f_d) \in \mathbf{L}^d$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\nu=1}^{N} \deg a_{d+1}^{(\nu)} = \frac{q^d}{q^d - 1} .$$

PROOF. We consider the sequence of random variables $\{X_\nu\}$ on the probability space $(\mathbf{L}^d, m_d)$ by $X_\nu(f_1, \dots, f_d) = \deg a_{d+1}^{(\nu)}$. From (8), we have

$$E(X_\nu) = \sum_{j=1}^{d} \sum_{n=1}^{\infty} \frac{1}{q^j} n \frac{q-1}{q^{(n-1)d}}$$
$$= \frac{q^d}{q^d - 1}$$

By the strong law of large numbers, we have the conclusion. $\square$

Now we put

$$\gamma := \frac{q^d}{q^d - 1} .$$

LEMMA 6. *Let*

$$w(\nu) := \max_{\tau_s < \nu} s ,$$

then there exists $\alpha > 0$ *such that*

$$\lim_{\nu\to\infty} \frac{w(\nu)}{\nu} = \alpha \quad a.e.$$

PROOF. For a fix $s \geq 1$, we put

$$A_l := \{(f_1, \dots, f_d) \in \mathbf{L}^d : \tau_{s+1} - \tau_s = l\}, \quad \text{for } l \geq d ,$$

and $\{Y_s\}$ is the sequence of random variables on $(\mathbf{L}^d, m_d)$ defined by $Y_s(f_1, \dots, f_d) = \tau_{s+1} - \tau_s$. Then, we have

$$E(Y_s) = \sum_{l=d}^{\infty} l \cdot m_d(A_l)$$
$$= d + \sum_{l=d}^{\infty} m_d(Y_s > l) .$$

Here we have

$$m_d(Y_\nu > l) < \sum_{k=1}^{d} \left(1 - \frac{(q-1)q^{d-k}}{q^d - 1}\right)^l ,$$

and have

$$E(Y_s) < d + \sum_{l=d}^{\infty} \sum_{k=1}^{d} \left(1 - \frac{(q-1)q^{d-k}}{q^d - 1}\right)^l = d + \alpha_0 < \infty .$$

It is easy to see that $\{Y_s\}_{s \le 1}$ is an independent and identically distributed sequence. The law of large numbers implies

$$\lim_{S \to \infty} \frac{1}{S} \sum_{s=1}^{S} Y_s = d + \alpha_0 \quad a.e. \tag{17}$$

Since

$$\tau_{S+1} = \sum_{s=1}^{S} (\tau_{s+1} - \tau_s) = \sum_{s=1}^{S} Y_s$$

and

$$\tau_{w(\nu)} < \nu < \tau_{w(\nu)+1} ,$$

we have

$$\frac{S}{\sum_{s=1}^{S} Y_s} \le \frac{w(\nu)}{\nu} < \frac{S}{\sum_{s=1}^{S-1} Y_s}$$

when $w(\nu) = S$. From (17), we have the assertion of the Lemma with

$$\alpha = \frac{1}{d + \alpha_0} . \qquad \square$$

PROPOSITION 3. *For a.e. $(f_1, \dots, f_d) \in \mathbf{L}^d$, there exists a positive constant $C_1 = C_1(\varepsilon)$ such that*

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_1}{q^{\nu\alpha(1-\varepsilon)}} \quad \text{for any } \varepsilon > 0, \quad 1 \le i \le d .$$

PROOF. We fix $\varepsilon > 0$. For a.e. $(f_1, \dots, f_d) \in \mathbf{L}^d$, from Lemma 6,

$$\alpha - \varepsilon < \frac{w(\nu)}{\nu} < \alpha + \varepsilon$$

for sufficiently large ν, equivalently,

$$\nu\alpha - \nu\varepsilon < w(\nu) < \nu\alpha + \nu\varepsilon \,.$$

Then,

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| \leq \frac{1}{q^{w(\nu)}} \leq \frac{1}{q^{\nu(\alpha-\varepsilon)}}$$

for sufficiently large ν. □

THEOREM 5. *For a.e. $(f_1, \dots, f_d) \in \mathbf{L}^d$, there exists a positive constant $C_2 = C_2(\varepsilon)$ such that*

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_2}{|B_0^{(\nu)}|^{1+\frac{\alpha}{\gamma}(1-\varepsilon)}} \quad \textit{for any} \quad \varepsilon > 0, \quad 1 \leq i \leq d \,.$$

PROOF. We fix $\varepsilon > 0$. From Proposition 3,

$$|B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_1}{q^{\nu\alpha(1-\frac{\varepsilon}{2})}} \,.$$

Since

$$\deg B_0^{(\nu)} = \sum_{i=1}^{\nu} \deg a_{d+1}^{(i)} \,,$$

from Proposition 2, we have

$$\begin{aligned} q^{\nu\alpha(1-\frac{\varepsilon}{2})} &= q^{\nu\gamma\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})} \\ &\geq (|B_0^{(\nu)}|^{(1-\frac{\varepsilon}{2})})^{\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})} \\ &= |B_0^{(\nu)}|^{\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})^2} \end{aligned}$$

for sufficiently large ν. Then there exists a positive constant C_2 such that

$$\begin{aligned} |B_0^{(\nu)}| \left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| &< \frac{C_2}{|B_0^{(\nu)}|^{\frac{\alpha}{\gamma}(1-\frac{\varepsilon}{2})^2}} \\ &\leq \frac{C_2}{|B_0^{(\nu)}|^{\frac{\alpha}{\gamma}(1-\varepsilon)}} \,, \end{aligned}$$

which means

$$\left| f_i - \frac{B_i^{(\nu)}}{B_0^{(\nu)}} \right| < \frac{C_2}{|B_0^{(\nu)}|^{1+\frac{\alpha}{\gamma}(1-\varepsilon)}} \,.$$

□

3. Rational functions

In this section, we study the number of $\left(\frac{B_1}{B_0}, \dots, \frac{B_d}{B_0}\right)$ with $B_i \in \mathbf{F}_q[X]$, $\deg B_i < \deg B_0 = n \geq 1,\ 1 \leq i \leq d$.

DEFINITION 2. For $(B_0, B_1, \dots, B_d) \in \mathbf{F}_q[X]^{d+1}$ with

$$(B_0, B_1, \dots, B_d) = 1 \quad \text{and} \quad \deg B_i < \deg B_0 \quad \text{for } 1 \leq i \leq d\,,$$

we denote by

$$L = L(B_0, B_1, \dots, B_d)$$

the length of the expansion by the MJPA.

DEFINITION 3. We put

$$E_\nu(n) = \#\left\{ (B_0, B_1, \dots, B_d) \in \mathbf{F}_q[X]^{d+1} \ :\ \begin{array}{l}(B_0, B_1, \dots, B_d) = 1, L = \nu, \\ \max\limits_{1 \leq i \leq d} \deg B_i < \deg B_0 = n\end{array}\right\}$$

and

$$E(n) = \#\left\{ (B_0, B_1, \dots, B_d) \in \mathbf{F}_q[X]^{d+1} \ :\ \begin{array}{l}(B_0, B_1, \dots, B_d) = 1 \\ \max\limits_{1 \leq i \leq d} \deg B_i < \deg B_0 = n\end{array}\right\}.$$

THEOREM 6. *We have*

$$E_\nu(n) = \binom{n-1}{\nu-1} q^n (q^d - 1)^\nu \quad \textit{and} \quad E(n) = (q^d - 1) q^{(d+1)n - d}\,.$$

PROOF. For $(B_0, B_1, \dots, B_d) \in \mathbf{F}_q[X]^{d+1}$, if $L = \nu$, then B_0 is determined by ν polynomials $a_{d+1}^{(1)}, \dots, a_{d+1}^{(\nu)}$. Recall that $\deg B_0^{(\nu)} = n = \sum_{i=1}^{\nu} \deg a_{d+1}^{(i)}$. Then, the number of choices of $\deg a_{d+1}^{(i)}$, $1 \leq i \leq \nu$, is equal to $\binom{n-1}{\nu-1}$. Put $n_i = \deg a_{d+1}^{(i)}$ for $1 \leq i \leq \nu$, then the number of possible choices of $\{a_{d+1}^{(i)}\}$ is $(q-1)q^{n_i}$. So when we fix positive integers $n_1, \dots, n_\nu$ with $\sum_{i=1}^{\nu} n_i = n$, the number of possible choices of $\{a_{d+1}^{(i)} : 1 \leq i \leq d\}$ is $(q-1)^\nu q^n$. Consequently the number of all choices of polynomials $a_{d+1}^{(1)}, \dots, a_{d+1}^{(\nu)}$ is equal to

$$\binom{n-1}{\nu-1} (q-1)^\nu q^n\,.$$

Since the number of possible choices of $\{a_j^{(i)} : 1 \leq j \leq d\}$ is $q^{d-\kappa(i)}$, the one of $\{a_j^{(i)} : 1 \leq j \leq d\,,\ 1 \leq \kappa(i) \leq d\}$ is

$$\sum_{\kappa(i)=1}^{d} q^{d-\kappa(i)} = \frac{q^d - 1}{q - 1}\,.$$

Therefore

$$E_\nu(n) = \binom{n-1}{\nu-1}(q-1)^\nu q^n \Bigg(\sum_{\kappa(i)=1}^{d} q^{d-\kappa(i)} \Bigg)^\nu$$
$$= \binom{n-1}{\nu-1} q^n (q^d-1)^\nu .$$

From the definition, it is clear that

$$\begin{aligned} E(n) &= \sum_{\nu=1}^{n} E_\nu(n) \\ &= \sum_{\nu=1}^{n} \binom{n-1}{\nu-1} q^n (q^d-1)^\nu \\ &= (q^d-1) q^n q^{d(n-1)} \\ &= (q^d-1) q^{(d+1)n-d} . \end{aligned}$$

□

DEFINITION 4. We put

$$\hat{E}(n) = \#\left\{ (B_0, B_1, \ldots, B_d) \in \mathbf{F}_q[X]^{d+1} \; : \; \begin{array}{l} (B_0, B_1, \ldots, B_d) = 1, \\ \max\limits_{1\le i\le d} \deg B_i \le \deg B_0 = n \end{array} \right\} .$$

THEOREM 7. *We have*

$$\hat{E}(n) = (q^d - 1) q^{(d+1)n} .$$

PROOF. For $(B_0, B_1, \ldots, B_d) \in \mathbf{F}_q[X]^{d+1}$ satisfying

$$\deg B_i < \deg B_0 = n \quad \text{for } 1 \le i \le d \quad \text{and} \quad (B_0, B_1, \ldots, B_d) = 1 ,$$

we consider q polynomials $\hat{B}_i$ of the form

$$\hat{B}_i = c\, B_0 + B_i \quad \text{with} \quad c \in \mathbf{F}_q .$$

It is clear that

$$(B_0, \hat{B}_i) = 1 \quad \text{and} \quad \deg \hat{B}_i = n$$

unless $c = 0$. Hence for each $(B_0, B_1, \ldots, B_d) \in \mathbf{F}_q[X]^{d+1}$, we get q^d vectors $(\hat{B}_1, \ldots, \hat{B}_d)$ which satisfies

$$(B_0, \hat{B}_1, \ldots, \hat{B}_d) = 1 \quad \text{and} \quad \deg \hat{B}_i \le n \quad \text{for } 0 \le i \le d .$$

Then

$$\begin{aligned}\hat{E}(n) &= q^d\, E(n) \\ &= q^d (q^d - 1) q^{(d+1)n-d} \\ &= (q^d - 1) q^{(d+1)n} . \end{aligned}$$ □

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