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On the Structure of Strictly Complete Valuation Rings

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Introduction

The purpose of this paper is to determine the structure of strictly complete discrete valuation rings of finite dimension with equal characteristic, and moreover study the finite extensions of these rings. First, we begin with recalling the definition of linear topologies associated with valuation rings and complete valuation rings (see [7]).

For a valuation ring A, we consider the linear topology on QA with fundamental system of neighborhoods of 0 :

$$\Sigma_A = \{a\mathfrak{m}(A) \mid a \in A,\ a \neq 0\}\,.$$

This topology is said to be the A-topology on QA. Here QA is the quotient field of A and $\mathfrak{m}(A)$ is the unique maximal ideal of A.

For a valuation ring A, the completion

$$\hat{A} = \text{proj.}\lim A/\mathfrak{a} \quad (\mathfrak{a} \in \Sigma_A)$$

with respect to the A-topology is also a valuation ring (see [7, Theorem 1]). If $A \cong \hat{A}$ holds naturally, then the valuation ring A is said to be complete. Here we introduce the notion of strictly complete valuation rings as follows:

DEFINITION. A valuation ring A is said to be strictly complete, if the valuation rings $A/\mathfrak{p}$ are complete for any $\mathfrak{p} \in \mathit{Spec}\ A$.

For valuation rings of dimension one, the strictly completeness is equivalent to the completeness. However, for $n \geqq 2$, there exists a complete valuation ring of dimension n, which is not strictly complete. See Example 2. The main results are stated as follows:

THEOREM 1. *Let A be an equal characteristic strictly complete discrete valuation ring of dimension n and $K = QA$.*

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(i) *Then there exist a subfield k of A and $t_1, \cdots, t_n \in A$ such that*

$$A = k \oplus \bigoplus_{i=1}^{n} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]], \quad K = k((t_n)) \cdots ((t_1)),$$

and t_i is transcendental over $k((t_n)) \cdots ((t_{i+1}))$ for any $i \in \{1, \cdots, n\}$.

(ii) *There exists an additive valuation* $\mathrm{ord}_A : K^\times \to \mathbf{Z}^n$ *of K corresponding to A which satisfies* $\mathrm{ord}_A(t_1^{m_1} \cdots t_n^{m_n}) = (m_1, \cdots, m_n)$ *for any* $(m_1, \cdots, m_n) \in \mathbf{Z}^n$.

(iii) *If the residue field $A/\mathfrak{m}(A)$ is a perfect field of characteristic p $(p \neq 0)$, then the coefficient field k of A is determined uniquely.*

THEOREM 2. *Suppose that A is an equal characteristic strictly complete discrete valuation ring of dimension n and $K = QA$. For a finite extension K'/K of fields, let A' denote the integral closure of A in K'. Then*

(i) *A' is also an equal characteristic strictly complete discrete valuation ring of dimension n and $K' = QA'$. Moreover we obtain $A = K \cap A'$.*

(ii) *Let e be the ramification index of A'/A and f the relative degree of A'/A, in other words*

$$e = e(A'/A) = (K'^\times/A'^\times : K^\times/A^\times),$$
$$f = f(A'/A) = [A'/\mathfrak{m}(A') : A/\mathfrak{m}(A)].$$

If we put

$$M = \begin{bmatrix} \mathrm{ord}_{A'}(t_1) \\ \vdots \\ \mathrm{ord}_{A'}(t_n) \end{bmatrix} \in M(n, \mathbf{Z})$$

for $t_1, \cdots, t_n$ determined in Theorem 1, (i), *then*

$$e = \det M, \quad ef = [K' : K].$$

(iii) *A' is a free A-module of rank $[K' : K]$.*

(iv) *If the extension $A/\mathfrak{m}(A) \hookrightarrow A'/\mathfrak{m}(A')$ is separable, then for any coefficient field k of A, the algebraic closure k' of k in K' is a coefficient field of A'.*

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1. Here we shall prove Theorem 1.

First we consider the case that $\dim A = 1$. The following results are well-known (see [4, Theorem 28.3]).

LEMMA 1. (i) *Let A be an equal characteristic complete discrete valuation ring of dimension one. Then there exist a subfield k of A and $t \in A$ such that $A = k[[t]]$.*

(ii) *For a field k and an indeterminate t, we put $A = k[[t]]$, $K = k((t))$. For a finite extension K'/K of fields, let A' denote the integral closure of A in K'. Then there exist a subfield k' of A' and $t' \in A'$ such that $A' = k'[[t']]$, $K' = k'((t'))$. If the extension $A/\mathfrak{m}(A) \hookrightarrow A'/\mathfrak{m}(A')$ is separable, then we can take $k \subset k'$.*

LEMMA 2. *If a valuation ring A is strictly complete, then both the valuation rings $A/\mathfrak{p}$ and $A_{\mathfrak{p}}$ are also strictly complete for any $\mathfrak{p} \in Spec\ A$.*

The proof is easy from the definition of strictly complete valuation rings and [7, Corollary to Lemma 5].

PROOF OF THEOREM 1. (i) We shall prove by induction on $n = \dim A$. For $n = 1$, it is easy from Lemma 1, (i). Let A be an equal characteristic strictly complete discrete valuation ring of dimension n. Then we can write $Spec\ A = \{\mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$, where $\mathfrak{p}_{i-1} \subset \mathfrak{p}_i$ $(1 \leqq i \leqq n)$. By Lemma 2, $A_{\mathfrak{p}_{n-1}}$ is an equal characteristic strictly complete discrete valuation ring of dimension $n-1$. Thus, by the assumption of induction, there exist a subfield k_1 of $A_{\mathfrak{p}_{n-1}}$ and $t_1, \cdots, t_{n-1} \in A_{\mathfrak{p}_{n-1}}$ such that $A_{\mathfrak{p}_{n-1}} = k_1 \oplus \bigoplus_{i=1}^{n-1} t_i k_1((t_{n-1})) \cdots ((t_{i+1}))[[t_i]]$. Then by

$$\mathfrak{p}_{n-1} = \mathfrak{m}(A_{\mathfrak{p}_{n-1}}) = \bigoplus_{i=1}^{n-1} t_i k_1((t_{n-1})) \cdots ((t_{i+1}))[[t_i]]\,,$$

we have $t_1, \cdots, t_{n-1} \in A$. Since $A/\mathfrak{p}_{n-1}$ is an equal characteristic complete discrete valuation ring of dimension one, we can write $A/\mathfrak{p}_{n-1} = k[[t_n]]$, by Lemma 1, (i). Noting that $Q(A/\mathfrak{p}_{n-1}) = A_{\mathfrak{p}_{n-1}}/\mathfrak{m}(A_{\mathfrak{p}_{n-1}}) = k_1$, we obtain $A = k \oplus \bigoplus_{i=1}^{n} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]]$.

(ii) is easy from [7, Example 3].

(iii) is proved by the similar method to the case that $\dim A = 1$.

COROLLARY TO THEOREM 1. *Any equal characteristic strictly complete discrete valuation ring of finite dimension has a coefficient field.*

REMARK. There exists an equal characteristic strictly complete discrete valuation ring of finite dimension which has at least two coefficient fields. See Example 1. On the other hand, if $n \geqq 2$, then there exists an equal characteristic complete discrete valuation ring of dimension n which has no coefficient field. See Example 2, (ii).

EXAMPLE 1. Let k be a field and t_1, t_2 indeterminates. Then the coefficient field of the equal characteristic complete discrete valuation ring $A = k\widehat{[t_1, t_2]}_{(t_1)}$ of dimension one is not determined uniquely, even if we assume that the coefficient field contains k.

EXAMPLE 2. Let k be a field and $t_1, \cdots, t_n$ $(n \geqq 2)$ indeterminates.

(i) Then both the valuation rings

$$A = t_1 k(t_n, \cdots, t_2)[[t_1]] \oplus \bigoplus_{i=2}^{n} t_i k(t_n, \cdots, t_{i+1})[t_i]_{(t_i)} \oplus k\,,$$

$$B = \bigoplus_{i=1}^{n-1} t_i k(t_n)((t_{n-1})) \cdots ((t_{i+1}))[[t_i]] \oplus k[t_n]_{(t_n)}$$

are equal characteristic complete discrete of dimension n, but these are not strictly complete. Note that k is a coefficient field of A and B.

(ii) Suppose that k is not algebraically closed. Take an irreducible polynomial $p \in k[t_n]$, $\deg p \geqq 2$, and put

$$A = t_1 k(t_n, \cdots, t_2)[[t_1]] \oplus \bigoplus_{i=2}^{n-1} t_i k(t_n, \cdots, t_{i+1})[t_i]_{(t_i)} \oplus k[t_n]_{(p)}\,,$$

$$B = \bigoplus_{i=1}^{n-1} t_i k(t_n)((t_{n-1})) \cdots ((t_{i+1}))[[t_i]] \oplus k[t_n]_{(p)}\,.$$

Then both A and B are equal characteristic complete discrete valuation rings of dimension n, but these are not strictly complete. Moreover both A and B have no coefficient field.

2. Here we shall prove that any equal characteristic strictly complete discrete valuation ring of finite dimension is a Henselian ring.

For a field K, let $Zar\ K$ denote the set of valuation rings of K. Then the set $Zar\ K$ has a structure of local ringed spaces. For an extension $K \subset L$ of fields, the mapping

$$Zar_{L|K} : \begin{array}{ccc} Zar\ L & \longrightarrow & Zar\ K \\ \Psi & & \Psi \\ B & \longmapsto & K \cap B \end{array}$$

is a morphism of local ringed spaces (see [5]).

LEMMA 3. *Let A be a valuation ring and $K = QA$. For a finite extension L/K of fields, we put $B = N_{L/K}^{-1}(A)$. Here $N_{L/K} : L \to K$ is the norm mapping.*

(i) *The following two conditions are equivalent*:

(1) *B is a valuation ring of L and satisfies $A = K \cap B$, that is, $B \in Zar_{L|K}^{-1}(A)$.*

(2) *$x \in B \Rightarrow 1 + x \in B$.*

(ii) *$A \neq K \Longleftrightarrow B \neq L$.*

The proof is easy.

LEMMA 4. *The valuation ring* $A = k \oplus \bigoplus_{i=1}^{n} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]]$ *is a Henselian ring. In other words, for an indeterminate* t *over* A *and for any* $f \in A[t]$, *the following statement holds; if there exist* g_0, $h_0 \in A[t]$ *such that*

(a_0) $f - g_0 h_0 \in \mathfrak{m}(A)[t]$

(b_0) g_0 *is monic and* $\deg g_0 \geqq 1$

(c_0) $a g_0 + b h_0 - 1 \in \mathfrak{m}(A)[t]$ *for some* $a, b \in A[t]$,

then there exist $g, h \in A[t]$ *such that*

(a) $f = gh$

(b) g *is monic and* $\deg g = \deg g_0$

(c) $g - g_0, h - h_0 \in \mathfrak{m}(A)[t]$.

PROOF. Put $\mathfrak{p}_r = \bigoplus_{i=1}^{r} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]]$ for $r \in \{0, \cdots, n\}$. Then $\mathfrak{p}_{r-1} \subset \mathfrak{p}_r$ $(1 \leqq r \leqq n)$ and $Spec\ A = \{\mathfrak{p}_0, \mathfrak{p}_1, \cdots, \mathfrak{p}_n\}$. By the repeated use of the fact that any complete valuation ring of dimension one is Henselian, there exist $g_r, h_r \in A[t]$ such that

(a_r) $f - g_r h_r \in \mathfrak{p}_{n-r}[t]$

(b_r) g_r is monic and $\deg g_r = \deg g_0$

(c_r) $g_r - g_0, h_r - h_0 \in \mathfrak{m}(A)[t]$,

for any $r \in \{0, \cdots, n\}$. Then $g = g_n$ and $h = h_n$ satisfy the conditions (a), (b) and (c).

COROLLARY 1. *Let* $f = a_0 t^n + a_1 t^{n-1} + \cdots + a_{n-1} t + a_n \in A[t]$. *If* $a_0 \in \mathfrak{m}(A)$, $a_0 \neq 0$ *and there exists* $i \in \{1, \cdots, n-1\}$ *such that* $a_i \in A^\times$, *then* f *is reducible in* $A[t]$.

COROLLARY 2. *Put* $K = k((t_n)) \cdots ((t_1))$. *Then*

$$f(0) \in A \implies f(t) \in A[t]$$

for any irreducible polynomial $f(t)$ *in* $K[t]$.

COROLLARY 3. *Put* $K = k((t_n)) \cdots ((t_1))$. *Then for any finite extension* L/K *of fields, the integral closure* B *of* A *in* L *is a valuation ring of* L *and* $B = N_{L/K}^{-1}(A)$. *Therefore* B *is the unique valuation ring of* L *that satisfies* $A = K \cap B$, *that is,* $Zar_{L|K}^{-1}(A) = \{B\}$.

3. Here we shall prove Theorem 2, (i).

First we consider the relationship between the completion with respect to the linear topologies and the Cauchy's completion.

LEMMA 5. *Suppose that a separable linear topology of a ring* A *has a fundamental system of neighborhoods* Σ *of* 0 *consisting of ideals in* A. *Let* $C(A)$ *denote the set of Cauchy sequences in* A, $C_1(A)$ *the set of convergent sequences in* A *and* $C_0(A)$ *the set of all sequences which converge to* 0. *Moreover we put* $\hat{A} = \text{proj. lim } A/\mathfrak{a}$ $(\mathfrak{a} \in \Sigma)$ *and define a ring homomorphism* $\varphi : C(A) \to \hat{A}$ *as follows*: *Take* $(a_i)_{i=0}^{\infty} \in C(A)$. *Then for any* $\mathfrak{a} \in \Sigma$, *there exists an integer* $n_\mathfrak{a}$ *such that* $i, j \geqq n_\mathfrak{a} \Rightarrow a_i - a_j \in \mathfrak{a}$. *Here we put*

$$\varphi((a_i)_{i=0}^{\infty}) = (a_{n_\mathfrak{a}} \bmod \mathfrak{a})_{\mathfrak{a} \in \Sigma} \in \hat{A}\,.$$

Then

(i) *the diagram*

$$\begin{array}{ccccccccc} 0 & \to & C_0(A) & \to & C(A) & \to & \hat{A} & & & \text{(exact)} \\ & & \| & & \uparrow & & \uparrow & & & \\ 0 & \to & C_0(A) & \to & C_1(A) & \to & A & \to & 0 & \text{(split exact)} \end{array}$$

commutes.

(ii) *If $A \cong \hat{A}$, then φ is surjective and $C_1(A) = C(A)$.*

(iii) *If A satisfies the first countability axiom, in other words if A is metrizable, then φ is surjective. Therefore $C(A)/C_0(A) \cong \hat{A}$ and*

$$A \cong \hat{A} \iff C_1(A) = C(A)\,.$$

The proof is easy.

COROLLARY. *If A is metrizable, then $\hat{A}$ is the completion of A as metric spaces.*

LEMMA 6. *Let A be a valuation ring, $K = QA$, L/K an extension of fields and $B \in Zar_{L|K}^{-1}(A)$. We introduce the following three conditions for the family $\Sigma_1 = \{K \cap b\mathfrak{m}(B) \mid b \in B, b \neq 0\}$ and $\Sigma_2 = \{a\mathfrak{m}(B) \mid a \in A, a \neq 0\}$:*

(3) *The linear topology on K defined by Σ_1 coincides with the A-topology.*

(4) *The linear topology on L defined by Σ_2 is separable.*

(5) *The linear topology on L defined by Σ_2 coincides with the B-topology.*

Then

(i) (3) $\Leftarrow$ (4) $\Leftrightarrow$ (5).

(ii) *If $A \neq K$, then* (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5).

(iii) *If $A = K$ and $B \neq L$, then* (3) *holds but* (4) *and* (5) *do not hold.*

The proof is easy.

LEMMA 7. *Let A be a valuation ring, $K = QA$ and L/K an extension of fields.*

(i) *If $B \in Zar_{L|K}^{-1}(A)$ and the extension $A \subset B$ is integral, then (3), (4) and (5) hold.*

(ii) *If the extension L/K is finite and $B = N_{L/K}^{-1}(A)$ satisfies the condition* (1), *then* (3), (4) *and* (5) *hold.*

The proof is easy.

EXAMPLE 3. Let A be a valuation ring and $K = QA$. For an indeterminate t over K, we put $L = K(t)$.

(i) If $B = A[t]_{\mathfrak{m}(A)[t]}$, then (3), (4) and (5) hold.

(ii) If $A \neq K$ and $B = A \oplus tK[t]_{(t)}$, then (3), (4) and (5) do not hold.

(ii′) If $A = K$ and $B = K[t]_{(t)}$, then (3) holds but (4) and (5) do not hold.

LEMMA 8. *Let A be a complete valuation ring, $K = QA$, L/K an extension of fields and $B \in Zar_{L|K}^{-1}(A)$. Assume that the linear topology on L defined by Σ_2 is separable.*

Take linearly independent elements $w_1, \cdots, w_r \in L$ *over* K, *and put* $b_n = \sum_{i=1}^{r} a_{ni} w_i$ *for sequences* $(a_{ni})_{n=0}^{\infty}$ $(1 \leqq i \leqq r)$ *in* K.

(i) $(b_n)_{n=0}^{\infty}$ *is a Cauchy sequence in* L, *if and only if* $(a_{ni})_{n=0}^{\infty}$ *are Cauchy sequences in* K *for all* $i \in \{1, \cdots, r\}$.

(ii) $(b_n)_{n=0}^{\infty}$ *converges to* 0, *if and only if* $(a_{ni})_{n=0}^{\infty}$ *converge to* 0 *for all* $i \in \{1, \cdots, r\}$.

The proof is easy.

LEMMA 9. *Let* A *be a valuation ring,* $K = QA$, L/K *a finite extension of fields and* $B \in Zar_{L|K}^{-1}(A)$.

(i) *If the linear topology on* L *defined by* Σ_2 *is metrizable, then*

$$A \text{ is complete} \implies B \text{ is complete}\,.$$

(ii) *If* $\dim A < +\infty$ *and the extension* $A \subset B$ *is integral, then*

$$A \text{ is strictly complete} \implies B \text{ is strictly complete}\,.$$

PROOF. (i) is induced from Lemma 5 and Lemma 8.

(ii) Take any $\mathfrak{q} \in Spec\ B$ and put $\mathfrak{p} = A \cap \mathfrak{q} \in Spec\ A$. Then the extension $A/\mathfrak{p} \hookrightarrow B/\mathfrak{q}$ is integral and the extension $Q(A/\mathfrak{p}) \hookrightarrow Q(B/\mathfrak{q})$ is finite. By Lemma 6 and Lemma 7, the linear topology on $Q(B/\mathfrak{q})$ defined by

$$\Sigma_2' = \{a \bmod \mathfrak{p} \cdot \mathfrak{m}(B/\mathfrak{q}) \mid a \in A, a \notin \mathfrak{p}\}$$

is metrizable. Therefore $B/\mathfrak{q}$ is complete by (i), and hence B is strictly complete.

Then the proof of Theorem 2, (i) is complete from Theorem 1, (i), Corollary 3 to Lemma 4 and Lemma 9, (ii).

4. Here we shall prove Theorem 2, (ii), (iii), (iv).

LEMMA 10. *For a field* k *and indeterminates* $t_1, \cdots, t_n$, *we put*

$$A = k \oplus \bigoplus_{i=1}^{n} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]]\,, \quad K = k((t_n)) \cdots ((t_1))\,.$$

For a finite extension K'/K *of fields, let* A' *denote the integral closure of* A *in* K'. *Moreover, following Theorem 1, (i) and Theorem 2, (i), we write*

$$A' = k' \oplus \bigoplus_{i=1}^{n} t_i' k'((t_n')) \cdots ((t_{i+1}'))[[t_i']]\,, \quad K' = k'((t_n')) \cdots ((t_1'))\,.$$

(i) $M = \begin{bmatrix} \mathrm{ord}_{A'}(t_1) \\ \vdots \\ \mathrm{ord}_{A'}(t_n) \end{bmatrix} \in M(n, \mathbf{Z})$ *is an upper triangular matrix and all the diagonal elements are positive.*

(ii) *Let $e_1, \cdots, e_n$ denote the diagonal elements of M. Then*

$$e_i = e(k'((t'_n)) \cdots ((t'_{i+1}))[[t'_i]]/k((t_n)) \cdots ((t_{i+1}))[[t_i]])$$

for any $i \in \{1, \cdots, n\}$.

PROOF. Put $\mathfrak{p}_r = \bigoplus_{i=1}^{r} t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]]$ for $r \in \{0, \cdots, n\}$.

(i) Since $t_i k((t_n)) \cdots ((t_{i+1}))[[t_i]] \subset \mathfrak{p}_i \subset \mathfrak{m}(A')$ for any $i \in \{1, \cdots, n\}$, we have $\mathrm{ord}_{A'}(t_i) > \sum_{j=i+1}^{n} m_j \mathrm{ord}_{A'}(t_j)$ for any $m_{i+1}, \cdots, m_n \in \mathbf{Z}$. This implies (i).

(ii) is induced from $(A/\mathfrak{p}_{i-1})_{(\mathfrak{p}_i/\mathfrak{p}_{i-1})} = k((t_n)) \cdots ((t_{i+1}))[[t_i]]$.

PROOF OF THEOREM 2, (ii). By Lemma 10, (i), we have $\det M = e_1 \cdots e_n$. By $\mathrm{ord}_{A'}(K^\times) = \mathbf{Z}^n M$, we have $e = (\mathbf{Z}^n : \mathbf{Z}^n M) = \mathrm{card}(\mathbf{Z}^n/\mathbf{Z}^n M) = e_1 \cdots e_n$. Therefore we obtain $e = \det M$. Moreover, by the theory of formal power series in one variable and Lemma 10, (ii), we get

$$[K' : K] = e_1[k'((t'_n)) \cdots ((t'_2)) : k((t_n)) \cdots ((t_2))] = \cdots = e_1 \cdots e_n[k' : k] = ef .$$

LEMMA 11. *Suppose that A is an equal characteristic strictly complete discrete valuation ring of dimension n and $K = QA$. For a finite extension K'/K of fields, let A' denote the integral closure of A in K'. Moreover, following Theorem* 1, (i) *and Theorem* 2, (i), *we write*

$$A' = k' \oplus \bigoplus_{i=1}^{n} t'_i k'((t'_n)) \cdots ((t'_{i+1}))[[t'_i]] , \quad K' = k'((t'_n)) \cdots ((t'_1)) .$$

Take $\alpha_1, \cdots, \alpha_f \in A'$ that satisfy

$$A'/\mathfrak{m}(A') = \bigoplus_{s=1}^{f} A/\mathfrak{m}(A)\ \alpha_s \bmod \mathfrak{m}(A') .$$

Then

$$K' = \bigoplus_{r_1=0}^{e_1-1} \cdots \bigoplus_{r_n=0}^{e_n-1} \bigoplus_{s=1}^{f} K\alpha_s t_1'^{r_1} \cdots t_n'^{r_n} ,$$

$$A' = \bigoplus_{r_1=0}^{e_1-1} \cdots \bigoplus_{r_n=0}^{e_n-1} \bigoplus_{s=1}^{f} A\alpha_s t_1'^{r_1} \cdots t_n'^{r_n} .$$

Here $e_1, \cdots, e_n$ are the diagonal elements of the matrix M defined in Theorem 2, (ii) and f is the relative degree of A'/A.

PROOF. Since the set $\{\alpha_s t_1'^{r_1} \cdots t_n'^{r_n} \mid 1 \leqq s \leqq f,\ 0 \leqq r_i \leqq e_i - 1\ (1 \leqq i \leqq n)\}$ is linearly independent over K, Theorem 2, (ii) implies Lemma 11.

Then the proof of Theorem 2, (iii) is obvious from Lemma 11.

PROOF OF THEOREM 2, (iv). Using Lemma 1, (ii), we can prove Theorem 2, (iv) by induction on $n = \dim A$.

REMARK. (i) The converse of Theorem 2, (iv) does not hold. See Example 4.

(ii) In Theorem 2, there exists a case that the extension $K \subset K'$ is Galois but the extension $A/\mathfrak{m}(A) \hookrightarrow A'/\mathfrak{m}(A')$ is not separable. Moreover, under the notation in Lemma 10, there exists a case that we can not take $k((t_n))\cdots((t_{i+1})) \subset k'((t'_n))\cdots((t'_{i+1})) \subset K'$ for some $i \in \{1, \cdots, n\}$. See Example 5.

EXAMPLE 4. Suppose that k is a field of characteristic p, which is not perfect. Put $A = k[[t]]$, $K = k((t))$. Moreover, for $a_0 \in k - k^p$, let K' denote the splitting field of a polynomial $T^p - a_0 \in k[T] \subset K[T]$ over K and A' the integral closure of A in K'. If we put $k' = k(\sqrt[p]{a_0})$, then $A' = k'[[t]]$, $K' = k'((t))$ and k' is the algebraic closure of k in K'. But the extension $A/\mathfrak{m}(A) \hookrightarrow A'/\mathfrak{m}(A')$ is purely inseparable of degree p.

EXAMPLE 5. Let k be a field of characteristic p ($p \neq 0$).

(i) Put $A = k[[t]]$, $K = k((t))$. Moreover, for $a_0 \in k^\times$, let K' denote the splitting field of a polynomial $T^p - t^{p-1}T - a_0 \in A[T] \subset K[T]$ over K and A' the integral closure of A in K'. Then K' is a cyclic extension of degree p over K and k is algebraically closed in K'. Therefore the following three conditions are equivalent:

(a) $a_0 \notin k^p$.

(b) The extension $A/\mathfrak{m}(A) \hookrightarrow A'/\mathfrak{m}(A')$ is purely inseparable of degree p.

(c) The valuation ring A' has no coefficient field which contains k.

(ii) Put $A = k[[t_2]] \oplus t_1 k((t_2))[[t_1]]$, $K = k((t_2))((t_1))$. Let K' denote the splitting field of a polynomial $T^p - t_1^{p-1}T - t_2 \in A[T] \subset K[T]$ over K and A' the integral closure of A in K'. Then K' is a cyclic extension of degree p over K and

$$f = e_1 = 1\,, \quad e_2 = p\,.$$

Here e_1 and e_2 are the diagonal elements of the matrix M. Therefore $k' = k$, but we can not take $k((t_2)) \subset k'((t'_2))$.

COROLLARY TO THEOREM 2. (i) *Suppose that there exist a coefficient field k of A and a coefficient field k' of A' such that $k \subset k'$. Then k' is the algebraic closure of k in K' and $k' \otimes_k K = k'K = k'((t_n))\cdots((t_1))$. Therefore*

$$e = [K' : k'K], \quad f = [k' : k] = [k'K : K]\,.$$

(ii) *The diagram*

$$\begin{array}{ccccccccl} 1 & \to & A^\times & \to & K^\times & \to & \mathbf{Z}^n & \to & 0 \quad \text{(split exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & A'^\times & \to & K'^\times & \to & \mathbf{Z}^n & \to & 0 \quad \text{(split exact)} \end{array}$$

commutes. Here down arrows are the inclusion mappings and $\gamma \mapsto \gamma M$ ($\gamma \in \mathbf{Z}^n$).

(iii) *The diagram*

$$\begin{array}{ccccccccc} 1 & \to & A'^{\times} & \to & K'^{\times} & \to & \mathbf{Z}^n & \to & 0 \quad \text{(split exact)} \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \to & A^{\times} & \to & K^{\times} & \to & \mathbf{Z}^n & \to & 0 \quad \text{(split exact)} \end{array}$$

commutes. Here down arrows are the norm mappings and $\gamma \mapsto \gamma f \tilde{M}$ $(\gamma \in \mathbf{Z}^n)$, *and* $\tilde{M}$ *denotes the adjugate matrix of* M.

PROOF. (i) is induced from Theorem 2, (ii).

(ii) is easy from the fact that $\mathrm{ord}_{A'}(x) = \mathrm{ord}_A(x)M$ for any $x \in K^{\times}$.

(iii) Since both the mappings $\mathrm{ord}_{A'}$ and $\mathrm{ord}_A \circ N_{K'/K}$ are additive valuations of K' corresponding to A', there exists $M_0 \in M(n, \mathbf{Z})$ such that $\mathrm{ord}_{A'}(x)M_0 = \mathrm{ord}_A(N_{K'/K}(x))$ for any $x \in K'^{\times}$. Here we assume $x \in K^{\times}$, then by (ii), we have $MM_0 = [K' : K]E_n$. By $\tilde{M}M = eE_n$ and Theorem 2, (ii), we obtain $M_0 = f\tilde{M}$, and hence $\mathrm{ord}_{A'}(x)f\tilde{M} = \mathrm{ord}_A(N_{K'/K}(x))$.

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