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Geometry of Reduced Sextics of Torus Type

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Abstract. In [7], we gave a classification of the configurations of singularities of irreducible sextic of torus type. In this paper, we give a classification of the configurations of singularities on reducible sextics of torus type. We determine the component types and the geometry of the components for each configuration.

1. Introduction and statement of the result

In our previous paper [7], we have classified the configurations of the singularities on irreducible sextics of torus type. In this note, we classify the configuration of singularities of reducible sextics of torus type. We use the same notations as in [7].

We denote by Σ_{in} and Σ_{out} the configurations of the inner singularities and of the outer singularities respectively. For the classification of the configurations of the reduced sextics of torus type, it is less important to distinguish inner or outer singularities but what is more important is to know the singularities of the irreducible components and their intersections. We put $\Sigma_{red} = \Sigma_{in} \cup \Sigma_{out}$, and we call it *the reduced configuration*. Let $B_{i_1}, \cdots, B_{i_k}$ be the irreducible components of C. We call $\{\deg B_{i_1}, \cdots \deg B_{i_k}\}$ *the component type* of a reducible sextic C. In this note, we always assume that the curves $B_i, B_i', \cdots$ are irreducible and their degrees are the same with the indices. Thus, for example, $C = B_1 + B_1' + B_4$ implies that C has three components of degree 1, 1, 4. The configurations of the singularities of B_i is denoted by $\Sigma(B_i)$. We say that C has the maximal rank if C has only simple singularities and the total Milnor number is 19. We denote configurations with maximal rank by upper suffix mr, like $[A_{11}, 2A_2, D_4]^{mr}$. The classification of reduced sextics of torus type with only simple singularities is given in Theorem 1 and the classification for the other case is given in Theorem 2.

1.1. Reduced sextics with simple singularities. We first classify the reduced sextics with simple singularities.

THEOREM 1. *The classifications of the configurations of singularities on reducible sextics of torus type with only simple singularities are given as follows.*

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(1) $\Sigma_{in} = [A_5, 4A_2] : C = B_5 + B_1$ *and* $[A_5, 4A_2, 2A_1]_2$, $[A_5, 4A_2, 3A_1]_2$, $[A_5, 5A_2, 2A_1]_2$, $[A_5, 4A_2, A_3, 2A_1]_2$, $[A_5, 4A_2, 4A_1]$, $[A_5, 4A_2, A_3]_2$, $[A_5, 4A_2, A_3, A_1]_2$, $[A_5, 4A_2, A_3, 2A_1]_3$, $[A_5, 4A_2, D_4]_2$, $[A_5, 4A_2, D_5]_2$.

(2) $\Sigma_{in} = [2A_5, 2A_2]$:
  (a) $C = B_1 + B_5$: $[2A_5, 2A_2, 2A_1]_2$, $[2A_5, 2A_2, 3A_1]_1$, $[2A_5, 3A_2, 2A_1]_1$, $[2A_5, 2A_2, A_3]_2$, $[2A_5, 2A_2, A_3, A_1]_1$, $[2A_5, 2A_2, D_4]_1$, $[2A_5, 2A_2, D_5]_1^{mr}$.
  (b) $C = B_1 + B_1' + B_4$: $[2A_5, 2A_2, 3A_1]_2$, $[2A_5, 2A_2, 4A_1]$, $[2A_5, 2A_2, D_4]_2$.
  (c) $C = B_2 + B_4$: $[2A_5, 2A_2, 2A_1]_3$, $[2A_5, 2A_2, 3A_1]_3$, $[2A_5, 3A_2, 2A_1]_2$, $[2A_5, 2A_2, A_3]_3$, $[2A_5, 2A_2, A_3, A_1]_2$, $[2A_5, 2A_2, D_4]_3$, $[2A_5, 2A_2, D_5]_2^{mr}$.
  (d) $C = B_3 + B_3'$: $[2A_5, 2A_2, 3A_1]_4$, $[2A_5, 2A_2, A_3, A_1]_3$, $[3A_5, 2A_2]^{mr}$.

(3) $\Sigma_{in} = [E_6, A_5, 2A_2]$: $C = B_1 + B_5$ *and* $[E_6, A_5, 2A_2, 2A_1]_2$, $[E_6, A_5, 2A_2, 3A_1]$, $[E_6, A_5, 2A_2, A_3]_2$, $[E_6, A_5, 2A_2, A_3, A_1]^{mr}$.

(4) $\Sigma_{in} = [3A_5]$:
  (a) $C = B_1 + B_5$: $[3A_5, 2A_1]_1$, $[3A_5, A_3]_1$.
  (b) $C = B_2 + B_4$: $[3A_5, 2A_1]_2$, $[3A_5, A_3]_2$.
  (c) $C = B_1 + B_1' + B_4$: $[3A_5, 3A_1]_1$, $[3A_5, D_4]_1^{mr}$.
  (d) $C = B_3 + B_3'$: $[3A_5]_2$, $[3A_5, A_1]_2$, $[3A_5, A_2]_2$, $[3A_5, 2A_1]_3$, $[3A_5, A_1, A_2]$, $[3A_5, 2A_2]^{mr}$.
  (e) $C = B_1 + B_2 + B_3$: $[3A_5, 2A_1]$, $[3A_5, 3A_1]_1$, $[3A_5, A_2, 2A_1]^{mr}$, $[3A_5, A_3]_3$, $[3A_5, A_3, A_1]^{mr}$.
  (f) $C = B_1 + B_1' + B_1'' + B_3$: $[3A_5, 3A_1]_2$, $[3A_5, 4A_1]^{mr}$, $[3A_5, D_4]_2^{mr}$.
  (g) $C = B_2 + B_2' + B_2''$: $[3A_5, 3A_1]_3$, $[3A_5, D_4]_3^{mr}$.

(5) $\Sigma_{in} = [2A_5, E_6]$:
  (a) $C = B_1 + B_5$: $[E_6, 2A_5, 2A_1]_1$, $[E_6, 2A_5, A_3]_1^{mr}$.
  (b) $C = B_2 + B_4$: $[E_6, 2A_5, 2A_1]_2$, $[E_6, 2A_5, A_3]_2^{mr}$.
  (c) $C = B_1 + B_1' + B_4$: $[E_6, 2A_5, 3A_1]^{mr}$.

(6) $\Sigma_{in} = [A_8, A_5, A_2]$: $C = B_1 + B_5$ *and* $[A_8, A_5, A_2, 2A_1]_2$, $[A_8, A_5, A_2, 3A_1]$, $[A_8, A_5, A_2, A_3]_2$, $[A_8, A_5, A_2, A_3, A_1]^{mr}$, $[A_8, A_5, A_2, D_4]^{mr}$.

(7) $\Sigma_{in} = [A_{11}, 2A_2]$:
  (a) $C = B_2 + B_4$: $[A_{11}, 2A_2, 2A_1]_2$, $[A_{11}, 2A_2, 3A_1]_1$, $[A_{11}, 3A_2, 2A_1]^{mr}$, $[A_{11}, 2A_2, A_3]_2$, $[A_{11}, 2A_2, D_4]^{mr}$.
  (b) $C = B_3 + B_3'$: $[A_{11}, 2A_2, 3A_1]_2$, $[A_{11}, 2A_2, A_3, A_1]$.

(8) $\Sigma_{in} = [A_{11}, A_5]$:
  (a) $C = B_1 + B_5$: $[A_{11}, A_5, 2A_1]_1$, $[A_{11}, A_5, A_3]_1^{mr}$.
  (b) $C = B_2 + B_4$: $[A_{11}, A_5, 2A_1]_2$, $[A_{11}, A_5, A_3]_2^{mr}$.
  (c) $C = B_3 + B_3'$: $[A_{11}, A_5]_2$, $[A_{11}, A_5, A_1]_2$, $[A_{11}, A_5, A_2]_2$, $[A_{11}, A_5, 2A_1]_3$, $[A_{11}, A_5, A_2, A_1]^{mr}$.
  (d) $C = B_1 + B_2 + B_3$: $[A_{11}, A_5, 2A_1]$, $[A_{11}, A_5, 3A_1]^{mr}$, $[A_{11}, A_5, A_3]_3^{mr}$.

(9) $\Sigma_{in} = [A_{17}]$: $C = B_3 + B_3'$, $[A_{17}]_2$, $[A_{17}, A_1]_2$, $[A_{17}, 2A_1]^{mr}$, $[A_{17}, A_2]_2^{mr}$.

Further geometrical informations are explained in the proof in Section 3. In the above theorem, the lower index, like $[A_5, 4A_2, A_3, 2A_1]_2$, is to distinguish other component with the same weak Zariski configuration. The index 1 is reserved for the irreducible case if there exists an irreducible sextics. Thus the configuration which start from index 2, like $[A_5, 4A_2, A_3, 2A_1]_2$, implies that there is an irreducible sextics with the same configuration.

THEOREM 2. *The reduced configurations with at least one non-simple singularities are given by the following.*

(1) $B_{3,6} \in \Sigma_{in}$:
 (a) $C = B_1 + B_5 : [B_{3,6}, 3A_2, A_1]_2$, $[B_{3,6}, 3A_2, 2A_1]$, $[B_{3,6}, 4A_2, A_1]$, $[B_{3,6}, A_5, A_2, A_1]_1$, $[B_{3,6}, E_6, A_2, A_1]$, $[B_{3,6}, A_8, A_1]$.
 (b) $C = B_1 + B_1' + B_4 : [B_{3,6}, A_5, A_2, 2A_1]_1$.
 (c) $C = B_1 + B_2 + B_3 : [B_{3,6}, A_5, A_2, 2A_1]_2$.
 (d) $C = B_2 + B_4 : [B_{3,6}, A_5, A_2, A_1]_2$.
 (e) $C = B_2 + B_2' + B_2'' : [2B_{3,6}]$.
(2) $C_{3,7} \in \Sigma_{in}$:
 (a) $C = B_1 + B_5 : [C_{3,7}, 3A_2, A_1]_2$, $[C_{3,7}, 3A_2, 2A_1]$, $[C_{3,7}, A_5, A_2, A_1]_1$, $[C_{3,7}, E_6, A_2, A_1]$, $[C_{3,7}, A_8, A_1]$.
 (b) $C = B_2 + B_4 : [A_2, A_5, C_{3,7}, A_1]_2$.
 (c) $C = B_1 + B_1' + B_4 : [C_{3,7}, A_5, A_2, 2A_1]$.
(3) $C_{3,8} \in \Sigma_{in}$:
 (a) (a-1) $C = B_1 + B_5$ *and* $(B_5, O) = A_3 : [C_{3,8}, 3A_2]_2$, $[C_{3,8}, 3A_2, A_1]_1$, $[C_{3,8}, A_5, A_2]$, $[C_{3,8}, E_6, A_2]$, $[C_{3,8}, A_8]$.
 (a-2) $C = B_1 + B_5$ *and* $(B_5, O) = A_5 : [C_{3,8}, 3A_2, A_1]_2$.
 (b) $C = B_2 + B_4 : [C_{3,8}, A_5, A_2]$.
 (c) $C = B_1 + B_2 + B_3 : [C_{3,8}, A_5, A_2, A_1]_1$, $[C_{3,8}, A_5, A_2, A_1]_2$.
(4) $C_{3,9} \in \Sigma_{in} : C = B_1 + B_5$ *and* $[C_{3,9}, 2A_2, A_1]_2$, $[C_{3,9}, 2A_2, 2A_1]$, $[C_{3,9}, 3A_2, A_1]$, $[C_{3,9}, A_5, A_1]$, $[C_{3,9}, E_6, A_1]$.
(5) $C_{3,12} \in \Sigma_{in}$:
 (a) $C = B_1 + B_5 : [C_{3,12}, A_2, A_1]_1$.
 (b) $C = B_2 + B_4 : [C_{3,12}, A_2, A_1]_2$.
 (c) $C = B_1 + B_2 + B_3 : [C_{3,12}, A_2, 2A_1]$.
(6) $C_{6,6} \in \Sigma_{in}$:
 (a) $C = B_1 + B_5 : [C_{6,6}, 2A_2, A_1]$.
 (b) $C = B_1 + B_1' + B_4 : [C_{6,6}, 2A_2, 2A_1]_1$.
 (c) $C = B_3 + B_3' : [C_{6,6}, A_5]$.
 (d) $C = B_3 + B_2 + B_1$*:* $[C_{6,6}, A_5, A_1]$.
 (e) $C = B_1 + B_1' + B_2 + B_2' : [C_{6,6}, A_5, 2A_1]_2$.
(7) $C_{6,9} \in \Sigma_{in} : C = B_1 + B_5$, $[C_{6,9}, A_2, A_1]$.
(8) $C_{3,15} \in \Sigma_{in} : C = B_1 + B_5$, $[C_{3,15}, A_1]$.
(9) $B_{3,12} \in \Sigma_{in}$*:* $C = B_2 + B_2' + B_2''$, $[B_{3,12}]$.

(10) $C_{6,12} \in \Sigma_{in}$:
 (a) $C = B_3 + B_3'$: $[C_{6,12}]$.
 (b) $C = B_1 + B_2 + B_3$: $[C_{6,12}, A_1]$.
(11) $B_{4,6} \in \Sigma_{in}$: $C = B_3 + B_3'$ *and* $[B_{4,6}, A_5]$.
(12) $D_{4,7} \in \Sigma_{in}$:
 (a) $C = B_1 + B_5$: $[D_{4,7}, 2A_2]$.
 (b) $C = B_1 + B_2 + B_3$: $[D_{4,7}, A_5]$.
(13) $Sp_2 \in \Sigma_{in}$: $C = B_3 + B_3'$ *and* $[Sp_2]$.
(14) $B_{6,6} \in \Sigma_{in}$: $[B_{6,6}]$.

2. Preliminaries

2.1. Genus formula and the class formula. Let C be an irreducible plane curve of a given degree d. Then the genus formula is given as

$$g(C) = \frac{(d-1)(d-2)}{2} - \sum_{P \in \Sigma(C)} \delta(P) \geq 0, \quad \delta(P) = \frac{\mu(C,P) + r(C,P) - 1}{2}$$

where $\mu(C,P)$ and $r(C,P)$ is the Milnor number and the number of local irreducible components ([3]). Let $\delta^*(C) = \sum_{P \in \Sigma(C)} \delta(P)$. Using the above inequality, we have $\delta^*(C) \leq 6, 3, 1, 0$ respectively for $d = 5, 4, 3, 2$ for an irreducible curve C. Now assume that C is not irreducible. Let $C = B_{i_1} + \cdots + B_{i_k}$ be the irreducible decomposition of C with degree$(B_{i_k}) = i_k$. We define $\delta^*(C) = \sum_{j=1}^{k} \delta^*(B_{i_j})$.

PROPOSITION 3. *Assume that C is a reducible sextic and let $C = B_{i_1} + \cdots + B_{i_k}$ be the irreducible decomposition. Then*

$$\delta^*(C) \leq \begin{cases} 6, & \text{if} \quad C = B_5 + B_1, \\ 3, & \text{if} \quad C = B_4 + B_2, \quad \text{or} \ B_4 + B_1 + B_1', \\ 2, & \text{if} \quad C = B_3 + B_3', \\ 1, & \text{if} \quad C = B_3 + B_2 + B_1, \text{or} \ B_3 + B_1 + B_1' + B_1'', \\ 0, & \quad\quad \text{otherwise}. \end{cases}$$

The class formula describes the degree $n^*(C)$ of the dual curve and it is given by the following formula ([4]).

$$n^*(C) = d(d-1) - \sum_{P \in \Sigma(C)} (\mu(C,P) + m(C,P) - 1).$$

The number of flex points $i(C)$ counted with multiplicity is given by

$$i(C) = 3d(d-2) - \sum_{P \in \Sigma(C)} \text{flex defect(C, P)}.$$

For the definition of flex defect, we refer Oka [6]

2.2. Intersection singularities. Let C be a plane curve and let $C^1, \cdots, C^k$ be the irreducible components. Let P be a singular point of C. We say that P is *a proper singularity* if $P \in C^i - \bigcup_{j\neq i} C^j$ for some component C^i. Otherwise we say that (C, P) *an intersection singularity*. Assume that $C = C^1 \cup C^2$, for example, and $P \in C^1 \cap C^2$ and C^1, C^2 are non-singular at P and let ι be the local intersection number. Then $(C, P) \cong A_{2\iota-1}$. Assume further that C^1 is a line and $\iota \geq 3$. Then we say that C^1 is a flex tangent line of C^2.

PROPOSITION 4. *Assume that C passes through $O = (0,0)$ and C is defined by $f(x, y) = 0$ and assume that the Newton boundary $\Gamma(f)$ is non-degenerate. Let $\Delta_1, \cdots, \Delta_k$ be the faces of $\Gamma(f)$ and assume that $f_{\Delta_i}(x, y) = \prod_{j=1}^{\nu_i}(y^{a_i} - \alpha_j x^{b_i})$ with $\gcd(a_i, b_i) = 1$ and $\alpha_1, \cdots, \alpha_{\nu_i}$ are mutually distinct. Then C has $\sum_{i=1}^k \nu_i$ local irreducible components at O which are different from the coordinate axis $x = 0$ or $y = 0$ and the defining equations can be written as $(y^{a_i} - \alpha_j x^{b_i}) + (higher\ terms) = 0$.*

See for example [5].

EXAMPLE 5. 1. Consider $D_4 : y^2x - x^3 = 0$. Then D_4 can be an intersection singularity of three smooth components, $x = 0$, $y \pm x = 0$ where each two of them intersect transversely. Similarly $D_5 : y^2x + x^4 = 0$ can be interpreted as an intersection singularity of a line $x = 0$ and a cusp $y^2 + x^3 = 0$.

2. Consider the singularity $C_{3,p} :\ y^3 + y^2x^2 - x^p = 0$.

Case 1. Assume that p is odd. Then $C_{3,p}$ has two local irreducible components. One component is smooth and is defined by $L : y + x^2 + (\text{higher terms}) = 0$ and another component M is defined by $y^2 - x^{p-2} + (\text{higher terms}) = 0$ and it is an A_{p-3}-singularity and $I(L, M; O) = 4$.

Case 2. Assume that p is even and put $p = 2m$, $m \geq 4$. Then $C_{3,2m}$ has three smooth components L_1, L_2, L_3 where $L_1 : y + x^2 + (\text{higher terms}) = 0$ and L_2, L_3 are defined by $y \pm x^{m-1} + (\text{higher terms}) = 0$. Note that $(L_2 \cup L_3, O) \cong A_{2m-3}$ and $(L_1 \cup L_2, O) \cong A_3$.

We use the following notations for non-simple singularities as in [8].

$$\begin{cases} B_{p,q} :\ y^p + x^q = 0 \text{ (Brieskorn-Pham type)} \\ C_{p,q} :\ y^p + x^q + x^2y^2 = 0, \quad \frac{2}{p} + \frac{2}{q} < 1 \\ D_{4,7} :\ y^4 + x^3y^2 + ax^5y + bx^7 = 0, \quad a^2 - 4b \neq 0 \\ Sp_1 :\ (y^2 - x^3)^2 + (xy)^3 = 0 \\ Sp_2 :\ (y^2 - x^3)^2 - y^6 = 0\,. \end{cases}$$

Hereafter we only consider sextics of torus type $C : f_2(x, y)^3 + f_3(x, y)^2 = 0$. The notation $C_2 : f_2(x, y) = 0$ and $C_3 : f_3(x, y) = 0$ is used throughout the paper.

2.3. Weak Zariski k-ple. A k-ple of reduced plane curves $\{C^1, \cdots, C^k\}$ is called *a weak Zariski k-ple* if degree$(C^1) = \cdots =$ degree(C^k) and they have same reduced configuration of singularities and the topology of the pair $(\mathbf{P}^2, C^j)$ are all different. We call $\Sigma(C^j)$ *a weak Zariski configuration.* Note that $C^1, \cdots, C^k$ may have different component types. Artal has first observed such a pair for $[A_{17}]$ or some others in sextics [1]. It is obvious that a Zariski pair is a weak Zariski pair.

2.4. Sextics of linear torus. A sextics C of torus type is called *of linear torus type* if C can be defined by $f(x, y) = f_2(x, y)^3 + f_3(x, y)^2$ where $f_2(x, y) = (ax + by + c)^2$. We may assume that $f_2 = -y^2$ by a linear change of coordinates so that f is a product of cubic forms $f(x, y) = (f_3(x, y) + y^3)(f_3(x, y) - y^3)$. It is easy to observe that the inner singularities are on $y = f_3(x, 0) = 0$. In particular, there are at most three inner singularities and they are colinear.

PROPOSITION 6. *The possibility of inner configuration of sextics of linear torus type is either* $[3A_5]$, *or* $[A_{11}, A_5]$ *or* $[A_{17}]$ *for simple singularities and for non-simple singularities, we have* $[C_{6,6}, A_5]$, $[B_{4,6}, A_5]$, $[D_{4,7}, A_5]$, $[C_{6,12}]$, $[Sp_2]$ *and* $[B_{6,6}]$.

PROOF. Assume that $f_3(\alpha, 0) = 0$. Assume that α is a simple solution of $f_3(x, 0) = 0$ (respectively a solution of multiplicity 2 or 3). Put $P = (\alpha, 0)$. Then $I(y^2, C_3; P) = 2$ (resp. 4 or 6). Let C^1, C^2 be the cubic defined by $f_3(x, y) \pm y^3 = 0$. If C^1, C^2 are non-singular at P, then $P \in C$ is an intersection singularity, and (C, P) is isomorphic to A_5, A_{11} or A_{17} depending to the multiplicity.

Assume that P is an singular point of C^1 and C^2. Then (C, P) can not be E_6 as (C, P) is not irreducible and the assertion follows from the classification of [8] and Theorem 2. □

Assume that C is a sextics with $3A_5$ or $A_{11} + A_5$ or A_{17}. We denote the location of these singularities by $\{P_1, P_2, P_3\}$ (respectively, $\{P_1, P_2\}$ or $\{P_1\}$) which are assumed to be mutually distinct. We say that C is *of linear type* if there is a line $L \subset \mathbf{P}^2$ such that

$$L \cap C = \begin{cases} \{P_1, P_2, P_3\}, I(C, L; P_i) = 2\,, & (C, P_i) \cong A_5 \\ \{P_1, P_2\}, I(C, L; P_1) = 2, I(C, L; P_2) = 4\,, & (C, P_1) \cong A_5, (C.P_2) \cong A_{11} \\ \{P_1\}, I(C, L; P_1) = 6\,, & (C, P_1) \cong A_{17}\,. \end{cases}$$

The following is the converse of Proposition 6.

PROPOSITION 7. *Assume that C is a reduced sextics with* $3A_5$ *or* $A_{11} + A_5$ *or* A_{17}. *Assume that*

1. *C is a sextics of linear type or*
2. *C is a sextics of torus type which is a union of two cubics.*

Then C is of linear torus type.

We give a computational proof in Appendix in §6.

REMARK 8. There exists sextics of non-torus type which has the decomposition type $C = B_3 + B_3'$ with $3A_5$ or $A_{11} + A_5$ or A_{17} which are not colinear. In fact, in the space of sextics, the moduli of sextics with configuration $[3A_5]$, $[A_{11}, A_5]$ or $[A_{17}]$ consists of 4 components: irreducible non-torus sextics, irreducible sextics of torus type, non-torus sextics with two cubics components, sextics of linear torus type. The assertion is shown by Artal [1] for the configuration $[A_{17}]$.

3. Proof of Theorem 1

3.1. Elimination of other configurations. We assume that sextics have only simple singularities. A main step to the proof is to list the possible configurations, eliminating other configurations. This process can be done by fixing each inner configuration. The proof of the existence for the survived maximal configurations is given by constructing explicit examples (in next subsection), and for other configurations, we leave it to the reader. In the following, $B_i, B_i', \cdots$ are assumed to be an irreducible component of degree i. By [8], the possible inner configurations are the combinations of $A_2, A_5, A_8, A_{11}, A_{14}, A_{17}, E_6$.

First consider the case $\Sigma_{in}(C) = [6A_2]$. This implies $\delta^*(C) \geq 6$. Assume that C is not irreducible. As A_2 is an irreducible singularity, it is not possible unless $C = B_5 + B_1$. However there is no quintic B_5 with $6A_2$, as $n^*(B_5) = 2$. (The conics are self-dual.)

The configurations $\Sigma_{in} = [4A_2, E_6], [2A_2, 2E_6], [3A_2, A_8], [A_2, A_{14}]$ are impossible to be on a reducible sextic curve as $\delta^* \geq 7$. Now we consider the other cases.

1. Assume that $\Sigma_{in}(C) = [4A_2, A_5]$. Then $\delta^*(C) \geq 5$ and the only possibility is the case: $C = B_1 + B_5$. If this is the case, B_1 must be a flex tangent line of B_5 and $\Sigma(B_5) = [4A_2]$ generically. Note also $B_1 \cap B_5 = A_5 + 2A_1$ or $A_5 + A_3$ if the intersections are on their smooth points.

Notation. *Here the equality $B_1 \cap B_5 = A_5 + 2A_1$ implies that the intersection of B_1 and B_5 are three distinct points, and the equivalence classes of the intersection singularities of $B_1 \cup B_5$ are A_5 and two A_1 respectively. We use this abuse of notation throughout the paper.*

Under the assumption $\Sigma_{in}(C) = [4A_2, A_5]$, B_5 can take further $A_1, A_2, A_3, A_4, 2A_1$ by the genus formula. There are no quintic with $4A_2 + A_4$ or $5A_2 + A_1$. In fact, if there is such a quintic, $n^*(B_5) = 3$ in both cases. However this is impossible by the following well-known fact.

Fact 1. *The dual of an irreducible smooth* (*resp. nodal, or cuspidal*) *cubic B_3 is a 9 cuspidal sextic* (*respectively* 3 *cuspidal quartic or cuspidal cubic*).

Note that A_5 must be on $B_1 \cap B_5$. Assume first $B_1 \cap B_5$ is $A_5 + 2A_1$. The configurations corresponding to the degeneration of the quintic is: $[A_5, 4A_2, 2A_1]$, $[A_5, 4A_2, 3A_1]$, $[A_5, 5A_2, 2A_1]_2$, $[A_5, 4A_2, A_3, 2A_1]_1$ and $[A_5, 4A_2, 4A_1]$.

Assume that $B_1 \cap B_5 = A_5 + A_3$. Then we can insert to B_5 either A_1 or $2A_1$ but we can not insert any other singularity. Thus we have $[A_5, 4A_2, A_3]_2$, $[A_5, 4A_2, A_3, A_1]_2$ and

$[A_5, 4A_2, A_3, 2A_1]_2$. In fact, assume that $\Sigma(B_5) = [5A_2]$. Then $n^*(B_5) = 5$ and $i(B_5) = 5$. Thus 5 cuspidal quintics are self dual. However if $B_1 \cap B_5 = A_5 + A_3$, B_1 is a flex tangent which is also tangent at another point. This implies one A_2 of B_5^* has to be replaced by D_5 (= the dual singularity of $(B_5, B_1 \cap B_5)^*$) which is impossible.

The exceptional cases $[4A_2, A_5, D_4]$ and $[4A_2, A_5, D_5]$ are given when $\Sigma(B_5) = 4A_2 + A_1$ or $5A_2$ respectively and the line component B_1 passes through the last outer A_1 or A_2. Note that the sextics with one of the above configurations can be degenerated into one of $[A_5, 4A_2, A_3, 2A_1]_1$ or $[A_5, 4A_2, A_3, 2A_1]_2$ or $[A_5, 4A_2, D_5]$.

There are further degenerations $[A_5, 4A_2, A_3, 2A_1]_1 \to [A_5, C_{3,7}, A_2, A_1]$ (5.3-1), $[A_5, 4A_2, A_3, 2A_1]_3 \to [A_5, E_6, A_3, 2A_2, A_1]_2^{mr}$ (5.3-2) and $[4A_2, A_5, D_5]_2 \to [2A_5, 2A_2, D_5]_1^{mr}$ (5.3-3) . We will give later explicit examples of these degenerations in 5.3. So the existence of the above configurations follows from the existence of these three configurations $[A_5, 4A_2, A_3, 2A_1]_1$, $[A_5, 4A_2, A_3, 2A_1]_3$ and $[A_5, 4A_2, D_5]_2$.

Note that $[A_5, 4A_2, A_3, 2A_1]_i,\ i = 1, 2$ is an interesting weak Zariski configuration: Both has the same decomposition type $B_1 + B_5$ but $\Sigma(B_5) = [4A_2, A_3]$ and $B_1 \cap B_5 = A_5 + 2A_1$ (respectively $\Sigma(B_5) = [4A_2, 2A_1]$ and $B_1 \cap B_5 = A_5 + A_3$) for $[A_5, 4A_2, A_3, 2A_1]_1$ (resp. for $[A_5, 4A_2, A_3, 2A_1]_2$). To distinguish them, we put the index 1 or 2. The configurations $[A_5, 4A_2, 2A_1]$, $[A_5, 4A_2, 3A_1]$ are also weak Zariski configurations as there exist irreducible sextics with these configurations ([7]). Hereafter we do not list up the weak Zariski configurations. They can be read from the indices.

2. Now we consider the case $\Sigma_{in}(C) = [2A_2, 2A_5]$.

(a) Consider the component type $C = B_1 + B_5$. Then $\Sigma(B_5) = [2A_2, A_5]$ and $B_1 \cap B_5 = A_5 + 2A_1$ or $A_5 + A_3$. We can put at most one A_1 or A_2 in B_5. In the case $B_1 \cap B_5 = A_5 + A_3$, we assert that A_2 can not be inserted in B_5. In fact, assume that B_5 is a quintic with $3A_2 + A_5$. Then $n^*(B_5) = 5$ and the dual curve B_5^* has the same singularities, as $i(C) = 3$ and A_5 is self-dual ([6]). If $B_1 \cap B_5 = A_5 + A_3$, B_1 is a flex tangent and the dual singularity is D_5, but this is impossible as the dual curve B_5^* can not have $A_5 + 2A_2 + D_5$. However the configurations $[A_5, 2A_2, D_4]$ and $[A_5, 2A_2, D_5]_1^{mr}$ (see 5.1-3) are possible by putting the above extra A_1 or A_2 on $B_1 \cap B_5$. Note that we have a degeneration $[2A_5, 3A_2, 2A_1] \to [2A_5, 2A_2, D_5]_1^{mr}$

(b) Assume that $C = B_4 + B_1 + B_1'$. Then to have $2A_5 + 2A_2$, B_4 must have two cusps and B_1, B_1' must be flex tangents. Thus the configuration is generically $[2A_5, 2A_2, 3A_1]$. The configuration $[2A_5, 2A_2, D_4]$ is given when two lines B_1, B_1' intersect on B_4. Furthermore B_4 can have one more node (so $[2A_5, 2A_2, 4A_1]$, see 5.3-4) but it can not have three cusp. In fact, if B_4 has three cusps, B_4^* is a nodal cubic. This is impossible as B_4^* have at least two cusps.

(c) Assume that $C = B_2 + B_4$, $\Sigma(B_4) = [2A_2]$ and $B_2 \cap B_4 = 2A_5 + 2A_1$ generically. We can put either A_1 or A_2 on B_4. See 5.3-5. Consider the case $B_2 \cap B_4 = 2A_5 + A_3$. Then we can only insert A_1 into B_4. We can put A_2 into B_4 only on $B_2 \cap B_4$ so that we get $[2A_5, 2A_2, D_5]_2^{mr}$ (see 5.1-3). The case $\Sigma(B_4) = [3A_2]$ and $B_2 \cap B_4 = 2A_5 + A_3$ does not occur. In fact, assume that $B_2 \cap B_4 = 2A_5 + A_3$ and $\Sigma(B_4) = [3A_2]$. Note that the dual curve B_2^* is a conic and the dual B_4^* is cubic. Now the assumption implies that $B_2^* \cap B_4^* = 2A_5 + A_3$ which is impossible by Bezout theorem.

(d) Assume that $C = B_3 + B_3'$. Then the cubics are cuspidal and $B_3 \cap B_3' = 2A_5 + 3A_1$ (generic) or $2A_5 + A_3 + A_1$ or $3A_5$. This is the most difficult case to find explicit examples. See the next section for explicit examples (see 5.1-1). The case $[[2A_5, 2A_2], [A_5]]$ coincides with $[[3A_5], [2A_2]]$. This corresponds to the fact that this configuration has two torus expressions (see 5.1-6). Note that every configurations with $\Sigma_{in} = [2A_5, 2A_2]$ except $[2A_5, 2A_2, 4A_1]$ is a weak Zariski configuration. For example, $[2A_5, 2A_2, 3A_1]$ has 4 different cases.

3. Assume that $\Sigma_{in} = [E_6, A_5, 2A_2]$. Then $\delta^* \geq 5$ and the possibility is $C = B_1 + B_5$, $\Sigma(B_5) = [E_6, 2A_2]$ and $B_1 \cap B_5 = A_5 + 2A_1$ or $A_5 + A_3$. As there is no quintic with $[E_6, 3A_2]$ by the dual curve discussion, we can put at most one A_1. This gives the configurations in the list. The added A_1 can not be on B_1. This has to be checked by a direct computation or it also follows from Yang, [11], as $[E_6, A_5, 2A_2, D_4]$ does not exist. Note that $[E_6, A_5, 2A_2, 2A_1]$ and $[E_6, A_5, 2A_2, A_3]$ are weak Zariski configurations.

4. Assume that $\Sigma_{in} = [3A_5]$. In the case of (a)–(c) of No. 4 in Theorem 1, B_5 in (a), B_4 in (b) or (c) are rational. Thus the assertion is obvious except the existence. The case (a) is given by $\Sigma(B_5) = [2A_5]$ and $B_1 \cap B_5 = A_5 + 2A_1$ or $[A_5 + A_3]$. See 5.3-6. The case (b) is given by $C = B_2 + B_4$, $\Sigma(B_4) = [A_5]$ and $B_2 \cap B_4 = 2A_5 + 2A_1$ or $2A_5 + A_3$. See 5.3-7. The case (d), three A_5 are colinear by Proposition 7 and assuming they are on $y = 0$, the generic form is given by $f_3(x, y)^2 - y^6$, where $f_2 = -y^2$. Thus every configurations in (d) can be obtained by putting either A_1 or A_2 in the cubics. The cases (e), (f) are special cases of (d). In case (e), we can put only A_1 or A_2 in B_3. However we need to show that if $B_1 \cap B_2 = A_3$, B_3 can not be cuspidal. In fact, if such a sextics exists, it gives rank 20 configuration $[3A_5, A_3, A_2]$ which is known to be impossible ([9, 2]). The assertion of (f) is also easy to see as three line components are flex tangents and a nodal (respectively cuspidal) cubic has three flex points (resp. one flex point). The last configuration $[3A_5, D_4]$ is realized, when three line components intersect at a point.

The case (g) is the only non-trivial case. By Proposition 7, three A_5 can not be colinear. The normal form is given in 5.1. In this family the intersection of any two conic components gives $A_5 + A_1$. The maximal configuration $[3A_5, D_5]$ is given by $u = -1/4$ where three conics intersect at a point.

5. Assume that $\Sigma_{in} = [2A_5, E_6]$. Assume that $C = B_1 + B_5$ and $\Sigma(B_5) = [A_5, E_6]$ and therefore B_5 is rational. In the case $C = B_2 + B_4$ or $C = B_1 + B_1' + B_4$, $\Sigma(B4) = [E_6]$. In any case, B_4 is rational and the configurations have maximal ranks. Thus there are no further possibility. See 5.1-4,5.

6. Assume that $\Sigma_{in}(C) = [A_8, A_5, A_2]$. Then $C = B_1 + B_5$. The non-existence of $\Sigma(B_5) = A_8 + 2A_2$ with $\Sigma_{red} = [A_8, A_5, 2A_2, 2A_1]$ is checked by a direct computation.

This is also a result of Yang [11], as this is not in his table of maximal rank configuration. The $[A_8, A_5, 2A_2, A_3]$ does not exists as the rank is 20.

7. Assume that $\Sigma_{in}(C) = [A_{11}, 2A_2]$. It is easy to see that A_{11} must be an intersection singularity.

(a) Assume first $C = B_2 + B_4$. As B_4 has $2A_2$, it can take only A_1 or A_2. In the case $B_2 \cap B_4 = A_{11} + A_3$, it can be checked by computation that B_4 can not have further singularity. Namely $[A_{11}, A_3, 2A_2, A_1]$ does not appear from this series. This also follows from the connectedness of the moduli space of the sextics with the configuration $[2A_2, A_{11}, A_3, A_1]^{mr}$ (see [11]), as it exists for the component type $C = B_3 + B_3'$. See 5.3-8.

(b) Assume that $C = B_3 + B_3'$. Then two cubics are cuspidal and $B_3 \cap B_3' = A_{11} + 3A_1$ or $A_{11} + A_3 + A_1$. As the rank is bounded by 19, there are no possibility of $[A_{11}, A_5, 2A_2]$.

8. Assume that $\Sigma_{in} = [A_{11}, A_5]$.

(a) Assume that $C = B_1 + B_5$ and $\Sigma(B_5) = [A_{11}]$. Then B_4 is rational and we can not put any further singularity in B_5. As $B_1 \cap B_5 = A_5 + 2A_1$ or $A_5 + A_3$, the assertion is clear.

Assume that A_{11} is an intersection singularity. Then it implies either $C = B_2 + B_4$, $C = B_3 + B_3'$ or $C = B_1 + B_2 + B_3$.

(b) Assume that $C = B_2 + B_4$. Then $\Sigma(B_4) = [A_5]$ and $B_2 \cap B_4 = A_{11} + 2A_1$ or $A_{11} + A_3$ and the assertion is clear.

(c) Assume that $C = B_3 + B_3'$. We can make two cubics are tangent at two points with multiplicity 6 and 3 so that $B_3 \cap B_3' = A_{11} + A_5$. Now the assertion follows by putting A_1 or A_2 in the cubics. As the rank is 19 for $[A_{11}, A_5, A_2, A_1]$, we can not put A_2 in both cubics simultaneously.

(d) Assume that $C = B_3 + B_2 + B_1$. We can make them in the mutual position so that $B_3 \cap B_2 = A_{11}$, $B_1 \cap B_3 = A_5$ and $B_1 \cap B_2$ is either $2A_1$ or A_3 and B_3 is either smooth or nodal.

9. Assume that $\Sigma_{in} = [A_{17}]$. Then the only possibility is $C = B_3 + B_3'$ with $B_3 \cap B_3' = A_{17}$. The assertion is clear. See 5.1-17,18.

4. Proof of Theorem 2

In this section, we prove Theorem 2. As in the proof of Theorem 1, after eliminating non-existing configurations, we give a computational proof of existence and we give some non-trivial examples later. We first fix a non-simple inner singularity at the origin O and then we consider the possibility of inner configurations and component types.

1. Assume that $B_{3,6} \in \Sigma_{in}$. Recall that (C_2, O) is smooth, $(C_3, O) \cong A_1$ and $\iota := I(C_2, C_3; O) = 3$. Possible inner configurations are $[3A_2, B_{3,6}], [A_2, A_5, B_{3,6}]$, $[A_2, E_6, B_{3,6}]$, $[A_8, B_{3,6}]$, $[2B_{3,6}]$. We assume that $(C, O) \cong B_{3,6}$. First observe that has locally three smooth components C^1, C^2, C^3 with $I(C^i, C^j; O) = 2$ for $i \neq j$.

Thus if $B_{3,6}$ is an intersection singularity of two global components, say $C^1, C^2 \cup C^2$, $(C^2 \cup C^3; O) \cong A_3$ and $I(C^1, C^2 \cup C^3; O) = 4$.

(a) Assume that $\Sigma_{in} = [3A_2, B_{3,6}]$. If C has two components, the singularities $3A_2$, A_3 must be in a component. Then the unique possibility is the case $C = B_1 + B_5$ and $B_{3,6}$ is an intersection singularity of B_1 and B_5 and B_5 has $A_3 + 3A_2$ as singularities. Thus $I(B_1, B_5; O) = 4$, $B_1 \cap B_5 = B_{3,6} + A_1$ and B_5 can take further at most either one A_1 or A_2. It is easy to observe that C can not have three irreducible components (see 5.2-1).

(b) Assume that $\Sigma_{in} = [B_{3,6}, E_6, A_2]$ or $[B_{3,6}, A_8]$. Then by an easy consideration about δ^*-genus, $C = B_1 + B_5$ and $\Sigma(B_5)$ is $[A_2, E_6, A_3]$ or $[A_8, A_3]$ and B_5 is already rational. We get $\Sigma_{red} = [B_{3,6}, E_6, A_2, A_1]$, $[B_{3,6}, A_8, A_1]$. See 5.2-4.

(c) Assume that $\Sigma_{in} = [A_2, A_5, B_{3,6}]$. In this case, we have more possibilities of component types.

(1) Assume that $C = B_1 + B_5$. Then $\Sigma(B_5) = [A_2, A_5, A_3]$ and B_5 is rational.

(2) Assume that $C = B_2 + B_4$. Then $\Sigma(B_4) = A_3 + A_2$ and B_4 is thus rational and $B_2 \cap B_4 = B_{3,6} + A_5 + A_1$.

(3) Assume that $C = B_1 + B_1' + B_4$. B_4 is as above and $B_1 \cap B_4 = B_{3,6}$ and $B_1' \cap B_4 = A_5 + A_1$. See 5.2-2.

(4) Assume that C has a cubic component B_3. Then B_3 has an A_2-singularity and smooth at O. To make $B_{3,6}$, the other components can not be three lines. As an irreducible cubic can not have an A_3 singularity, the only possibility is $C = B_1 + B_2 + B_3$, $B_2 \cap B_3 = A_3 + A_5 + A_1$ and B_1 is tangent to B_2 and B_3 at O so that $(B_1 \cup B_2 \cup B_3, O) \cong B_{3,6}$ and $B_1 \cap B_3$ has one transverse intersection outside of O which gives an A_1 singularity. See 5.2-3.

(5) Finally for $\Sigma_{in} = [2B_{3,6}]$, it is already observed in [8] that $C = B_2 + B_2' + B_2''$.

2. Assume that $C_{3,7} \in \Sigma_{in}$. By the classification [8], C_2 is smooth, C_3 is nodal at O and $\iota = 3$. Recall that $C_{3,7}$ is an intersection singularity of a smooth component and a component with A_4. This implies that C must have a component of degree ≥ 4. The possibilities of Σ_{in} are $[3A_2, C_{3,7}]$, $[A_2, A_5, C_{3,7}]$, $[A_2, E_6, C_{3,7}]$, $[A_8, C_{3,7}]$ ([8]). In any case, as $\delta(C_{3,7}) = 6$, $C_{3,7}$ must be an intersection singularity. We assume that O is $C_{3,7}$-singularity.

(a) Assume that $\Sigma_{in} = [3A_2, C_{3,7}]$. Then $C = B_1 + B_5$. Note that $I(B_1, B_5; O) = 4$ by Example 5 in the section 2 and $B_1 \cap B_5 = C_{3,7} + A_1$. Then $\Sigma(B_5) \supset 3A_2 + A_4$, we can put A_1 in B_5. Note that a quintic B_5 can not have $4A_2 + A_4$ as, if so, we get $n^*(B_5) = 3$ which is a contradiction.

(b) Assume that $\Sigma_{in} = [A_2, E_6, C_{3,7}]$ or $[A_8, C_{3,7}]$. Then the possibility is $C = B_1 + B_5$ and $\Sigma(B_5) = A_2 + E_6 + A_4$ or $A_2 + E_6 + A_4$. In any case B_5 is rational and $B_1 \cap B_5 = C_{3,7} + A_1$ and we get $[C_{3,7}, E_6, A_2, A_1]$ and $[C_{3,7}, A_8, A_1]$. See 5.2-5.

(c) Assume that $\Sigma_{in} = [A_2, A_5, C_{3,7}]$.

(1) If $C = B_1 + B_5$, $\Sigma(B_5) = A_2 + A_5 + A_4$ and thus B_5 is rational. Thus $\Sigma_{red} = [C_{3,7}, A_5, A_2, A_1]$.

(2) Assume that $C = B_2 + B_4$. Then $\Sigma(B_4) = A_4 + A_2$ and thus B_4 is rational and $B_2 \cap B_4 = A_5 + C_{3,7} + A_1$ as $I(B_2, B_4; O) = 4$. Assume that $C = B_1 + B_1' + B_4$. Then B_4 is as above and $B_1 \cap B_4 = C_{3,7}$ and $B_1' \cap B_4 = A_5 + A_1$ and the corresponding configuration is $[C_{3,7}, A_5, A_2, 2A_1]$. See 5.2-6.

3. Assume that $C_{3,8} \in \Sigma_{in}$. Then C_2 is smooth, C_3 is nodal at O and $\iota = 3$. Assume that O is $C_{3,8}$ singularity defined by $y^3+y^2x^2-x^8+$(higher terms) $= 0$ for simplicity. Recall that it has three smooth components L_1, L_2, L_3 where $L_1 : y + x^2 +$ (higher terms) $= 0$ and $L_2, L_3 : y \pm x^3 +$ (higher terms) $= 0$. To make $C_{3,8}$ as an intersection singularity of two components, there are two ways.

(a-1) Assume that L_2 is a smooth component of C and $L_1 \cup L_3$ is another component. Then $I(L_2, L_1 \cup L_3; O) = 5$ and $(L_1 \cup L_3; O) = A_3$.

(a-2) Assume that L_1 is a smooth component of C and $L_2 \cup L_3$ is another component. Then $I(L_1, L_2 \cup L_3; O) = 4$ and $(L_2 \cup L_3; O) = A_5$.

Possible inner configurations are $[3A_2, C_{3,8}]$, $[A_2, A_5, C_{3,8}]$, $[A_2, E_6, C_{3,8}]$, $[A_8, C_{3,8}]$.

(1) Assume that $\Sigma_{in} = [3A_2, C_{3,8}]$ or $[A_2, E_6, C_{3,8}]$ or $[A_8, C_{3,8}]$. Then $C = B_1+B_5$.

(1-1) Assume first that B_1 corresponds to L_2 and B_5 corresponds to $L_1 \cup L_3$. Then $(B_5, O) \cong A_3$, $B_1 \cap B_5 = C_{3,8}$ and generically we have $\Sigma(B_5) = [3A_2, A_3]$, $[A_2, E_6, A_3]$, $[A_8, A_3]$ respectively. In the last two cases, B_5 is rational and it is easy to see that (a-2) does not occur. We get $\Sigma_{red} = [C_{3,8}, E_6, A_2]$ and $[C_{3,8}, A_8]$.

Assume that $\Sigma_{in} = [3A_2, C_{3,8}]$. Then $\Sigma(B_5) = [3A_2, A_3]$ and we can put further one A_1. Thus we get $\Sigma_{red} = [C_{3,8}, 3A_2]$, $[C_{3,8}, 3A_2, A_1]$. See 5.2-7. We assert that we can not put A_2 in B_5:

ASSERTION 9. Such a quintic B_5 with $\Sigma(B_5) = [4A_2, A_3]$ does not exist.

PROOF. Suppose that such a quintic exists. Then $n^*(B_5) = 4$. By the assumption, (C, O) has locally three components L_1, L_2, L_3 and B_5 has locally two components L_1, L_3. Recall that $I(L_2, L_3; O) = 3$. As we have assumed that $B_1 = L_2$, this implies that L_3 has a flex point at O. Other component L_1 has $I(L_2, L_1; O) = 2$. Assuming (x, y) is an affine coordinate system so that $y = 0$ be the equation of L_2, L_1 and L_2 are defined by $h_1(x, y) = (y + ax^2 +$ (higher terms)$) = 0$ and $h_3(x, y) = (y + bx^3 +$ (higher terms)$) = 0$ for some $a, b \neq 0$. Here h_1, h_2 are analytic functions defined in a neighborhood of O, though (x, y) are affine coordinates. By the following lemma, this implies that the dual singularity of (B_5, O) is a union of a cusp L_3^* and a smooth curve L_1^* which has the same tangent with the cusp. Thus the Milnor number of (B_5^*, O^*) is 7. (This implies A_3 is not generic in the sense of Puiseux order [6]). However a quartic can have at most 6 as the total Milnor number, which is a contradiction. □

LEMMA 10. *Let B_5 be a projective curve with a singularity at the origin whose defining function takes the form $h_1(x, y)h_3(x, y)$. Then the dual singularity (B^*, O^*) is locally defined* *by* $g(u, v) = 0$ *where*

$$
\begin{aligned}
g(u, v) &= (v + a'u^2 + (\textit{higher terms}))(v^2 + b'u^3 + (\textit{higher terms}))\,, \quad a', b' \neq 0 \\
&= v^3 + b'vu^3 + a'b'u^5 + (\textit{higher term}) = 0\,.
\end{aligned}
$$

Thus the dual singularity is E_7 *and the Milnor number is* 7.

PROOF. We use the parametrization $L_1 : x(t) = t,\ y(t) = -at^2 +$ (higher terms) and $L_3 : x(t) = t,\ y(t) = -bt^3 +$ (higher terms). Then the equations of the images by the Gauss map can be obtained by an easy computation (see [6]) and the assertion follows. □

(1-2) Now we consider the case $C = B_1 + B_5$ which corresponds to (a-2): $(B_5, O) \cong A_5$ and $B_1 \cap B_5 = C_{3,8} + A_1$. Then the unique possible inner configuration is $\Sigma_{in} = [C_{3,8}, 3A_2]$ and $\Sigma(B_5) = [3A_2, A_5]$ and B_5 is rational. This gives the configuration $[C_{3,8}, 3A_2, A_1]_2$. See 5.2-8.

(2) Now assume that $\Sigma_{in} = [C_{3,8}, A_5, A_2]$.

(2-1) Assume that $C = B_1 + B_5$. As $\Sigma(B_5) \supset \{A_5, A_2\}$, B_5 can not take another A_5. Thus the case (a-1) is the unique possibility and B_5 is rational with $[A_5, A_3, A_2]$. This gives the configuration $\Sigma_{red} = [C_{3,8}, A_5, A_2]$.

(2-2) Assume $C = B_2 + B_4$. As we have seen in (a-1) and (a-2), we need either A_3 or A_5 on B_4 to make $C_{3,8}$. Thus the only possibility is the case $\Sigma(B_4) = [A_3, A_2]$ ((a-1)) and $\Sigma_{in} = [A_2, A_5, C_{3,8}]$. In fact, this case is possible and $B_2 \cap B_4 = C_{3,8} + A_5$ as $I(B_2, B_4; O) = 5$ and $\Sigma_{red} = [A_2, A_5, C_{3,8}]$.

(2-3) Now we consider the case: C does not have any component of degree greater than 3. Only possible inner configuration is $[A_2, A_5, C_{3,8}]$ and the component type must be $\{1, 2, 3\}$ and the cubic component must be cuspidal. To make $C_{3,8}$, we need either A_3 or A_5 in the other union of components. We can make the components B_1, B_2, B_3 in two ways.

(a-1) $I(B_1, B_2; O) = 2$, $I(B_1, B_3; O) = 3$ and $I(B_2, B_3; O) = 2$ and $(B_2 \cup B_3, O) \cong A_3$: The corresponding configuration is denoted by $[C_{3,8}, A_5, A_2, A_1]_1$. This is a degeneration of $[C_{3,8}, A_5, A_2]_1$. See 5.2-9.

(a-2) $I(B_1, B_2; O) = 2$, $I(B_1, B_3; O) = 2$ and $I(B_2, B_3; O) = 3$ and $(B_2 \cup B_3, O) \cong A_5$: The corresponding configuration is denoted by $[C_{3,8}, A_5, A_2, A_1]_2$. This is a degeneration of $[C_{3,8}, A_5, A_2]_2$. See 5.2-10.

4. $C_{3,9} \in \Sigma_{in}$. Then C_2 is smooth and C_3 is nodal at O and $\iota = 3$ or 4. We assume that O is $C_{3,9}$-singularity as before. First we observe that $\mu(C_{3,9}) = 13$ and it must be an intersection singularity of a smooth component L and a component M with A_6. Note that $I(L, M; O) = 4$. There are two $C_{3,9}$ with different ι (=the intersection number $I(C_2, C_3; O)$).

(4-1) The case $\iota = 3$, the only possibility of the inner configuration is $\Sigma_{in}= [3A_2, C_{3,9}]$ by [8] which is given by $C = B_1 + B_5$ and $\Sigma(B_5) = 3A_2 + A_6$ and $B_1 \cap B_5 = C_{3,9} + A_1$, and $\Sigma_{red} = [3A_2, C_{3,9}, A_1]$.

(4-2) The other case is $\iota = 4$ and the possible inner configurations are $[2A_2, C_{3,9}]$, $[A_5, C_{3,9}]$, $[E_6, C_{3,9}]$.

a. Assume $\Sigma_{in} = [C_{3,9}, 2A_2]$. Then $\Sigma(B_5) = 2A_2 + A_6$ and we can put A_1 or A_2. This gives $\Sigma_{red} = [C_{3,9}, 2A_2, A_1], [C_{3,9}, 2A_2, 2A_1], [C_{3,9}, 3A_2, A_1]$.

b. In the cases $\Sigma_{in} = [A_5, C_{3,9}]$ or $[E_6, C_{3,9}]$, we have $\Sigma(B_5) = A_5 + A_6$ or $E_6 + A_6$ and therefore B_5 is already rational. Note that $I(B_1, B_5; O) = 4$ and thus $B_1 \cap B_5 = C_{3,9} + A_1$. Thus the possibilities for Σ_{red} are $[2A_2, C_{3,9}, A_1]$, $[2A_2, C_{3,9}, 2A_1]$, $[3A_2, C_{3,9}, A_1]$, $[A_5, C_{3,9}, A_1]$, $[E_6, C_{3,9}, A_1]$. See 5.2-12.

REMARK 11. We get the reduced configuration $[C_{39}, 3A_2, A_1]$ from two inner configurations $[C_{3,9}, 3A_2]$ and $[C_{3,9}, 2A_2]$. In fact, the moduli is the same and it has two different torus decompositions. An example is the following.

$$\mathrm{f}(x, y) := (y^2 + (x+1)y - x^2)^3 + \left(y^3 + \left(\frac{16}{3}x + 1\right)y^2 + (6x^2 + 3x)y + x^3\right)^2$$
$$= \frac{343}{27}\left(y^2 + \frac{15}{7}yx + \frac{3}{7}y + \frac{9}{7}x^2\right)^3 - \frac{1}{27}(27x^3 + 9yx + 60yx^2 + 9y^2 + 54y^2x + 17y^3)^2 .$$

5. $C_{3,12} \in \Sigma_{in}$. In this case, C_2 is smooth, C_3 is nodal at O and $\iota = 5$. Possible inner configuration is $[A_2, C_{3,12}]$. First note that $C_{3,12}$ has locally three smooth components L_1, L_2, L_3 which satisfies

$$I(L_1, L_2; O) = I(L_1, L_3; O) = 2, \quad I(L_2, L_3; O) = 5,$$
$$(L_1 \cup L_2; O), (L_1 \cup L_3; O) \cong A_3, \quad (L_2 \cup L_3; O) \cong A_9 .$$

If C has two components, it can be either $L_1 + (L_2 \cup L_3)$ or $L_2 + (L_1 \cup L_3)$.

Assume that $C = B_1 + B_5$. As $I(B_1, B_5; O) \le 5$, we must have $B_5 = L_2 \cup L_3$. Then $B_1 \cap B_5 = C_{3,12} + A_1$ and $\Sigma(B_5) = [A_2, A_9]$ and B_5 is rational. This gives the configuration $[C_{3,12}, A_2, A_1]_1$. See 5.2-13.

Assume that $C = B_2 + B_4$. Then as B_4 can not have A_9, we must have $B_4 = L_1 \cup L_3$ and $\Sigma(B_4) = [A_3, A_2]$ and $B_2 \cap B_4 = C_{3,12} + A_1$ as $I(L_2, L_1 \cup L_3; O) = 7$. Thus B_4 is rational and $\Sigma_{red} = [A_2, C_{3,12}, A_1]_2$. See 5.2-13.

Assume that $C_{3,12}$ is an intersection singularity of three global components. The intersection singularity of two of them have to make A_9. To make A_9, we need the intersection multiplicity 5. Thus the unique possibility is the case: $C = B_1 + B_2 + B_3$ with $(B_2 \cup B_3, O) \cong A_9$. Thus we may assume that $B_1 = L_1$, $B_2 = L_2$, $B_3 = L_3$, $B_2 \cap B_3 = A_9 + A_1$ and B_1 is tangent to B_2 at O so that $(B_1 \cup B_2 \cup B_3, O) \cong C_{3,12}$. The corresponding configuration is $[C_{3,12}, A_2, 2A_1]$. This is a degeneration of $[C_{3,12}, A_2, A_1]_i$, $i = 1, 2$.

6. $C_{6,6} \in \Sigma_{in}$. In this case, both of C_2 and C_3 are nodal at O and $\iota = 4$. Possible inner configurations are $[2A_2, C_{6,6}]$ and $[A_5, C_{6,6}]$. We assume $C_{6,6}$ singularity is at O and is locally defined by $y^6 - x^2y^2 + x^6 + (\text{higher terms}) = 0$ for simplicity. First note that $C_{6,6}$ has locally 4 smooth components L_1, L_2, K_1, K_2 such that $L_1, L_2 : x \pm y^2 + (\text{higher terms}) = 0$

and $K_1, K_2 : y \pm x^2 +$ (higher terms) $= 0$ and $I(K_1, K_2; O) = I(L_1, L_2; O) = 2$ and $I(L_i, K_j; O) = 1$. Note also that $(L_1 \cup L_2 \cup K_1, O) \cong D_6$ (same for any three components) and $(L_1 \cup L_2, O)$, $(K_1 \cup K_2, O) \cong A_3$.

(6-1) Assume that $C = B_1 + B_5$. Then $\Sigma(B_5) = 2A_2 + D_6$ and $\Sigma_{red} = [2A_2, C_{6,6}, A_1]$. As B_5 is already rational, the case $\Sigma_{in} = [C_{6,6}, A_5]$ does not occur.

The case $C = B_2 + B_4$ does not exist because B_4 can not have a D_6-singularity.

(6-2) Assume that $C = B_3 + B_3'$. Then this case is possible only if $\Sigma_{in} = [A_5, C_{6,6}]$ and two cubics are nodal at O with the same tangent cone so that $B_3 \cap B_3' = A_5 + C_{6,6}$ and $\Sigma_{red} = [A_5, C_{6,6}]$. Note that $I(L_1 \cup K_1, L_2 \cup K_2; O) = 6$. Another decomposition possibilities:

(6-3) Assume $C = B_4 + B_1 + B_1'$: $\Sigma(B_4) = [2A_2, A_1]$ and B_1 and B_1' are tangent to branches of A_1 of B_4 so that $(B_1 \cup B_1' \cup B_4, O) = C_{6,6}$ and $\Sigma_{red} = [2A_2, C_{6,6}, 2A_1]$. Two A_1's are the transverse intersection of B_1 or B_1' and B_4 outside of O. See 5.2-14. As $I(B_1, B_4; O) = 3$, the case $\Sigma_{in} = [C_{6,6}, A_5]$ does not exist.

(6-4) Assume $C = B_3 + B_2 + B_1$: B_3 is nodal and $\Sigma_{in} = [A_5, C_{6,6}]$. $B_1 \cap B_2 \cap B_3 = C_{6,6}$ and $B_2 \cap B_3 = A_5 + D_6$ and $\Sigma_{red} = [C_{6,6}, A_5]$.

(6-5) $C = B_2 + B_2' + B_1 + B_1'$: This can be understood as a degeneration of $B_3 + B_3'$ and $\Sigma_{red} = [A_5, C_{6,6}, 2A_1]$. See 5.2-15. The case $C = B_3 + B_1 + B_1' + B_1''$ can not make $C_{6,6}$.

7. $C_{6,9} \in \Sigma_{in}$. Possible inner configuration is $[A_2, C_{6,9}]$. Note that $C_{6,9}$ has two smooth components L_1, L_2 defined by $L_i : y + a_i x^2 +$ (higher terms) $= 0$, $a_i \neq 0$, $a_1 \neq a_2$, and one component K defined by $y^2 + bx^7 +$ (higher terms) $= 0$, $b \neq 0$ with A_6 singularity and $I(L_1, L_2; O) = 2$ and $I(L_i, K; O) = 2$. Note that $(L_2 \cup K; O) \cong D_9$. Thus the unique possibility is the case $C = B_1 + B_5$ with $\Sigma(B_5) = [A_2, D_9]$, $B_1 \cap B_5 = C_{6,9} + A_1$ and $\Sigma_{red} = [A_2, C_{6,9}, A_1]$.

8. $C_{3,15} \in \Sigma_{in}$. Then C_2 is smooth and C_3 is nodal at O. Note that $C_{3,15}$ has two components: a smooth component L and another component K with A_{12} singularity and $I(L, K; O) = 4$. Thus the unique possibility is the case $C = B_1 + B_5$, $\Sigma(B_5) = [A_{12}]$ and $B_1 \cap B_5 = C_{3,15} + A_1$. This case gives $\Sigma_{red} = [C_{3,15}, A_1]$. See 5.2-17.

9. $B_{3,12} \in \Sigma_{in}$: This case is unique and $C = B_2 + B_2' + B_2''$ and $\Sigma_{red} = [B_{3,12}]$. See 5.2-18 and [8].

10. $C_{6,12} \in \Sigma_{in}$. In this case, C_2 is a multiple line and C_3 is nodal at O. Note that $C_{6,12}$ has 4 smooth components L_1, L_2, K_1, K_2 with $I(L_1, L_2; O) = 2$, $I(K_1, K_2; O) = 5$ and $I(L_i, K_j; O) = 1$. Note also that

$$(L_1 \cup K_1; O) \cong A_1\,, \quad (L_1 \cup L_2; O) \cong A_3\,, \quad (K_1 \cup K_2; O) \cong A_9\,,$$
$$(L_1 \cup L_2 \cup K_1; O) \cong D_6\,, \quad (L_2 \cup K_1 \cup K_2, O) = D_{12}\,.$$

Thus $C = B_2 + B_4$ is not possible. If $C = B_1 + B_5$ is the case, $(B_5, O) \cong D_6$ and $B_5 = L_1 \cup L_2 \cup K_1$. But this is impossible as $5 \leq I(B_1, B_5; O) = 7$.

Assume that $C = B_3 + B'_3$. Then the cubics are nodal and they correspond to $L_i \cup K_i$, $i = 1, 2$ respectively and $I(B_3, B_3; O) = 9$. This case exists and $\Sigma_{red} = [C_{6,12}]$.

Suppose that B'_3 degenerate into $B_1 + B_2$. Then $B_1 \cap B_3 = D_6$ and $B_2 \cap B_3 = D_{12}$ and $B_1 \cap B_2 = 2A_1$. This also exists and $\Sigma_{red} = [C_{6,12}, A_1]$. See 5.2-16.

There are no other possibility of three components case. Note also that C can not have two line components. In fact, if it has two line components, we may assume that L_1, K_1 are the lines components. Put $C = L_1 + K_1 + J$, where J is the union of other components, degree$(J) = 4$ and $J = L_2 \cup K_2$. Then we have a contradiction $8 \geq I(L_1 \cup K_1, J; O) = I(L_1 \cup K_1, L_2 \cup K_2; O) = 9$.

11. $B_{4,6} \in \Sigma_{in}$. In this case, C_2 is a multiple line and C_3 has a cusp (or A_3). The possible inner configurations are $[2A_2, B_{4,6}]$ and $[A_5, B_{4,6}]$. Note that $B_{4,6}$ can be intersection singularity of two cuspidal components with intersection number 6. Thus the only possibility is that $C = B_3 + B'_3$ and it is easy to see that $[2A_2, B_{4,6}]$ does not exist as a reducible sextics. For $[A_5, B_{4,6}]$, we can take two cuspidal cubic B_3, B'_3 such that $B_3 \cap B'_3 = B_{4,6} + A_5$. There are no other possibility.

12. $D_{4,7} \in \Sigma_{in}$. Then the possible inner configurations are $[2A_2, D_{4,7}]$ and $[A_5, D_{4,7}]$. Recall that $D_{4,7}$ is defined by $y^4 + x^3y^2 + ayx^5 + bx^7 = 0$ with $a^2 - 4b \neq 0$ and $\mu(D_{4,7}) = 16$. It has three components L_1, L_2, K where K is cuspidal component of type $x^3 + y^2 +$ (higher terms) $= 0$ and L_1, L_2 are smooth components of type $y + \alpha x^2 +$ (higher terms) $= 0$ and thus $I(L_1, L_2; O) = 2$ and $I(L_i, K; O) = 3$. Thus $(L_1 \cup L_2; O) \cong A_3$ and $(L_1 \cup K, O) \cong E_7$. For $[2A_2, D_{4,7}]$, as an component have to support $2A_2$ and E_7, the only possibility is the case $C = B_1 + B_5$ and $\Sigma(B_5) = 2A_2 + E_7$ and so B_5 is rational and $B_1 \cap B_5 = D_{4,7}$.

Consider the case $\Sigma_{in} = [A_5, D_{4,7}]$. If $D_{4,7}$ is an intersection singularity of two components, this is only possible for $C = B_1 + B_5$, but then we can not make A_5. Thus C has three components and the unique possibility is $C = B_3 + B_2 + B_1$. In fact, this is possible if B_3 is cuspidal and $B_2 \cup B_3 = E_7 + A_5$ and $\Sigma_{red} = [A_5, D_{4,7}]$. See 5.2-19 and [8].

13. $Sp_2 \in \Sigma_{in}$. This case is studied by [8] and given by $C = B_3 + B'_3$, where both cubics are cuspidal with the same tangent cone and $I(B_3, B'_3; O) = 9$.

14. $B_{6,6}$ is possible only when $f_2(x, y), f_3(x, y)$ are homogeneous polynomials of degree 2 and 3 in x, y.

5. Examples

5.1. Examples I. Simple singularities. We give explicit examples of the sextics, mostly for the configurations with maximal rank, to confirm the existence of the configurations listed in Theorem 1. The most of the configurations in the list are easily computed starting from the normal form of the given inner configuration and component type. For example, if the component type is $\{1, 5\}$ or $\{2, 4\}$, take the intersection singularity at the origin and assuming the line component (respectively the conic component) is defined by $y = 0$ (resp. by $y - x^2 = 0$), we solve the equation $f(x,0) \equiv 0$ (resp. $f(x,x^2) \equiv 0$). We do not give the whole moduli description but it can be computed as in [7].

1. $C = B_3 + B_3'$ with $\Sigma(C) = [2A_5, 2A_2, 3A_1]_4 s \neq 1, \xrightarrow{s=1} \Sigma(C) = [2A_5, 2A_2, A_3, A_1]_3$, (2-(d)):

$$f := \frac{2187}{1372}s^2(y-x^2)^3 + \Big(\frac{1}{2}y^3 - \frac{9}{49}y^2\sqrt{21}s + 2y^2 - \frac{9}{98}y^2\sqrt{21}sx - \frac{9}{49}ys\sqrt{21} - 3yx^2 + 2xy - \frac{27}{98}xy\sqrt{21}s + \frac{9}{49}yx^2\sqrt{21}s + \frac{1}{2}x^2 - 2x^3 + \frac{45}{98}x^3\sqrt{21}s - \frac{9}{98}\sqrt{21}sx + \frac{9}{49}x^2s\sqrt{21}\Big)^2.$$

2. $C = B_1 + B_5$ with $\Sigma_{red} = [E_6, A_5, 2A_2, A_3, A_1]^{mr}$:

$$f := \Big(-\frac{3}{2}y^2 + \frac{3}{2}y - x^2\Big)^3 + \Big(-\frac{27}{16}y^3 + \Big(-\frac{9}{8}x + \frac{27}{8}\Big)y^2 + \Big(-\frac{27}{16}x^2 + \frac{9}{8}x - \frac{27}{16}\Big)y + x^3\Big)^2.$$

3. $[2A_5, 2A_2, D_5]_1^{mr}, C = B_1 + B_5$ and $[2A_5, 2A_2, D_5]_2^{mr}, C = B_2 + B_4$:

$$f := \Big(-y^2 + \frac{25}{16}xy - x^2 + x\Big)^3 + \Big(-y^3 + \frac{107}{32}xy^2 + \Big(-\frac{7843}{2048}x^2 + \frac{7}{2}x\Big)y + \frac{155}{128}x^3 - \frac{283}{128}x^2 + x\Big)^2,$$

$$f := \Big(\frac{1}{2}y^2 + xy - \frac{1}{4}x^2 - \frac{3}{2}x\Big)^3 + \Big(\frac{23}{16}y^3 + \Big(-\frac{2233}{1024}x - 1\Big)y^2 + \Big(\frac{3}{4}x^2 - \frac{7}{16}x\Big)y - \frac{1}{8}x^3 + \frac{697}{1024}x^2 + x\Big)^2.$$

4. $[E_6, 2A_5, A_3]_1^{mr}(C = B_1 + B_5)$ and $[E_6, 2A_5, A_3]_2^{mr}, C = B_2 + B_4$:

$$f := \Big(-\frac{1}{4}y^2 + \Big(2x + \frac{1}{2}\Big)y + \frac{1}{4}x^2 - \frac{1}{4}\Big)^3 + \Big(\frac{1}{8}y^3 + \Big(-\frac{13}{8}x - \frac{3}{8}\Big)y^2 + \Big(\frac{7}{8}x^2 + \frac{5}{4}x + \frac{3}{8}\Big)y + \frac{1}{8}x^3 + \frac{1}{8}x^2 - \frac{1}{8}x - \frac{1}{8}\Big)^2,$$

$$f := \Big(-\frac{7}{16}y^2 + \Big(\frac{1}{8}x + \frac{1}{2}\Big)y + \frac{1}{16}x^2 - \frac{1}{16}\Big)^3 + \Big(\frac{37}{128}y^3 + \Big(-\frac{13}{128}x - \frac{63}{128}\Big)y^2 + \Big(-\frac{17}{128}x^2 + \frac{5}{64}x + \frac{27}{128}\Big)y + \frac{1}{128}x^3 + \frac{1}{128}x^2 - \frac{1}{128}x - \frac{1}{128}\Big)^2.$$

5. $[E_6, 2A_5, 3A_1]^{mr}, C = B_4 + B_1 + B_1'$:

$$f := (-y^2 + 1 - x^2)^3 + (y^3 + (x-1)y^2 + (-x^2+1)y + x^3 + x^2 - x - 1)^2 .$$

6. $C = B_3 + B_3'$, $\Sigma(C) = [3A_5, 2A_2]$. This case has two torus decomposition: $[[3A_5], [2A_2]], [[2A_5, 2A_2], A_5]$.

$$f := -y^6 + \left(\frac{1}{12}x + 4x^3 + \frac{23}{24}y^3 - 7yx^2 - \frac{1}{24}y + x^2 + \frac{1}{12}y^2 - \frac{7}{6}yx + \frac{13}{12}y^2x\right)^2$$
$$= \frac{343}{1728}\left(y^2 - \frac{60}{7}yx - \frac{8}{7}y + \frac{48}{7}x^2 + \frac{12}{7}x + \frac{1}{7}\right)^3 - \frac{1}{1728}(-1 + 12y - 33y^2 + 22y^3$$
$$- 18x + 156yx - 282y^2x - 120x^2 + 552yx^2 - 288x^3)^2 .$$

7. $C = B_2 + B_2' + B_2''$ with $\Sigma(C) = [3A_5, 3A_1]_3$, ($u$: generic) or $[3A_5, D_4]_3^{mr}, u = -1/4$:

$$f := -(x^2 + y^2 - 1)^3u + (2u - 2x^2u - 2yu + 2yx^2u - y + yx^2 + 2y^2 - y^3)^2(u+1) .$$

8. $[3A_5, D_4]_i^{mr}, i = 1, 2, 3$ with component types (1,1,4), (1,1,1,3) and (2,2,2):

$$(1,1,4) : f := y(8x + 16 - 15y)(16x^4 - 32yx^3 + 32x^3 + 24y^2x^2 - 24yx^2$$
$$- 24y^3x + 48y^2x - 24yx + 16y - 39y^2 + 30y^3 - 7y^4) ,$$
$$(1,1,1,3) : f := y(x-y)(x+3y)(4x^3 - 23x^2y + 12x^2 - 48xy + 50xy^2$$
$$+ 12x + 4 + 48y^2 - 35y^3 - 24y) ,$$
$$(2,2,2) : f := (4y^2 + x^2 - 1)(2y^2 - 3xy - 3y + 2x^2 + 3x + 1)$$
$$(2y^2 + 3xy - 3y + 2x^2 - 3x + 1) .$$

9. $C = B_2 + B_4$ with $[A_{11}, 3A_2, 2A_1]^{mr}$:

$$f := (-y^2 + y - 9x^2)^3 + \left(y^3 + \frac{11}{8}y^2 + \left(-\frac{99}{8}x^2 + \frac{9}{4}x - \frac{1}{8}\right)y - \frac{81}{4}x^3 + \frac{9}{8}x^2\right)^2 .$$

10. $C = B_3 + B_3'$ and $\Sigma_{red} = [A_{11}, 2A_2, 3A_1]$:

$$f := -(262087 + 18817x^2 + 1351y^2 - 155578x + 94085y + 151316\sqrt{3} + 780y^2\sqrt{3}$$
$$- 89823x\sqrt{3} - 220762yx + 10864x^2\sqrt{3} - 127457yx\sqrt{3} + 54320y\sqrt{3})^3/$$
$$((97 + 56\sqrt{3})^3(2+\sqrt{3})^6) + \frac{1}{4}(-27246964 - 6500766x^2 - 1053390y^2$$
$$+ 24261189x - 14671830y - 10084y^3 - 15731042\sqrt{3} - 608175y^2\sqrt{3}$$
$$+ 14007204x\sqrt{3} + 2471685y^2x - 9224454yx^2 + 31450440yx + 524174x^3$$

$$- 5822y^3\sqrt{3} + 302632x^3\sqrt{3} - 3753219x^2\sqrt{3} + 18157920yx\sqrt{3} - 5325741yx^2\sqrt{3}$$
$$+ 1427028y^2x\sqrt{3} - 8470785y\sqrt{3})^2/((97 + 56\sqrt{3})^2(2 + \sqrt{3})^6)\,.$$

11. $C = B_3 + B_3'$ and $\Sigma_{red} = [A_{11}, 2A_2, A_3, A_1]^{mr}$

$$f := (-27y^2 - 2yx + 10x^2 - 14x)^3$$
$$+ \left(\frac{14}{3}yx - \frac{32}{3}yx^2 - \frac{14}{3}xy^2 - 63y^2 - \frac{98}{3}x + \frac{4}{3}x^3 + 18y^3 + \frac{70}{3}x^2\right)^2.$$

REMARK 12. In fact, $[A_{11}, 2A_2, 3A_1]$, $C = B_3 + B_3'$ degenerates into $[A_{11}, 2A_2, A_3, A_1]^{mr}$ but we could not find any good family with simple coefficients. The above examples factor into cubics over $\mathbf{Q}(\sqrt{3}, \sqrt{2})$ or $\mathbf{Q}(\sqrt{3})$ respectively.

12. $\Sigma_{red} = [A_{11}, A_5, A_3]_1^{mr}$ and $C = B_1 + B_5$.

$$f := (-y^2 + y - x^2)^3 + \left(\frac{17}{16}y^3 + \left(-x - \frac{33}{16}\right)y^2 + \left(\frac{1}{16}x^2 + x + 1\right)y - x^3\right)^2.$$

13. $\Sigma_{red} = [A_5, A_{11}, A_3]_2^{mr}$ and $C = B_2 + B_4$.

$$f := (-y^2 + y - x^2)^3 + \left(y^3 - \frac{3}{2}y^2 + \left(\frac{3}{2}x^2 + \frac{1}{2}\right)y - \frac{1}{2}x^2\right)^2,$$

14. $\Sigma_{red} = [A_{11}, A_5, A_3]_3^{mr}$ with $C = B_1 + B_2 + B_3$. An example is given by

$$f := -y^6 + ((-2 + 2x)y^2 + (1 + 2x^2 - 2x)y + x^3 - x^2)^2\,.$$

15. $\Sigma_{red} = [A_{11}, A_5, A_2, A_1]^{mr}$ with $C = B_3 + B_3'$ can be given by

$$f := -y^6 + \left(-\frac{23}{24}y^3 + \left(-x - \frac{1}{12}\right)y^2 + \left(7x^2 + x + \frac{1}{24}\right)y - 4x^3 - x^2\right)^2.$$

16. $[A_{11}, A_5, 2A_1]$ with $C = B_3 + B_2 + B_1$:

$$f := -y^6 + \left(\left(\left(-\frac{5}{4} + \frac{1}{4}s\right)x - 2\right)y^2 + \left(\frac{1}{4}sx^2 - x + 1\right)y + \frac{1}{4}x^3 + x^2 + x\right)^2.$$

Two degenerations: $[A_{11}, A_5, A_3]^{mr}$ at $s = -11$ and $[A_{11}, A_5, 3A_1]$ at $s = 5$.

17. $\Sigma_{red} = [A_{17} + 2A_1]$ with $C = B_3 + B_3'$:

$$f := -y^6 + (2y^3 - 2y^2 + (-6x^2 + 1)y + 4x^3)^2\,.$$

18. $\Sigma_{red} = [A_{17}, A_2]$ with $C = B_3 + B_3'$:

$$f := -y^6 + \left(2y^3 + (-x - 2)y^2 + \left(\frac{1}{4}x^2 + x + 1\right)y + x^3\right)^2.$$

5.2. Examples II. Non-simple singularities. We give explicit examples for some configurations. The normal form of $B_{3,6}$-singularity at O with $y = 0$ a linear component is given as

$$f_2 := a_{02}y^2 + (a_{11}x + a_{01})y - t^2x^2$$
$$f_3 := b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + b_{11}x)y + t^3x^3 .$$

The normal form of torus decomposition with $C = B_2 + B_4$ and $B_{3,6}$ at O where B_2 is defined by $y - x^2 = 0$ is given

$$f := \left(-t^2y^2 + (a_{11}x + a_{01})y - \frac{1}{4}\frac{(4t^2a_{01} + a_{11}{}^2)x^2}{t^2} \right)^3 + \frac{1}{64}(-8t^6y^3 + 12y^2t^4a_{11}x - 8y^2t^3b_{02} - 6x^2yt^2a_{11}{}^2 + 8x^2yt^3b_{02} - 8xyt^3b_{11} + x^3a_{11}{}^3 + 8x^3t^3b_{11})^2/t^6 .$$

In the following examples, those with a line or conic component can be easily derived from the above normal forms.

(1) $B_{3,6} \in \Sigma_{in}$: $\Sigma_{red} = [4A_2, B_{36}, A_1]$, $C = B_1 + B_5$, $\Sigma(B_5) = [4A_2, A_3]$:

$$f := (y^2 + (-5x + 1)y - x^2)^3 + \left(-\frac{3}{4}y^3 + \left(\frac{15}{4}x + \frac{3}{4}\right)y^2 + \left(-15x^2 + \frac{9}{4}x\right)y + x^3\right)^2 .$$

(2) $\Sigma_{red} = [A_2, A_5, B_{3,6}, 2A_1]$, $C = B_1 + B_1' + B_4$:

$$f := (-y^2 + y - 4x^2)^3 + (y^3 + (-4x - 1)y^2 + 4yx - 8x^3)^2 .$$

(3) $\Sigma_{red} = [A_2, A_5, B_{3,6}, 2A_1]$, $C = B_3 + B_2 + B_1$:

$$f := (-y^2 + (-3x + 1)y - x^2)^3 + \left(\frac{2}{3}y^3 + \left(\frac{13}{3}x - \frac{2}{3}\right)y^2 + \left(5x^2 - \frac{7}{3}x\right)y + x^3\right)^2 .$$

(4) $[A_8, B_{3,6}, A_1]$, $C = B_1 + B_5$ and $\Sigma(B_5) = [A_8, A_3]$:

$$f := (-y^2 + (-3x + 1)y - x^2)^3 + \left(-y^3 + \left(\frac{2}{3}x + 1\right)y^2 + \left(10x^2 - \frac{11}{3}x\right)y + x^3\right)^2 .$$

(5) $\Sigma_{red} = [C_{3,7}, A_8, A_1]$ with $C = B_1 + B_5$:

$$f := (-y^2 + y - x^2)^3 + (-2y^3 + (-3x + 2)y^2 + (-2x^2 + 3x)y + x^3)^2 .$$

(6) $\Sigma_{red} = [C_{3,7}, A_5, A_2, 2A_1]$ with $C = B_1 + B_1' + B_4$:

$$f := (-y^2 + y - x^2)^3 + \left(-\frac{9}{2}y^3 + \left(3x + \frac{9}{2}\right)y^2 - 3xy - x^3\right)^2 .$$

(7) $\Sigma_{red} = [3A_2, C_{3,8}, A_1]_1$, $C = B_1 + B_5$, $\Sigma(B_5) = [3A_2, A_3, A_1]$:

$$f := (y^2 + (-2x + 1)y - x^2)^3 + \left(y^3 + \left(-\frac{3}{2}x + 3\right)y^2 + \left(-3x^2 - \frac{3}{2}x\right)y + x^3\right)^2 .$$

(8) $\Sigma_{red} = [3A_2, C_{3,8}, A_1]_2$, $C = B_1 + B_5$, $\Sigma(B_5) = [3A_2, A_5]$:

$$f := (y^2 + (-x+1)y - x^2)^3 + (y^3 + (x+1)y^2 + 3xy + x^3)^2 .$$

(9) $C = B_1 + B_2 + B_3$, $B_2 \cap B_3 = A_3 + A_5 + A_1$ and $\Sigma_{red} = [C_{3,8}, A_5, A_2, A_1]_1$:

$$f := \left(-y^2 + \left(x + \frac{3}{4}\right)y - x^2 \right)^3 + \left(y^3 + \left(-\frac{3}{2}x + \frac{3}{4}\right)y^2 - \frac{9}{8}yx + x^3 \right)^2 .$$

(10) $C = B_1 + B_2 + B_3$, $B_2 \cap B_3 = A_5 + A_5$ and $\Sigma_{red} = [C_{3,8}, A_5, A_2, A_1]_2$:

$$f := \left(-\frac{1}{16}y^2 + (x-3)y - x^2 \right)^3 + \left(\frac{1}{64}y^3 - \frac{3}{8}xy^2 + (3x^2 - 9x)y + x^3 \right)^2 .$$

(11) $C = B_1 + B_5$, $\Sigma(B_5) = [A_8, A_3]$ and $\Sigma_{red} = [A_8, C_{3,8}]$:

$$f : = (-y^2 + (-3x+1)y - x^2)^3 + \left(-\frac{27}{8}y^3 + \left(-\frac{69}{8}x + \frac{27}{8}\right)y^2 + \left(\frac{9}{8}x^2 - \frac{3}{2}x\right)y + x^3 \right)^2 .$$

(12) $\Sigma_{red} = [C_{3,9}, E_6, A_1]$ with $C = B_1 + B_5$:

$$f := (-y^2 + y - x^2)^3 + (y^2x + (-2x^2 - x)y + x^3)^2 .$$

(13) $[C_{3,12}, A_2, A_1]_1 \xrightarrow{u \to 1} [C_{3,12}, A_2, A_1]_3$:

$$f := (-uy^2 + y - x^2)^3 + (y^3 + y^2 + (-x^2 + x)y - x^3)^2$$

and $[C_{3,12}, A_2, A_1]_2 \xrightarrow{u \to 1} [C_{3,12}, A_2, A_1]_3$:

$$f := (-y^2 + uy - ux^2)^3 + (y^3 + y^2 + (-x^2 + x)y - x^3)^2 .$$

(14) $\Sigma_{red} = [C_{6,6}, 2A_2, A_1]_1$ with $C = B_1 + B_5 \xrightarrow{s=0} [C_{6,6}, 2A_2, 2A_1]_1$ with $C = B_1 + B_1' + B_4$:

$$f := (y^2 + xy - x^2)^3 + (y^3 + ((-7+s)x + 2)y^2 + (x^2 - x)y + x^3)^2 .$$

(15) $\Sigma_{red} = [C_{6,6}, A_5, 2A_1]$ with $C = B_1 + B_1' + B_2 + B_2'$:

$$f := -y^6 + (y^3 + xy^2 + (x^2 + x)y + 2x^3 + x^2)^2 .$$

(16) $C = B_3 + B_3'$ with $[C_{6,12}]$ for $s \neq 1$ and $C = B_3 + B_2 + B_1$ with $[C_{6,12}, A_1]$ for $s = 1$ are given by:

$$f := -y^6 + (y^3 + (x+1)y^2 + (x^2 + x)y + 2sx^3)^2 .$$

(17) $\Sigma_{red} = [C_{3,15}, A_1]$ with $C = B_1 + B_5$:

$$f := ((x+1)y - x^2)^3 + (y^3 + (x+1)y^2 + xy - x^3)^2 .$$

(18) $\Sigma_{red} = [B_{3,12}]$:

$$f := (-y^2 + y - x^2)^3 + (y^3 - 3y^2 + 3yx^2)^2$$
$$= (-x^2 + 3y^2 + 2y^2\sqrt{3} + y)(x^2 - y - 3y^2 + 2y^2\sqrt{3})(x^2 - y)\,.$$

(19) $C = B_1 + B_5$, $\Sigma_{red} = [D_{4,7}, 2A_2] \overset{s=0}{\longrightarrow} C = B_1 + B_2 + B_3, [D_{4,7}, A_5]$:

$$f := (sxy - x^2)^3 + (-y^3 + (x+1)y^2 + yx^2 + x^3)^2\,.$$

5.3. Non-trivial degenerations. We give some non-trivial degenerations. We do not give the whole moduli description but it can be computed as in [7].

(1) $\Sigma_{red} = [A_5, 4A_2, A_3, 2A_1]_2$, $C = B_1 + B_5$ and $\Sigma(B_5) = [4A_2, A_3]$ can not degenerate into any further simple configuration, but it degenerates into $C_0 = B_1 + B_1' + B_4$ with $\Sigma_{red} = [A_5, C_{3,7}, A2, 2A_1]$ as $t \to 0$. For $t = 0$, A_5 and $C_{3,7}$ are at O and $(0, 1)$.

$$\mathrm{f}_t(x, y) = ((-1 - t^2)y^2 + (-4tx + 1)y - x^2)^3 + \left(\left(-t^3 - \frac{3}{2}t + 1\right)y^3 + \left((-3 - 6t^2)x + \frac{3}{2}t - 2\right)y^2 + \left(-\frac{15}{2}tx^2 + 3x + 1\right)y + x^3\right)^2.$$

(2) $\Sigma_{red} = [A_5, 4A_2, A_3, 2A_1]_3$, $C = B_1 + B_5$, $\Sigma(B_5) = [4A_4, 2A_1]$ degenerates into $C_0 = B_1 + B_5$, with $\Sigma_{red} = [A_5, E_6, A_3, 2A_2, A_1]^{mr}$ when $t \to 0$.

$$\mathrm{f}_t(x, y) = (-y^2 - y^2t^2 - 3yx - 4yxt + y - x^2)^3 + \frac{1}{64}(8y^3t^4 + 48y^2xt^3 + 8y^3t^3 + 60yx^2t^2 - 12y^2t^2 + 84y^2xt^2 + 12y^3t^2 - 12y^2t + 12y^3t + 132yx^2t + 60y^2xt - 24yxt - 8x^3t - 6y^2 - 8x^3 - 24yx + 72yx^2 + 24y^2x + 3y + 3y^3)^2/(1+t)^2\,.$$

(3) $\Sigma_{red} = [A_5, 4A_2, D_5]_2$, $C = B_1 + B_5$ degenerates into $\Sigma_{red} = [2A_5, 2A_2, D_5]_1^{mr}$: For $t = 0$, two A_5's are at $\{(0,0), (0,1)\}$.

$$f := \left(-y^2 + \left(\left(-\frac{25}{16} + \frac{1}{16}t\right)x + 1\right)y - x^2\right)^3 + \frac{1}{4294967296}(6428160y^3 + 65536x^3t^2 + 65536yt^2 - 1179648yt - 11736576y^2 + 1552896y^2t - 41472y^2t^2 + 1536y^2t^3 - 36y^3t^4 - 2586y^3t^3 + y^3t^5 - 491103y^3t + 66204y^3t^2 + 5308416y + 4128768yxt - 229376yxt^2 - 6144yx^2t^3 + 329728yx^2t^2 - 4442112yx^2t + 96y^2xt^4 - 4992y^2xt^3 + 186432y^2xt^2 - 3602304y^2xt - 18579456yx + 5308416x^3 + 20329056y^2x - 1179648x^3t + 17750016x^2y)^2/(-9+t)^4\,.$$

(4) $[2A_5, 2A_2, 4A_1], C = B_4 + B_1 + B_1' \to [3A_5, 4A_1]^{mr}, a = 0, B_3 + B_1 + B_1' + B_1''$:

$$f := (yxa^2 - x^2a + ax - 4yxa - y^2)^3 + \left(\frac{1}{2}y^2xa^3 - \frac{1}{2}a^3x^2y - \frac{1}{2}x^2a^2 + \frac{3}{2}yx^2a^2 - \frac{7}{2}y^2xa^2 + \frac{1}{2}x^3a^2 + yxa^2 - \frac{1}{2}x^2a + 2yx^2a - 3yxa + \frac{1}{2}ax + 7y^2xa - 4y^2x + \frac{1}{2}x - 3yx^2 - yx - y^3 - \frac{1}{2}x^3\right)^2 .$$

The curve $f(x, y, 0)$ has three line components and a nodal cubic component.

$$f(x, y, 0) = \frac{1}{4}x(x + 1 + 2y)(x - 1 + 4y)(4y^3 + 8y^2x + 6yx^2 + 2yx + x^3 - x)$$

(5) $C_u = B_2 + B_4$, $\Sigma_{red} = [2A_5, 3A_2, 2A_1]_2 \to [3A_5, A_2, 2A_1]^{mr}$ with $C_0 = B_1 + B_2 + B_3$.

$$\mathrm{f}_2 := -y^2 + y^2t_1u - \frac{1}{2}y^2u^2 - xyt_1 + xyu - \frac{1}{4}x^2{t_1}^2 + \frac{1}{2}x^2t_1u - \frac{1}{4}x^2u^2 - xt_1u + \frac{1}{2}xu^2 ,$$

$$\mathrm{f}_3 := y^3 + \frac{3}{8}x^3t_1u^2 + \frac{3}{2}y^3u^2 + \frac{9}{4}x^2{t_1}^2u - \frac{1}{2}y^2xu^3 + \frac{3}{4}x^2yu^2 - \frac{3}{2}xyu^2 - \frac{9}{2}y^3t_1u - \frac{3}{2}x^2yt_1u + \frac{9}{2}xyt_1u + \frac{15}{8}y^2xt_1u^2 - \frac{9}{4}y^2x{t_1}^2u - \frac{9}{4}u^2y^2t_1 + 3uy^2{t_1}^2 - 3ux{t_1}^2 - \frac{1}{2}xu^3 - \frac{1}{8}x^3u^3 + \frac{1}{2}x^2u^3 + \frac{1}{2}y^2u^3 - \frac{3}{8}x^3{t_1}^2u + \frac{9}{4}u^2xt_1 - \frac{15}{8}x^2t_1u^2 - y^2{t_1}^3 + y^2x{t_1}^3 + x{t_1}^3 + \frac{1}{8}x^3{t_1}^3 + 3y^3{t_1}^2 - x^2{t_1}^3 + \frac{3}{2}y^2xt_1 - 3xy{t_1}^2 + \frac{3}{4}x^2y{t_1}^2 - \frac{3}{2}y^2xu ,$$

with $t_1 := u + \frac{1}{2}\sqrt{u^2 - 6}$. For $t_1 = 0$, this degenerates into:

$$f(x, y, 0) = \frac{3}{32}(9x^3 - 36x^2 + 36x - 18x^2y\sqrt{-6} + 36xy\sqrt{-6} - 36y^2x - 36y^2 - 20y^3\sqrt{-6})(x - 1 - \sqrt{-6}y)(x - y^2) .$$

(6) $[3A_5, A_3]_1, C = B_1 + B_5 \xrightarrow{u\to 1} [3A_5, D_4], C = B_1 + B_1 + B_4$:

$$\mathrm{f} := (-y^2 - 3xy - x^2 + x)^3 + \frac{1}{16}(-4y^3u - 14xy^2u + xy^2 + 4xy^2u^2 - 10x^2yu + 3x^2y + 4x^2yu^2 + 4xyu - 4xyu^2 - 2ux^3 + x^3 + u^2x^3 + 2x^2u - x^2 - 2x^2u^2 + xu^2)^2/u^2 .$$

(7) $[3A_5, A_3]_2, C = B_2 + B_4 \longrightarrow [3A_5, D_4]_1^{mr} (a = 1/4), [3A_5, D_4]_2^{mr} (a = -1/12)$:

$$\mathrm{f}(x, y) := \left(-y^2 a - \frac{1}{2}y^2 + \frac{1}{2}y - ax^2 + a \right)^3 + \left(\frac{5}{4}y^3 a + \frac{3}{8}y^3 - \frac{1}{4}y^2 a - \frac{1}{2}y^2 + \frac{5}{4}yax^2 - \frac{5}{4}ya + \frac{1}{8}y - \frac{1}{4}ax^2 + \frac{1}{4}a \right)^2 .$$

(8) A degeneration $[A_{11}, 2A_2, A_3] s \neq 1 \to [A_{11}, 2A_2, D_4], s = 1$ with $C = B_2 + B_4$.

$$\mathrm{f} := \left(-y^2 + y - \frac{8}{3}\frac{x^2}{s} \right)^3 + \left(y^3 - \frac{5}{4}y^2 + \left(\frac{10}{3}\frac{x^2}{s} + x + \frac{3}{8}s \right) y - \frac{8}{3}\frac{x^3}{s} - x^2 \right)^2 .$$

6. Appendix

Proof of Proposition 7. The proof is computational.

1. Assume that C is a sextics of linear type with $3A_5$. This is a special case of the result of Tokunaga, [10]. We assume that L is defined by $y = 0$. We start from the generic polynomial of degree 6:

$$\mathrm{f}(x, y) = \sum_{i+j \le 6} a_{i,j} x^i y^j .$$

By the action of $PGL(3, \mathbf{C})$, we may assume that three A_5's are at $\{P_1 = (-1, 0), P_2 = (0, 0), P_3 = (1, 0)\}$ and the tangent cones at P_1, P_3 are given by $x = \pm 1$. These condition says

(1) $f(x, 0) = x^2(x^2 - 1)^2$ and
(2) $f_x(P_i) = f_y(P_i) = 0$ for $i = 1, 2, 3$ and
(3) $f_{x,y}(P_i) = f_{y,y}(P_i) = 0, i = 1, 3$.

Eliminating coefficients from $f(x, y)$ using these equations, then we eliminate further coefficients using the assumptions $(C, P_i) \cong A_5$. Then we get a normal form of sextics with four parameters a_{04}, a_{06}, t, s. Then we apply the degeneration method to the family $fu := f - uy^6$ (see [7]). Finally we find that $f(x, y)$ has a torus expression:

$$\mathrm{f}(x, y) = \tau y^6 + \Big(-ty^2 - sy^2 - yx^2 + xty^2 + y - xsy^2 + x^3 - x - y^3 st + \frac{1}{2}y^3 a_{04} - \frac{1}{2}y^3 s^2 - \frac{1}{2}y^3 t^2 \Big)^2 ,$$

$$\tau = a_{06} + \frac{1}{2}s^2 a_{04} - t^3 s - s^3 t - \frac{3}{2}s^2 t^2 - \frac{1}{4}t^4 + t a_{04} s - \frac{1}{4}s^4 - \frac{1}{4}a_{04}{}^2 + \frac{1}{2}a_{04} t^2 .$$

The proof for the cases $\Sigma(C) \supset A_{11}+A_5, A_{17}$ are similar. In the case of $A_{11}+A_5$, we assume that A_{11} is at O and A_5 is at $(1,0)$. We assume that the tangent cone at $(1,0)$ is $x=1$. In the case A_{17}, we assume that A_{17} is at O. For the condition of $(C,O) \cong A_{11}$ (respectively for $\cong A_{17}$) at the origin, we assume that the normal form is given by the change of coordinates $y_1 := y+\sum_{i=2}^{5} t_i x^i$ (resp. $y_1 := y+\sum_{i=2}^{8} t_i x^i$). Let $f'(x,y)=f(x,y_1)$ and let $c_0 = f'(x,0)$ and c_1 be the coefficients of y in $f'(x,y)$. As we assume that $f'(x,y) = ay^2+byx^6+cx^{12}+$(higher terms)s (resp. $f'(x,y)=ay^2+byx^9+cx^{18}+$(higher terms)s), normal forms are obtained by solving the equalities: $Coeff(c_0,x,j)=0$ for $j\le 11$ and $Coeff(c_1,x,k)=0$ for $k\le 5$ (resp. $Coeff(c_0,x,j)=0$ for $j\le 17$ and $Coeff(c_1,x,k)=0$ for $k\le 8$). Then we find the torus expressions by the degeneration method. For the case $A_{11}+A_5$, $f(x,y)=u_{11}y^6+f_3(x,y)^2$ where

$$\begin{aligned} u_{11} = -\frac{1}{4}(&4a_{04}{t_2}^9 t_4{t_3}^2 + {t_2}^4{t_4}^4 - 2a_{04}{t_2}^{10}{t_4}^2 - 2{t_2}^8 a_{04}{t_3}^4 + 6{t_2}^2{t_4}^2{t_3}^4 \\ &- 4{t_2}^3{t_4}^3{t_3}^2 - 4{t_3}^6 t_4 t_2 + {a_{04}}^2{t_2}^{16} + {t_3}^8 - 4a_{06}{t_2}^{14})/{t_2}^{14}, \\ \mathrm{f}_3(x,y) = \frac{1}{2}(&6{t_2}^3y^2t_3xt_4 - 2{t_2}^7x^2 - 2{t_2}^4y^2t_4 - 2{t_2}^6yx + 2{t_2}^3y^2{t_3}^2 \\ &- 2{t_2}^3y^2{t_3}^2x + 2{t_2}^4y^2xt_4 - 4{t_2}^2y^2{t_3}^3x + 2{t_2}^6y - 2{t_2}^4y^2xt_5 - y^3{t_2}^2{t_4}^2 \\ &+ 2t_2y^3{t_3}^2t_4 - y^3{t_3}^4 + 2{t_2}^7x^3 + {t_2}^8y^3a_{04} - 2{t_2}^5yt_3x + 2{t_2}^5yt_3x^2)/{t_2}^7 \end{aligned}$$

and $f(x,y)=u_{17}y^6+h_3(x,y)^2$ for the case A_{17} where

$$\begin{aligned} u_{17} = -\frac{1}{4}(&24{t_6}^2{t_3}^6{t_5}^2{t_4}^2 - 24{t_4}^4{t_6}^2{t_3}^5t_5 + 8a_{04}{t_3}^{13}t_4t_6t_5 - 8t_4{t_6}^3{t_3}^7t_5 \\ &- 8a_{04}{t_3}^{12}{t_5}^2{t_4}^2 + 8a_{04}{t_4}^4{t_3}^{11}t_5 + {a_{04}}^2{t_3}^{20} - 2a_{04}{t_4}^6{t_3}^{10} - 2a_{04}{t_3}^{14}{t_6}^2 \\ &+ 4{t_4}^3{t_6}^3{t_3}^6 + 6{t_4}^6{t_6}^2{t_3}^4 - 4a_{04}{t_3}^{12}{t_4}^3t_6 + {t_3}^8{t_6}^4 + 4{t_4}^9t_6{t_3}^2 - 4a_{06}{t_3}^{18} \\ &+ 24{t_4}^8{t_5}^2{t_3}^2 - 8{t_4}^{10}t_5t_3 + 48{t_4}^5t_6{t_3}^4{t_5}^2 - 24{t_4}^7t_6{t_3}^3t_5 + {t_4}^{12} \\ &- 32t_6{t_3}^5{t_5}^3{t_4}^3 + 16{t_4}^4{t_5}^4{t_3}^4 - 32{t_4}^6{t_5}^3{t_3}^3)/{t_3}^{18}, \\ \mathrm{h}_3(x,y) := \frac{1}{2}(&-4{t_5}^2{t_3}^2y^3{t_4}^2 + 2{t_5}^2{t_3}^5y^2x + 4t_6t_5{t_3}^3y^3t_4 - 10t_5{t_3}^4y^2x{t_4}^2 \\ &+ 4t_5{t_3}^5y^2t_4 + 4t_5t_3y^3{t_4}^4 - 2t_5{t_3}^7yx^2 - 2{t_3}^9x^3 + 2{t_3}^6x^2{t_4}^2y \\ &- 2{t_3}^6y^2xt_7 + 6t_6{t_3}^5y^2xt_4 + 4{t_3}^3xy^2{t_4}^4 - 2{t_3}^7xyt_4 - y^3{t_6}^2{t_3}^4 \\ &- 2t_6{t_3}^2y^3{t_4}^3 - y^3{t_4}^6 - 2t_6{t_3}^6y^2 - 2{t_3}^4y^2{t_4}^3 + 2{t_3}^8y + y^3{t_3}^{10}a_{04})/{t_3}^9. \end{aligned}$$

2. Next we consider the case $C=B_3+B_3'$ and C is a sextics of torus type and assume that $\Sigma_{in}(C)=[3A_5]$ or $[A_{11},A_5]$ or $[A_{17}]$ and let P_1, P_2, P_3 be the corresponding singular points. (In the case of $[A_{11},A_5]$ or $[A_{17}]$, $P_2=P_3$ and $P_1=P_2=P_3$ respectively.) We

show that three P_1, P_2, P_3 must be colinear. We start from the expression:

$$\begin{aligned} \mathrm{f}(x,y) &:= f_{31}(x,y) f_{32}(x,y) \\ &= (a_{03}y^3 + (a_{12}x + a_{02})y^2 + (a_{21}x^2 + a_{11}x + a_{01})y + a_{30}x^3 + a_{20}x^2 + a_{10}x + a_{00}) \\ &\quad (b_{03}y^3 + (b_{12}x + b_{02})y^2 + (b_{21}x^2 + b_{11}x + b_{01})y + b_{30}x^3 + b_{20}x^2 + b_{10}x + b_{00}) \,. \end{aligned}$$

Assume that C is defined by $f_2(x,y)^3 + f_3(x,y)^2 = 0$ and let C_2 and C_3 be the conic and the cubic defined by $f_2 = 0$ and $f_3 = 0$. Let $P_i \in C$ be an inner singularity. Recall that by [8] we have the equivalence

($\star$) $(C, P_i) \cong A_{6j-1} \Leftrightarrow I(C_2, C_3; P_i) = 2j$ and C_3 is smooth at P_i for $j = 1, 2, 3$.

In particular, if C_2 is smooth at P_i, ($\star$) implies that C_2 and C_3 are tangent at P_i. Again by an easy computation, we can see that there are no cases when P_1, P_2, P_3 are not colinear. We give a recipe of the computation. Assuming that P_1, P_2, P_3 are not colinear. To each P_i, we associate its tangent cone direction ℓ_i and we identify ℓ_i and a line in $\mathbf{P}^2$.

There are two cases.

(a) C_2 is a smooth conic, or

(b) C_2 is a union of two distinct lines L_1, L_2.

In the case of (a), we may assume that $P_1 = (-1, 1)$, $P_2 = O = (0, 0)$ and $P_3 = (1, 1)$ and $\ell_1 = \{y + 2x + 1 = 0\}$, $\ell_3 = \{y - 2x + 1 = 0\}$. Then the conic must be defined by $y - x^2 = 0$. Thus $\ell_2 = \{y = 0\}$. Here we used the next easy lemma.

LEMMA 13. *Let $(C, \{P_1, P_2, P_3\})$ and $(C', \{P_1', P_2', P_3'\})$ two smooth conics with three points on the respective conic. Then they are isomorphic by an action of a matrix $A \in PGL(3, \mathbf{C})$.*

Thus we need to have the equations

$$(\star) : \begin{cases} f_{31}(P_i) = f_{32}(P_i) = 0 \,, & i = 1, 2, 3 \,, \\ (f_{3j,x} - 2f_{3j,y})(P_1) = 0 \,, \quad (f_{3j,x} + 2f_{3j,y})(P_3) = 0 \,, & f_{3j,y}(O) = 0, \ j = 1, 2 \,. \end{cases}$$

The last condition says that two cubic are tangent to $y = x^2$ at P_1, P_3. Let $R(x)$ and $S(y)$ be the resultant of f_{31} and f_{32} with respect to y-variable and x-variable respectively. The above equality implies that $(x^2 - 1)^2 x^2 | R(x)$ and $y^2(y^2 - 1)^2 | S(y)$. Eliminating coefficients using these equalities, we consider the further condition for P_1, P_2, P_3 to be A_5-singularities. This is given by the condition $x^3(x^2 - 1)^3 | R(x)$ and $y^3(y^2 - 1)^3 | S(y)$. At the end of calculation, we find that there are no such f_{31}, f_{32} which corresponds to a reducible sextics.

We consider the case (b). Assume that C_2 is a union of two lines L_1, L_2. Then we can see that L_i are tangent to the cubic C_3 and the intersection $L_1 \cap L_2$ is also on C_3 so that this makes the third A_5. In this case, we may assume that P_1, P_2, P_3 be as above but ℓ_1 is $y = -x$ and ℓ_3 is $y = x$. The $\star$ should be replaced by

$$(\star) : \begin{cases} f_{31}(P_i) = f_{32}(P_i) = 0 \,, & i = 1, 2, 3 \,, \\ (f_{3j,x} - f_{3j,y})(P_1) = 0 \,, \quad (f_{3j,x} + f_{3j,y})(P_3) = 0 \,, & j = 1, 2 \,. \end{cases}$$

Then we consider the A_5-condition to see that there exists no such sextics.

The case $\Sigma_{in} = [A_{11}, A_5]$, we take A_5 at P_1 and A_{11} at O and the $(\star)$-condition is replaced by

$$(\star) : \begin{cases} f_{31}(P_i) = f_{32}(P_i) = 0\,, \quad i = 1, 2\,, \quad (f_{3j,x} - 2f_{3j,y})(P_1) = 0\,, \quad j = 1, 2\,, \\ x^4 \mid f_{3j}(x, x^2),\, j = 1, 2\,. \end{cases}$$

The last condition says that the intersection multiplicity of each cubic and the conic $y - x^2 = 0$ at O is 4. The reason that we have chosen the conic $y - x^2 = 0$ is to make the last condition to be easier to be used.

The case $\Sigma_{in} = [A_{17}]$, we take A_{17} at P_2, and the the torus type condition is

$$(\star) : f_{31}(0, 0) = f_{32}(0, 0) = 0\,, \quad x^6 \mid f_{3j}(x, x^2),\, j = 1, 2\,.$$

In any cases, one conclude that there does not exist any solution which corresponds to a reduced sextics.

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