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On the Beurling Convolution Algebra II

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Abstract. We treat the extended rings according to Beurling. Especially, Theorems VII and X in [1] are extended to the case of n-dimensional euclidean space $\mathbf{R}^n$.

1. Introduction

Beurling [1] considered a class of functions on $\mathbf{R}^1$, each member of which is the Fourier transform of an integrable function. The purpose of this paper is to extend his results to the class of functions on $\mathbf{R}^n$.

Let us start to describe notations, definitions and theorems, which we shall ask for. According to Beurling [1], we consider a normed family Ω of strictly positive functions $\omega(x)$ on $\mathbf{R}^n$ which are measurable with respect to the ordinary Lebesgue measure dx, and furthermore, together with the norm $N(\omega)$, satisfy the following conditions:

(I) For each $\omega \in \Omega$, $N(\omega)$ takes a finite value,

$$0 < \int \omega dx \leq N(\omega)\,.$$

(II) If λ is a positive number and $\omega \in \Omega$, then $\lambda\omega \in \Omega$ and

$$N(\lambda\omega) = \lambda N(\omega)\,.$$

(III) If $\omega_1, \omega_2 \in \Omega$, then the sum $\omega_1 + \omega_2$ as well as the convolution $\omega_1 * \omega_2$ are also in Ω and

$$N(\omega_1 + \omega_2) \leq N(\omega_1) + N(\omega_2)\,,$$

$$N(\omega_1 * \omega_2) \leq N(\omega_1)N(\omega_2)\,.$$

(IV) Ω is complete under the norm N in the sense that for any sequence $\{\omega_n\}_1^\infty \subset \Omega$ such that $\sum_1^\infty N(\omega_n) < \infty$, $\omega = \sum_1^\infty \omega_n$ is in Ω and

$$N(\omega) \leq \sum_1^\infty N(\omega_n)\,.$$

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We associate with each $\omega \in \Omega$, the Banach space $L^2_{\omega^{-1}}$ of measurable functions F on $\mathbf{R}^n$ with the finite norms

$$\|F\|_{L^2_{\omega^{-1}}} = \left(\int_{\mathbf{R}^n} \omega dx \int_{\mathbf{R}^n} \frac{|F|^2}{\omega} dx\right)^{1/2} .$$

From these spaces, we define a family of functions $A^2 = A^2(\mathbf{R}^n, \Omega)$ by

$$A^2 = \bigcup_{\omega \in \Omega} L^2_{\omega^{-1}}$$

and a norm

$$\|F\| = \|F\|_{A^2} = \inf_{\omega \in \Omega} \|F\|_{L^2_{\omega^{-1}}} .$$

Beurling [1] proved that in this norm, A^2 is the Banach algebra under the addition and the convolution.

Let us consider the algebras A^2 which are generated by some particularly simple families of Ω. First let $\Omega = \Omega(\mathbf{R}^n)$ be a set of positive, summable and non-increasing functions $\omega(|x|)$ with the norm

$$N(\omega) = \int_{\mathbf{R}^n} \omega dx .$$

Next let us consider the subfamily Ω_1 of Ω consisting of functions with the property:

$$\omega(0) = \lim_{x \to 0} \omega(x) < \infty .$$

The norm in Ω_1 is defined as

$$N(\omega) = \omega(0) + \int_{\mathbf{R}^n} \omega dx .$$

Since the sets Ω and Ω_1 satisfy conditions (I)–(IV), we can define the Banach algebras $A^2 = A^2(\mathbf{R}^n, \Omega)$ and $\mathcal{A}^2 = \mathcal{A}^2(\mathbf{R}^n, \Omega_1)$, respectively. The ring of Fourier transforms f of $F \in A^2$ is denoted by $\tilde{A}^2$ and its norm by $\|f\| = \|f\|_{\tilde{A}^2} = \|F\|_{A^2}$. The ring $\tilde{\mathcal{A}}^2$ is defined similarly.

Let us also introduce the following notation

$$\eta(\alpha) = \eta(\alpha, f) = \sqrt{\left(\frac{1}{2\pi}\right)^n \int_{\mathbf{R}^n} |\Delta^n_\alpha f(t)|^2 dt} ,$$

where $\Delta^n_\alpha f$ is the difference along the vector α, that is

$$\Delta^n_\alpha f(t) = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} f(t + (n-k)\alpha) ,$$

and

$$A(f) = \int_{\mathbf{R}^n} \eta(\alpha, f) \frac{d\alpha}{|\alpha|^{3n/2}} .$$

We proved in [2] the following theorems being the extention of Theorems III, VIII and IX in [1] to the n-dimensional euclidean space $\mathbf{R}^n$.

THEOREM I. *A function f belongs to the ring $\tilde{A}^2$ if and only if*:

(a) *f is continuous*,

(b) $\lim_{|t|\to\infty} f(t) = 0$,

(c) $A(f) < \infty$.

Under these conditions, f is represented by the Fourier transform of some $F \in A^2$, and the following inequalities hold:

$$c_n \|F\|_{A^2} \le A(f) \le d_n \|F\|_{A^2} , \tag{1.1}$$

provided $f \neq 0$, where c_n and d_n are positive constants independent of f.

THEOREM II. *The space $\tilde{\mathcal{A}}^2$ is the intersection of A^2 and L^2, and the norms in these spaces satisfy the inequalities*

$$\|F\|_{\mathcal{A}^2} > \|F\|_{A^2} , \quad \|F\|_{\mathcal{A}^2} > \|F\|_{L^2} , \quad \|F\|_{\mathcal{A}^2} < \|F\|_{A^2} + \|F\|_{L^2} .$$

THEOREM III. *A function f belongs to the ring $\tilde{\mathcal{A}}^2$ if and only if*:

(a) *f is continuous*,

(b) $f \in L^2$,

(c) $A(f) < \infty$.

Under these conditions the following inequalities hold:

$$c_n \|F\|_{\mathcal{A}^2} < A(f) + (1/\sqrt{2\pi})^n \|f\|_{L^2} < (d_n + 1) \|F\|_{\mathcal{A}^2} \tag{1.2}$$

provided $f \neq 0$, where constants c_n and d_n are those of Theorem I.

Beurling [1] also introduced a new ring. To the normed ring $\tilde{A}^2$ we may adjoin each function h with property that $g \in \tilde{A}^2$ implies $gh \in \tilde{A}^2$. By the closed graph theorem, we have

$$\|gh\| \le m\|g\| \quad (m < \infty) , \tag{1.3}$$

and define the norm of h as the least number m satisfying (1.3). By the completion of $\tilde{A}^2$ with respect to this norm we obtain a new ring, so called the extended ring which is denoted by $ex\tilde{A}^2$. The norm in $ex\tilde{A}^2$ is denoted by $\|h\|_{ex}$. By $M(h)$ we mean the supremum norm of h. Then we have

$$M(h) \le \|h\|_{ex} .$$

The extended ring $ex\tilde{\mathcal{A}}^2$ is defined similarly.

For this extended ring $ex\tilde{A}^2$ of functions on $\mathbf{R}^1$ Beurling [1] proved the following theorems.

THEOREM IV. *The extended normed ring* $ex\tilde{A}^2$ *consists of all continuous functions h of the form*

$$h(t) = c + f(t)\,, \tag{1.4}$$

where c is a constant and $f \in \tilde{A}^2$.

THEOREM V. *A bounded continuous function h on* $\mathbf{R}^1$ *belongs to* $ex\tilde{A}^2$ *if and only if*

$$K(h) = \sup_{\psi \in \mathcal{C}} \int_0^1 \sqrt{\psi(\alpha)} \frac{d\alpha}{|\alpha|^{3/2}} < \infty\,, \tag{1.5}$$

where

$$\mathcal{C} = \left\{ \psi(\alpha) = \sum_{m=-\infty}^{\infty} \tau_m {\eta_m}^2(\alpha, h); \quad \tau_m > 0 \quad \sum_{m=-\infty}^{\infty} \tau_m \leq 1 \right\}.$$

Our goal is to extend Theorems IV and V to the n-dimensional case.

2. The extended ring $ex\tilde{A}^2$

We begin with the following interesting theorem which extend Theorem IV on the real line $\mathbf{R}^1$ to the case of $n-$dimensional euclidian space $\mathbf{R}^n$.

THEOREM 1. *The extended normed ring* $ex\tilde{A}^2$ *consists of all continuous functions of the form*

$$h(t) = c + f(t)\,, \tag{2.1}$$

where c is a constant and $f \in \tilde{A}^2$.

PROOF. The non-trivial part of this theorem is that $ex\tilde{A}^2$ may not contain functions other than function (2.1). In the proof, assuming $h \in ex\tilde{A}^2$, we shall insert in the formula $A(h)$, the sequence of functions $\{g_m\}$ such that $\|g_m\| \leq 1$ and converges uniformly to 1 on each compact set as $m \to \infty$, Applying Fatou's lemma, (1.1) and the inequality $\|gh\| \leq$

$\|g\|\|h\|_{ex}$ ($\forall g \in \tilde{A}^2$), we have

$$\begin{aligned} A(h) &= \int \frac{d\alpha}{|\alpha|^{3n/2}} \sqrt{\left(\frac{1}{\pi}\right)^n \int \lim_{m\to\infty} |\Delta_\alpha^n g_m h|^2 dt} \\ &\le \varliminf_{m\to\infty} \int \frac{d\alpha}{|\alpha|^{3n/2}} \sqrt{\left(\frac{1}{\pi}\right)^n \int |\Delta_\alpha^n g_m h|^2 dt} \\ &= \varliminf_{m\to\infty} A(g_m h) \le d_n \varliminf_{m\to\infty} \|g_m h\| \\ &\le d_n \varliminf_{m\to\infty} \|g_m\|\|h\|_{ex} \le d_n \|h\|_{ex} < \infty . \end{aligned}$$

Since h is assumed to be continuous, be the same way as in the proof of Theorem I, there exists $F \in A^2 \subset L^1$ such that

$$\Delta_\alpha^n h(t) = \int e^{-itx}(e^{-i\alpha x} - 1)^n F(x)dx . \tag{2.2}$$

Let us denote its Fourier transform by

$$f(t) = \int e^{-itx} F(x)dx = \hat{F}(t) . \tag{2.3}$$

Then we have

$$\Delta_\alpha^n f(t) = \int e^{-itx}(e^{-i\alpha x} - 1)^n F(x)dx , \tag{2.4}$$

so by (2.2) and (2.4)

$$\Delta_\alpha^n (h(t) - f(t)) = 0 \quad (\forall t, \alpha \in \mathbf{R}^n) . \tag{2.5}$$

Now we shall prove the following lemma for the sake of completeness.

LEMMA 2. *Let $g(t)$ be a bounded continuous function. If g satisfies the condition*

$$\Delta_\alpha^n g(t) = 0 \quad (\forall t, \alpha \in \mathbf{R}^n) , \tag{2.6}$$

then $g(t)$ is a constant.

PROOF. It is known that if a continuous function $g(t_1)$ of $t_1 \in \mathbf{R}^1$ satisfy $\Delta_{\alpha_1}^n g(t_1) = 0$ $(\forall t_1, \alpha_1 \in \mathbf{R}^1)$, then it is a polynomial of at most degree $n-1$.

Let us write $t = (t_1, t')$ with $t' = (t_2, \cdots, t_n)$ and $\alpha = (\alpha_1, 0, \cdots, 0)$. We consider condition (2.6) as a function of t_1 and α_1 for any fixed $t' = (t_2, \cdots, t_n)$. Then we can show that $g(t_1, t')$ is a polynomial of t_1 at most degree $n-1$. Furthermore since $g(t_1, t') = O(1)$ as $|t_1| \to \infty$, we can conclude that $g(t_1, t')$ is a constant as for t_1 and so $g(t_1, t') = g(0, t')$ for any $t' = (t_2, \cdots, t_n)$. Next let us write $t = (0, t_2, t'')$ with $t'' = (t_3, \cdots, t_n)$ and $\alpha = (0, \alpha_2, 0, \cdots, 0)$. Let us consider condition (2.6) as a function of t_2 and α_2 for any fixed $t'' = (t_3, \cdots, t_n)$. Let us also remark that $g(0, t_2, t'') = O(1)$ as $|t_2| \to \infty$. Then we can conclude as before that $g(0, t_2, t'')$ is a constant for t_2 and so $g(t) = g(0, 0, t'')$ for any

$t'' = (t_3, \cdots, t_n)$. Continuing these arguments, we can conclude that $g(t)$ is nothing but a constant $g(0) = g(0, 0, \cdots, 0) = c$, say. Thus we have proved Lemma 2.

Now if we apply this lemma to function $g(t) = h(t) - f(t)$, we can prove by condition (2.5) that $h(t) - f(t)$ is nothing but a constant and we can show that $h(t)$ has representation (2.1). Since $f(t)$ is continuous, $f(t) \to 0$ ($|t| \to \infty$) by (2.3) and $A(f) = A(h) < \infty$ by (2.1), we have $f \in \tilde{A}^2$ by Theorem I. The remaining part of the theorem is clear.

We shall observe the basic properties of $ex\tilde{A}^2$.

The $ex\tilde{A}^2$ is a normed ring of numerical functions under the pointwise addition and multiplication. Furthermore $ex\tilde{A}^2$ is complete and separable.

It follows from definition that $\tilde{A}^2 \subset ex\tilde{A}^2$, and

$$\|f\|_{ex} \le \|f\| \le \frac{d_n}{c_n}\|f\|_{ex} \quad \text{for } f \in \tilde{A}^2 . \tag{2.7}$$

This shows that two norms $\|\cdot\|_{ex}$ and $\|\cdot\|$ are equivalent on $\tilde{A}^2$. The following propositions are immediate corollaries to Theorem 1.

PROPOSITION 1. *$h \in ex\tilde{A}^2$ if and only if*

(a) *h is continuous*,

(b) $\lim_{|t|\to\infty} h(t) = c$ (*a constant c*),

(c) $A(h) < \infty$.

PROPOSITION 2. *If $h \in ex\tilde{A}^2$, then $h = c + f$ with a constant c and $f \in \tilde{A}^2$. Among these functions, we have the inequalities*

$$\|h\|_{ex} \le |c| + \|f\| , \tag{2.8}$$

$$|c| \le \|h\|_{ex} , \quad \|f\| \le 2\frac{d_n}{c_n}\|h\|_{ex} . \tag{2.9}$$

The $ex\tilde{A}^2$ satisfies the principle of contraction under some additional condition.

PROPOSITION 3. *Let h be a continuous function and a contraction of the series $\sum_{\nu=1}^{N} h_\nu$, where each h_ν belongs to $ex\tilde{A}^2$. Suppose that*

$$\lim_{|t|\to\infty} h(t) = c . \tag{i}$$

Then we have $h \in ex\tilde{A}^2$ and

$$\|h\|_{ex} \le \left\{1 + 2\left(\frac{d_n}{c_n}\right)^2\right\} \sum_{\nu=1}^{N} \|h_\nu(t)\| , \tag{2.10}$$

where constants c_n and d_n are those of Theorem I.

PROOF. Let us write by hypothesis

$$\text{(ii)} \qquad |h(t)| \le \sum_{\nu=1}^{N} |h_\nu(t)|\,,$$

and

$$\text{(iii)} \qquad |\Delta_\alpha^n h(t)| \le \sum_{\nu=1}^{N} |\Delta_\alpha^n h_\nu(t)|\,.$$

Let us also write $h_\nu = c_\nu + f_\nu$ with a constant c_ν and $f_\nu \in \tilde{A}^2$ $(\nu = 1, 2, \cdots, N)$. Then we have by the use of the properties (i) and (ii) as $|t| \to \infty$

$$|c| \le \sum_{\nu=1}^{N} |c_\nu|\,.$$

Hence, writing $h = c + f$, we have by the use of property (iii)

$$|\Delta_\alpha^n f(t)| \le \sum_{\nu=1}^{N} |\Delta_\alpha^n f_\nu(t)|\,,$$

$$\eta(\alpha, f) \le \sum_{\nu=1}^{N} \eta(\alpha, f_\nu)\,,$$

and so

$$A(f) \le \sum_{\nu=1}^{N} A(f_\nu)\,.$$

Since $h(t)$ is continuous and so is $f(t)$, we see that $f \in \tilde{A}^2$ by Theorem I and

$$\|f\| \le \frac{d_n}{c_n} \sum_{\nu=1}^{N} \|f_\nu\|\,.$$

Therefore we have $h \in ex\tilde{A}^2$ and

$$\begin{aligned} \|h\|_{ex} &\le |c| + \|f\| \\ &\le \sum_{\nu=1}^{N} |c_\nu| + \left(\frac{d_n}{c_n}\right) \sum_{\nu=1}^{N} \|f_\nu\| \\ &\le \left\{1 + 2\left(\frac{d_n}{c_n}\right)^2\right\} \sum_{\nu=1}^{N} \|h_\nu\|_{ex}\,. \end{aligned}$$

PROPOSITION 4. *Let h be a continuous function and let $\{h_m\}$ be a sequence of continuous functions such that each function is a contraction of the series $\sum_{\nu=1}^{N} h_\nu$, where each h_ν belongs to $ex\tilde{A}^2$. Suppose that*

(i´) $$\lim_{|t|\to\infty} h(t) = c\,, \qquad \lim_{|t|\to\infty} h_m(t) = c_m \quad (m = 1, 2, 3, \cdots)\,.$$

Then h and $\{h_m\}$ belong to $ex\tilde{A}^2$. Moreover, if

$$\lim_{m\to\infty} M(h_m - h) = 0\,,$$

then

$$\lim_{m\to\infty} \|h_m - h\|_{ex} = 0\,.$$

PROOF. h and $\{h_m\}$ belong to $ex\tilde{A}^2$ by Proposition 3. We put $h = c + f$ and $h_m = c_m + f_m$ $(m = 1, 2, 3, \cdots)$. By hypothesis, h satisfies properties (ii) and (iii) of Poroposition 3 and $\{h_m\}$ satisfies

(ii´) $$|h_m(t)| \le \sum_{\nu=1}^{N} |h_\nu(t)| \quad (m = 1, 2, 3, \cdots)$$

and

(iii´) $$|\Delta_\alpha^n h_m(t)| \le \sum_{\nu=1}^{N} |\Delta_\alpha^n h_\nu(t)| \quad (m = 1, 2, 3, \cdots)\,.$$

If $h_\nu = c_\nu + f_\nu$ with a constant c_ν and $f_\nu \in \tilde{A}^2$ $(\nu = 1, 2, \cdots, N)$, then we have by (i´)

$$\lim_{|t|\to\infty} (h_m(t) - h(t)) = c_m - c \quad (m = 1, 2, 3, \cdots)$$

and by (ii), (ii´) and (iii), (iii´),

$$|h_m(t) - h(t)| \le 2\sum_{\nu=1}^{N} |h_\nu(t)| \quad (m = 1, 2, 3, \cdots)\,,$$

$$|\Delta_\alpha^n (h_m(t) - h(t))| \le 2\sum_{\nu=1}^{N} |\Delta_\alpha^n h_\nu(t)| \quad (m = 1, 2, 3, \cdots)\,.$$

Tracing the same lines as the proof of Theorem 2 in [2], we see that $\lim_{m\to\infty} M(h_m - h) = 0$ implies $\lim_{m\to\infty} A(f_m - f) = 0$ and so $\lim_{m\to\infty} \|f_m - f\| = 0$. On the other hand it is clear that $\lim_{m\to\infty} M(h_m - h) = 0$ implies $\lim_{m\to\infty} |c_m - c| = 0$. Therefore we have

$$\|h_m - h\|_{ex} \le |c_m - c| + \|f_m - f\| \to 0 \quad (m \to \infty)\,.$$

In the study of $ex\tilde{\mathcal{A}}^2$, it is convenient to set

$$c = h(\infty) = \lim_{|t|\to\infty} h(t)\,, \tag{2.19}$$

and adjoin $t = \infty$ as an ideal point of $\mathbf{R}^n$, the so-called one-point compactification.

3. The extended ring $ex\tilde{\mathcal{A}}^2$

Let us provide a short account of basic properties of $\tilde{\mathcal{A}}^2$ and $ex\tilde{\mathcal{A}}^2$.

(1) $\tilde{\mathcal{A}}^2 = \tilde{A}^2 \cap L^2$ and the inequalities

$$\|f\|_{\tilde{\mathcal{A}}^2} > \|f\|_{\tilde{A}^2}\,,$$

$$\|f\|_{\tilde{\mathcal{A}}^2} > \left(\frac{1}{\sqrt{2\pi}}\right)^n \|f\|_{L^2}\,,$$

$$\|f\|_{\tilde{\mathcal{A}}^2} < \|f\|_{\tilde{A}^2} + \left(\frac{1}{\sqrt{2\pi}}\right)^n \|f\|_{L^2}\,,$$

hold for $f \in \tilde{\mathcal{A}}^2,\ f \neq 0$.

(2) The normed ring $\tilde{\mathcal{A}}^2$ is complete and separable, and it satisfies the uniform contraction principle with a certain constant $k = 1 + \frac{d_n}{c_n}$.

(3) $\tilde{\mathcal{A}}^2$ has the property

$$\sup_{\|g\|\le 1} \frac{|g(t)|}{\|g\|} = 1 \quad (\forall t \in \mathbf{R}^n)\,. \tag{3.1}$$

We observe at once that (3.1) implies

$$M(g) \le \|g\|\,. \tag{3.2}$$

Now we can conclude that if each function in a sequence of continuous functions $\{g_m\}$ is a contraction of the series $\sum_{\nu=1}^{N} f_\nu$ for each $f_\nu \in \tilde{\mathcal{A}}^2$, then $M(g_m) \to 0$ $(m \to \infty)$ and $\|g_m\| \to 0$ $(m \to \infty)$ are equivalent.

(4) The set of all functions $f \in \tilde{\mathcal{A}}^2$ which satisfy $\Delta_\alpha{}^k f(t) = O(|\alpha|^k)$, $(|\alpha| \le 1, k = 1, 2, 3, \cdots)$ is dense in $\tilde{\mathcal{A}}^2$.

This is proved as follows. Let us write $f = \hat{F}$, $F \in \mathcal{A}^2$ and let us write

$$F_N(x) = \begin{cases} F(x) & (|x| \le N)\,, \\ 0 & (|x| > N)\,. \end{cases}$$

Then $F_N \in \mathcal{A}^2$ and we have

$$\|F - F_N\|^2 = \inf_{\omega\in\Omega_1}\left(\omega(0) + \int \omega dx\right)\int_{|x|>N} \frac{|F|^2}{\omega} dx \to 0 \quad (N \to \infty)\,.$$

Therefore if we write $f_N = \hat{F_N}$, then $f_N \in \tilde{\mathcal{A}}^2$ and

$$\|f - f_N\| = \|F - F_N\| \to 0 \quad (N \to \infty)\,.$$

On the other hand, if we take N sufficiently large and fix it, then we have

$$f_N(t) = \int e^{-itx} F_N(x)dx = \int_{|x|\le N} e^{-itx} F(x)dx\,,$$

$$\Delta_\alpha^k f_N(t) = \int_{|x|\le N} e^{-itx}(e^{-i\alpha x} - 1)^k F(x)dx\,.$$

Since $e^{-i\alpha x} - 1 = O(|\alpha|)$ $(|\alpha| \le 1, |x| \le N)$, we have

$$|\Delta_\alpha^k f_N(t)| \le O(|\alpha|^k) \int_{|x|\le N} |F(x)|dx$$

$$\le O(|\alpha|^k)\left(\omega(0) + \int \omega dx\right)^{\frac{1}{2}} \left(\int \frac{|F|^2}{\omega} dx\right)^{\frac{1}{2}}.$$

Taking the infimum for $\omega \in \Omega_1$ on the right hand side, we have

$$|\Delta_\alpha^k f_N(t)| \le O(|\alpha|^k)\|f\| \quad (|\alpha| \le 1, k = 1, 2, 3, \cdots)\,.$$

(5) Example of a function in $\tilde{\mathcal{A}}^2$.

Let $\chi_N(t)$ be the characteristic function of the set $E_N = \{t; |t| \le N\}$ and $\rho(t)$ the mollifier due to Friedrichs. Now let us write

$$\gamma(t) = \chi_N * \rho(t)\,. \tag{3.3}$$

Then we have

$$\gamma(t) = \int_{|s|\le N} \chi_N(s)\rho(t - s)ds\,. \tag{3.4}$$

Therefore we have

$$\Delta_\alpha^n \gamma(t) = \int_{|s|\le N} \chi_N(s)\Delta_\alpha^n \rho(t - s)ds\,.$$

By an elementary calculation, we obtain the estimate

$$|\Delta_\alpha^n \rho(t)| \le C_n |\alpha|^n \sum_{k=1}^{n} P_k(t)\rho_k(t + \Theta_k \alpha)\,, \tag{3.5}$$

where

$$P_k(t) = 1 + |t| + \cdots + |t|^k\,,$$

$$\rho_k(t) = \begin{cases} \dfrac{1}{(1 - |t|^2)^{2k}} e^{-\frac{1}{1-|t|^2}} & (|t| \le 1)\,, \\ 0 & (|t| > 1)\,, \end{cases}$$

$|\alpha| \leq 1,\ 0 < \Theta_k < k$ and C_n is a constant depending only on n.

Applying the Minkowski inequality of integral type, picking the term of highest degree singularity and absorbing other terms into it, we have

$$\eta(\alpha, \gamma) \leq \frac{C_n' |\alpha|^n}{(2\pi)^{n/2}} \int_{|s| \leq N} ds \left(\int_{|t| \leq 1} \frac{1}{(1 - |t|^2)^{2n}} e^{-\frac{2}{1-|t|^2}} dt \right)^{\frac{1}{2}} .$$

This implies that

$$\int_{|s| \leq N} ds = \frac{|\sum_{n-1}|}{n} N^n \quad (n \geq 2),$$

$$\int_{|t| \leq 1} \frac{1}{(1 - |t|^2)^{2n}} e^{-\frac{2}{1-|t|^2}} dt = \int_{\sum_{n-1}} d\sigma \int_0^1 \frac{r^{n-1}}{(1 - r^2)^{2n}} e^{-\frac{2}{1-r^2}} dr$$

$$\leq \int_{\sum_{n-1}} d\sigma \int_0^1 \frac{r}{(1 - r^2)^{2n}} e^{-\frac{1}{1-r^2}} dr = \frac{1}{2} \left| \sum\nolimits_{n-1} \right| \int_1^\infty u^{2n-2} e^{-u} du ,$$

where $\sum_{n-1} = \{x \in \mathbf{R}^n ; |x| = 1\}$.

Then we have

$$A_1(r) = \int_{|\alpha| \leq 1} \eta(\alpha, r) \frac{d\alpha}{|\alpha|^{3n/2}} \leq C_n'' N^n \int_{|\alpha| \leq 1} \frac{d\alpha}{|\alpha|^{n/2}}$$

$$= C_n''^2 \frac{|\sum_{n-1}|}{n} N^n < \infty .$$

On the other hand, $\gamma(t)$ is continuous and belongs to L^2. Thus we can conclude that $\gamma(t) \in \tilde{\mathcal{A}}^2$ by Theorem III.

(6) To the normed ring $\tilde{\mathcal{A}}^2$ we may adjoin each function h with a property that $g \in \tilde{\mathcal{A}}^2$ implies $gh \in \tilde{\mathcal{A}}^2$. By the closed graph theorem, we have

$$\|gh\| \leq m\|g\| \quad (m < \infty), \tag{3.6}$$

and we define the norm of h as the least number m satisfying (3.6). We observe at once that (3.1) and (3.6) imply that

$$M(h) \leq \|h\|_{ex} . \tag{3.7}$$

Next if we take the function $\gamma(t) \in \tilde{\mathcal{A}}^2$ in (5), then by the definition of $ex\tilde{\mathcal{A}}^2$ we have

$$h(t) = \gamma(t)h(t) \in \tilde{\mathcal{A}}^2 \quad (|t| < N - 1),$$

where N is a positive integer, so $h(t)$ is continuous by Theorem III.

(7) Since each function $f \in \tilde{\mathcal{A}}^2 = \tilde{A}^2 \cap L^2$ is square summable, it follows that

$$\eta(\alpha, f) \leq \left(\frac{2}{\sqrt{2\pi}} \right)^n \|f\|_{L^2} . \tag{3.8}$$

Since the amount of the integral in $A(f)$ on the range $1 \le |\alpha|$ is therefore no longer significant, we put according to Beurling [1]

$$A_1(f) = \int_{|\alpha|\le 1} \eta(\alpha, f)\frac{d\alpha}{|\alpha|^{3n/2}} \tag{3.9}$$

and observe that

$$A_1(f) < A(f) < A_1(f) + \frac{e_n}{(\sqrt{2\pi})^n}\|f\|_{L^2}, \tag{3.10}$$

where $e_n = \frac{2^{n+1}}{n}\left|\sum_{n-1}\right|$.

Combining this with the inequalities in Theorems I and II, we have

$$\|f\| < \frac{1}{c_n}A_1(f) + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\}\frac{1}{(\sqrt{2\pi})^n}\|f\|_{L^2}, \tag{3.11}$$

and similarly,

$$\|f\| > \frac{1}{d_n}A_1(f),\ \|f\| > \frac{1}{(\sqrt{2\pi})^n}\|f\|_{L^2}. \tag{3.12}$$

Under these preparations, we shall prove the following lemma.

LEMMA 3. *Let $h(t)$ be a continuous and bounded function such that*

$$|\Delta_\alpha^k h(t)| \le C_k|\alpha|^k M(h) \quad (|\alpha| \le 1, k = 1, 2, \cdots, n-1), \tag{3.13}$$

where $C_k \le C_n$ $(1 \le k \le n-1)$ and C_n is a constant depending only on n.

Let us define

$$\xi(\alpha, g, h) = \sqrt{\left(\frac{1}{2\pi}\right)^n \int |g(t)|^2|\Delta_\alpha^n h(t)|^2 dt} \tag{3.14}$$

for any $g \in \tilde{\mathcal{A}}^2$ and

$$\xi(h) = \sup_{\|g\|\le 1}\int_{|\alpha|\le 1}\xi(\alpha, g, h)\frac{d\alpha}{|\alpha|^{3n/2}}. \tag{3.15}$$

Then $h \in ex\tilde{\mathcal{A}}^2$ is equivalent to $\xi(h) < \infty$.

PROOF. For any $g \in \tilde{\mathcal{A}}^2$ it is easy to see the formula

$$\begin{aligned}\Delta_\alpha^n gh &= \sum_{k=0}^{n}\binom{n}{k}\Delta_\alpha^k g\,\Delta_\alpha^{n-k}T_{k\alpha}h \\ &= g\,\Delta_\alpha^n h + \sum_{k=1}^{n}\binom{n}{k}\Delta_\alpha^k g\,\Delta_\alpha^{n-k}T_{k\alpha}h\end{aligned}$$

holds, where $T_{k\alpha}h(t) = h(t + k\alpha)$ $\quad (k = 1, 2, \cdots, n)$.

Then we have

$$\left|\int_{|\alpha|\le 1}\xi(\alpha,g,h)\frac{d\alpha}{|\alpha|^{3n/2}}-A_1(gh)\right|$$

$$\le M(h)\sum_{k=1}^{n}C_k\binom{n}{k}\int_{|\alpha|\le 1}\frac{d\alpha}{|\alpha|^{3n/2-(n-k)}}\sqrt{\left(\frac{1}{2\pi}\right)^n\int|\Delta_\alpha^k g(t)|^2dt}\,.$$

Here let us write

$$\eta_k(\alpha)=\sqrt{\left(\frac{1}{2\pi}\right)^n\int|\Delta_\alpha^k g(t)|^2dt}\quad (k=1,2,\cdots,n)\,,$$

and

$$A_{1,k}(g)=\int_{|\alpha|\le 1}\eta_k(\alpha)\frac{d\alpha}{|\alpha|^{3n/2-(n-k)}}\quad (k=1,2,\cdots,n)\,.$$

The estimation of $A_{1,k}(g)$.

(i) The case where $1\le k<n/2$.

Since $g\in L^2$, we have

$$\eta_k(\alpha)\le\frac{2^k}{\left(\sqrt{2\pi}\right)^n}\|g\|_{L^2}\,,$$

and so

$$\begin{aligned}A_{1,k}(g)&\le\frac{2^k}{\left(\sqrt{2\pi}\right)^n}\|g\|_{L^2}\int_{|\alpha|\le 1}\frac{d\alpha}{|\alpha|^{n/2+k}}\\&=\frac{2^k}{n/2-k}\frac{|\sum_{n-1}|}{\left(\sqrt{2\pi}\right)^n}\|g\|_{L^2}\le\frac{2^k|\sum_{n-1}|}{n/2-k}\|g\|\,.\end{aligned}$$

(ii) The case where $n/2\le k\le n-1$.

For any $\omega\in\Omega_0$ we consider the ω^* of Lemma in [2]. That is, ω^* is a majorant of ω such that $|x|^a\omega^*(|x|)$ is decreasing and $|x|^b\omega^*(|x|)$ is increasing with $a<n<b$, where constants a and b are determined later. Then we have

$$\begin{aligned}A_{1,k}(g)&=\int_{|\alpha|\le 1}\eta_k(\alpha)\frac{d\alpha}{|\alpha|^{3n/2-(n-k)}}\\&=\int_{|\alpha|\le 1}\frac{\eta_k(\alpha)}{\omega^*(1/|\alpha|)^{1/2}|\alpha|^{n/2-(n-k)}}\frac{\omega^*(1/|\alpha|)^{1/2}d\alpha}{|\alpha|^n}\\&\le\left(\int_{|\alpha|\le 1}\frac{\eta_k^2(\alpha)}{\omega^*(1/|\alpha|)|\alpha|^{-n+2k}}d\alpha\right)^{1/2}\left(\int_{|\alpha|\le 1}\frac{\omega^*(1/|\alpha|)}{|\alpha|^{2n}}d\alpha\right)^{1/2}.\end{aligned}$$

As for the second integral of the last formula, we have

$$\int_{|\alpha|\le 1}\frac{\omega^*(1/|\alpha|)}{|\alpha|^{2n}}d\alpha=\int_{\sum_{n-1}}d\sigma\int_0^1\frac{\omega^*(1/r)}{r^{n+1}}dr$$
$$=\int_{\sum_{n-1}}d\sigma\int_1^\infty\omega^*(s)s^{n-1}ds=\int_{|x|\ge 1}\omega^*(|x|)dx\,.$$

As for the first integral of the last formula, let us write $g=\hat{G}$, $G\in\mathcal{A}^2=A^2\cap L^2$. Applying the Plancherel theorem, we have

$$\int_{|\alpha|\le 1}\frac{\eta_k^2(\alpha)}{\omega^*(1/|\alpha|)|\alpha|^{-n+2k}}d\alpha$$
$$=\int_{|\alpha|\le 1}\frac{d\alpha}{\omega^*(1/|\alpha|)|\alpha|^{-n+2k}}2^{2k}\int|G(x)|^2\sin^{2k}\left(\frac{\alpha x}{2}\right)dx$$
$$=2^{2k}\int|G(x)|^2dx\int_{|\alpha|\le 1}\frac{\sin^{2k}(\alpha x/2)}{\omega^*(1/|\alpha|)|\alpha|^{-n+2k}}d\alpha\,.$$

In estimating the inner integral, let us write $\alpha=rs, s\in\sum_{n-1}$, $r=|\alpha|$, then $d\alpha=r^{n-1}drd\sigma$, where $d\sigma$ is area element of $\sum_{n-1}$. Furthermore let us write $\rho=|x|r$, then $dr=d\rho/|x|$. Then we have

$$I_k=\int_{|\alpha|\le 1}\frac{\sin^{2k}(\alpha x/2)}{\omega^*(1/|\alpha|)|\alpha|^{-n+2k}}d\alpha=\int_0^1dr\int_{\sum_{n-1}}\frac{\sin^{2k}(rsx/2)}{\omega^*(1/r)r^{-n+2k}}r^{n-1}d\sigma$$
$$=\int_0^{|x|}d\rho\int_{\sum_{n-1}}\frac{\sin^{2k}(\rho sx/(2|x|))}{\omega^*(|x|/\rho)}\left(\frac{\rho}{|x|}\right)^{2n-2k-1}\frac{d\sigma}{|x|}$$
$$=\frac{|x|^{-2n+2k}}{\omega^*(|x|)}\int_0^{|x|}\rho^{2n-2k-1}d\rho\int_{\sum_{n-1}}\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)}\sin^{2k}(\rho sx/(2|x|))d\sigma\,.$$

Here in the case of $|x|\le 1$, since $|x|^b\omega^*(|x|)$ $(n<b)$ is increasing and $0<\rho<1$, we have $|x|^b\omega^*(|x|)\le(|x|/\rho)^b\omega^*(|x|/\rho)$ and so

$$\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)}\le\frac{1}{\rho^b}\,,\quad \sin^{2k}(\rho sx/(2|x|))\le\rho^{2k}\,.$$

Therefore we have

$$\int_0^{|x|}\rho^{2n-2k-1}d\rho\int_{\sum_{n-1}}\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)}\sin^{2k}(\rho sx/(2|x|))d\sigma$$
$$\le|\textstyle\sum_{n-1}|\displaystyle\int_0^{|x|}\rho^{2n-b-1}d\rho=\frac{|\sum_{n-1}|}{2n-b}|\rho|^{2n-b}\Big|_{\rho=0}^{|x|}\,.$$

Now if we set $b = 2k + 1/2$, then $2n - b > 3/2$. Then we have

$$\begin{aligned} I_k &\le \frac{|x|^{-2n+2k}}{\omega^*(|x|)} \frac{|\sum_{n-1}|}{2n-(2k+1/2)} |x|^{2n-(2k+1/2)} \\ &\le \frac{2|\sum_{n-1}|}{3} \frac{|x|^{3/2}}{\omega^*(|x|)} \le \frac{2|\sum_{n-1}|}{3} \frac{1}{\omega^*(|x|)} . \end{aligned}$$

Next in the case of $|x| > 1$, we decompose I_k into

$$\begin{aligned} &\frac{|x|^{-2n+2k}}{\omega^*(|x|)} \left(\int_0^1 + \int_1^{|x|} \right) \rho^{2n-2k-1} d\rho \int_{\sum_{n-1}} \frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \sin^{2k}(\rho s x/(2|x|)) d\sigma \\ &= I_{k,1} + I_{k,2} . \end{aligned}$$

As for $I_{k,1}$, using the fact that $|x|^b \omega^*(|x|)$ with $b = 2k + 1/2$ is increasing, we have

$$I_{k,1} \le |\textstyle\sum_{n-1}| \tfrac{1}{\omega^*(|x|)} .$$

As for $I_{k,2}$, since $|x|^a \omega^*(|x|)$ $(a < n)$ is decreasing and $\rho > 1$, we have $|x|^a \omega^*(|x|) \le (|x|/\rho)^a \omega^*(|x|/\rho)$ and so

$$\frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \le \frac{1}{\rho^a}, \quad \sin^{2k}(\rho s x/(2|x|)) \le 1 .$$

Then we have

$$\begin{aligned} &\int_1^{|x|} \rho^{2n-2k-1} d\rho \int_{\sum_{n-1}} \frac{\omega^*(|x|)}{\omega^*(|x|/\rho)} \sin^{2k}(\rho s x/(2|x|)) d\sigma \\ &\le |\textstyle\sum_{n-1}| \displaystyle\int_1^{|x|} \rho^{2n-2k-1} \frac{d\rho}{\rho^a} = |\textstyle\sum_{n-1}| \displaystyle\left[\frac{\rho^{2n-2k-a}}{2n-2k-a} \right]_{\rho=1}^{|x|} . \end{aligned}$$

Now if we set $a = n - k - 1/2$, then $2n - 2k - a \ge 3/2$, from which we can deduce

$$\begin{aligned} I_{k,2} &\le \frac{|x|^{-2n+2k}}{\omega^*(|x|)} \frac{|\sum_{n-1}|}{(n-k)+1/2} |x|^{(n-k)+1/2} \\ &\le \frac{2|\sum_{n-1}|}{3} \frac{|x|^{-(n-k)+1/2}}{\omega^*(|x|)} \le \frac{2|\sum_{n-1}|}{3} \frac{1}{\omega^*(|x|)} . \end{aligned}$$

Therefore we have

$$I_k = I_{k,1} + I_{k,2} \le \frac{5|\sum_{n-1}|}{3} \frac{1}{\omega^*(|x|)} .$$

By these estimates, we have

$$A_{1,k}^2(g) \le 2^{2k+1} |\textstyle\sum_{n-1}| \displaystyle\int_{|x|\ge 1} \omega^*(|x|) dx \int \frac{|G(x)|^2}{\omega^*(|x|)} dx .$$

Since $b = 2k + 1/2$, $a = n - k - 1/2$ and $n/2 \leq k \leq n - 1$, we have

$$\frac{b(2n-a)}{(n-a)(b-n)} = \frac{(2k+\frac{1}{2})(n+k+\frac{1}{2})}{(k+\frac{1}{2})(-n+2k+\frac{1}{2})}$$
$$\leq \frac{(2n-\frac{3}{2})(2n-\frac{1}{2})}{\frac{1}{2}(\frac{n}{2}+\frac{1}{2})} \leq \frac{(4n-3)(4n-1)}{n+1} \leq 8\,.$$

Applying Lemma in [2], we have

$$A_{1,k}^2(g) \leq 2^{2k+4}|\textstyle\sum_{n-1}|\left(\omega(0) + \displaystyle\int \omega dx\right)\displaystyle\int \frac{|G(x)|^2}{\omega}dx\,,$$

hence

$$A_{1,k}(g) \leq \sqrt{2^{2k+4}\left|\textstyle\sum_{n-1}\right|}\|g\|\,.$$

(iii) The case where $k = n$. Since

$$A_{1,n}(g) = A_1(g) \leq d_n\|g\|\,,$$

we have

$$\left|\int_{|\alpha|\leq 1} \xi(\alpha, g, h)\frac{d\alpha}{|\alpha|^{3n/2}} - A_1(gh)\right|$$
$$\leq M(h)\sum_{k=1}^{n} C_k\binom{n}{k}A_{1,k}(g) \leq CM(h)\|g\|\,,$$

where C is a constant depending on n.

Now let us write

$$a_1(h) = \sup_{\|g\|\leq 1} A_1(gh)$$

and take the supremum with respect to $g \in \tilde{\mathcal{A}}^2$ with $\|g\| \leq 1$ in the above inequality. Then we have

$$|\xi(h) - a_1(h)| \leq CM(h)\,.$$

On the other hand from (3.11) and (3.12) we have

$$\frac{1}{d_n}A_1(gh) \leq \|gh\| \leq \frac{1}{c_n}A_1(gh) + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\}\frac{1}{(\sqrt{2\pi})^n}\|gh\|_{L^2}\,.$$

Taking the supremum with respect to $g \in \tilde{\mathcal{A}}^2$ with $\|g\| \leq 1$, we have

$$\frac{1}{d_n}a_1(h) \leq \|h\|_{ex} \leq \frac{1}{c_n}a_1(h) + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\}\frac{1}{(\sqrt{2\pi})^n}M(h)\,.$$

Combining these inequality, we have

$$\frac{1}{d_n}(\xi(h) - CM(h)) \leq \|h\|_{ex}$$
$$\leq \frac{1}{c_n}(\xi(h) + CM(h)) + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\} \frac{1}{(\sqrt{2\pi})^n} M(h)\,.$$

Therefore the equivalence of $h \in ex\tilde{\mathcal{A}}^2$ and $\xi(h) < \infty$ has been proved.

(8) From Lemma 3, we have easily derived the following properites.

Let us assume that $h(t)$ is continuous and bounded. Under this condition, we have

(i) If $\Delta_\alpha^k h(t) = O(|\alpha|^k)(|\alpha| \leq 1, k = 1, 2, \cdots, n)$, then $\xi(h) < \infty$, so that $h \in ex\tilde{\mathcal{A}}^2$.

(ii) If $h = f + g$ with $f \in \tilde{\mathcal{A}}^2$ and $\Delta_\alpha^k g(t) = O\left(|\alpha|^k\right)(|\alpha| \leq 1, k = 1, 2, \cdots, n)$, then $h \in ex\tilde{\mathcal{A}}^2$. In particular, if $h = f + c$ with a constant c, then $h \in ex\tilde{\mathcal{A}}^2$.

Here it should be noted the difference between (3.13) of Lemma 3 and the assumption in (i).

The following theorem is the extention of the interesting Theorem V on the real line $\mathbf{R}^1$ to the case of $n-$dimensional euclidian space $\mathbf{R}^n$.

THEOREM 2. *Let us suppose that h is a bounded continuous function on* $\mathbf{R}^n$ *and satisfies the same condition as* (3.13) *of Lemma* 3. *Then h belongs to* $ex\tilde{\mathcal{A}}^2$ *if and only if*

$$K(h) = \sup_{\psi \in \mathcal{C}} \int_{|\alpha| \leq 1} \sqrt{\psi(\alpha)} \frac{d\alpha}{|\alpha|^{3n/2}} < \infty\,, \tag{3.16}$$

where

$$\mathcal{C} = \left\{\psi(\alpha) = \sum_{m=0}^{\infty} \tau_m {\eta_m}^2(\alpha, h);\, \tau_m > 0,\ \sum_{m=0}^{\infty} \tau_m \leq 1\right\} \tag{3.17}$$

and

$${\eta_m}^2(\alpha, h) = \frac{1}{(\sqrt{2\pi})^n} \int_{m \leq |t| \leq m+1} |\Delta_\alpha^n h(t)|^2 dt \quad (m = 0, 1, 2, \cdots)\,. \tag{3.18}$$

The proof can be done by following the same lines as Beurling [1] through several lemmas.

LEMMA 4. *Let* $\{E_i\}_{i=1}^{\infty}$ *be a sequence of closed sets in* $\mathbf{R}^n$ *such that the distance between* E_i *and* E_j *is larger than n for* $i \neq j$. *Let f belong to* $\tilde{\mathcal{A}}^2$ *and have the expansion* $\sum_{i=1}^{\infty} f_i$ *where each* f_i *is continuous and vanishes outside* E_i. *Then each* f_i *belongs to* $\tilde{\mathcal{A}}^2$ *and satisfies*

$$\sqrt{\sum_{i=1}^{\infty} \|f_i\|^2} \leq k_1 \|f\|\,, \tag{3.19}$$

where

$$k_1 = \sqrt{2\left\{\left(\frac{d_n}{c_n}\right)^2 + \left(1+\frac{e_n}{c_n}\right)^2\right\}}\,. \tag{3.20}$$

PROOF. For $|\alpha| \le 1$ we have by assumption,

$$\Delta_\alpha^n f_i(t)\overline{\Delta_\alpha^n f_j(t)} = 0 \quad (i \neq j)\,,$$

from which it follows that

$$\sum_{i=1}^{\infty} \eta^2(\alpha, f_i) = \eta^2(\alpha, f)\,.$$

By the Schwartz inequality we have

$$A_1^2(f_i) = \left(\int_{|\alpha|\le 1} \eta(\alpha, f_i)\frac{d\alpha}{|\alpha|^{3n/2}}\right)^2 \le A_1(f)\int_{|\alpha|\le 1} \frac{\eta^2(\alpha, f_i)}{\eta^2(\alpha, f)}\frac{d\alpha}{|\alpha|^{3n/2}}\,.$$

Then taking the summation on both sides, we have

$$\sum_{i=1}^{\infty} A_1^2(f_i) \le A_1^2(f) < \infty\,.$$

Similarly, we have by assumption

$$f_i(t)\overline{f_j(t)} = 0 \quad (i \neq j)\,,$$

hence

$$\sum_{i=1}^{\infty} \|f_i\|_{L^2}^2 = \|f\|_{L^2}^2 < \infty\,.$$

Therefore we can conclude that each f_i belongs to $\tilde{\mathcal{A}}^2$. Now applying inequality (3.11) yields

$$\|f_i\|^2 < \frac{2}{c_n^2}A_1^2(f_i) + 2\left(1+\frac{e_n}{c_n}\right)^2\frac{1}{(2\pi)^n}\|f_i\|_{L^2}^2\,.$$

From (3.12) we obtain

$$\begin{aligned}\sum_{i=1}^{\infty}\|f_i\|^2 &\le \frac{2}{c_n^2}A_1^2(f) + 2\left(1+\frac{e_n}{c_n}\right)^2\frac{1}{(2\pi)^n}\|f\|_{L^2}^2 \\ &\le 2\left\{\left(\frac{d_n}{c_n}\right)^2 + \left(1+\frac{e_n}{c_n}\right)^2\right\}\|f\|^2 = k_1^2\|f\|^2\,.\end{aligned}$$

LEMMA 5. *There is a constant* k_2 *such that for any* $g \in \tilde{A}^2$

$$\sqrt{\sum_{m=0}^{\infty} b_m^2} \leq k_2 \|g\| , \tag{3.21}$$

where b_m *is the maximum of* $|g(t)|$ *on the set* $\{t \in \mathbf{R}^n; m \leq |t| \leq m+1\}$ $(m = 0, 1, 2, \cdots)$.

PROOF. We begin by constructing the function $\gamma_m(t)$ according to Example in (5). Let us write

$$\gamma_m(t) = \chi_m(t) * \rho(t) \quad (m = 0, 1, 2, \cdots) ,$$

where $\chi_m(t) = 1$ $(m \leq |t| \leq m+1)$; $\chi_m(t) = 0$ $(|t| < m$ or $m+1 < |t|)$ and $\rho(t)$ is the mollifier. Then we have

$$\gamma_m(t) = \int \chi_m(t)\rho(t-s)ds = \int_{m \leq |t| \leq m+1, |t-s| \leq 1} \rho(t-s)ds .$$

Now let us write

$$\gamma^{[j]}(t) = \sum_{i=0}^{\infty} \gamma_{4ni+j}(t) \quad (j = 0, 1, 2, \cdots, 4n-1) .$$

Then we have

$$|\Delta_\alpha^k \gamma^{[j]}(t)| = \left| \sum_{i=0}^{\infty} \Delta_\alpha^k \gamma_{4ni+j}(t) \right| \leq \int_{|t-s| \leq 1} |\Delta_\alpha^k \rho(t-s)|ds .$$

By applying the estimation in Example of (5), the right hand side of the above formula is less than

$$\begin{aligned} C_k|\alpha|^k \sum_{l=1}^{k} \int_{|s| \leq 1} & P_l(s)\rho_l(s + \Theta_l\alpha)ds \\ & \leq C'_k|\alpha|^k \int_{|s| \leq 1} \frac{1}{(1-|s|^2)^{2l}} e^{-\frac{1}{1-|s|^2}} ds \\ & \leq C''_k|\alpha|^k \quad (\forall t, |\alpha| \leq 1, k = 0, 1, 2, \cdots, n) , \end{aligned}$$

where $j = 0, 1, 2, \cdots, 4n-1$. Furthermore, by Lemma 3 we have

$$\begin{aligned} \xi(\alpha, g, \gamma^{[j]}) &= \sqrt{\frac{1}{(2\pi)^n} \int |g(t)|^2 |\Delta_\alpha^n \gamma^{[j]}(t)|^2 dt} \\ &\leq \frac{C_n}{(\sqrt{2\pi})^n} |\alpha|^n \|g\|_{L^2} , \end{aligned}$$

hence

$$\xi(\gamma^{[j]}) = \sup_{\|g\|\le 1} \int_{|\alpha|\le 1} \xi(\alpha, g, \gamma^{[j]}) \frac{d\alpha}{|\alpha|^{3n/2}}$$
$$\le \frac{C_n}{(\sqrt{2\pi})^n} \|g\|_{L^2} \int_{|\alpha|\le 1} \frac{d\alpha}{|\alpha|^{n/2}} = C'_n \|g\|_{L^2} < \infty\,,$$

from which we can conclude that $\gamma^{[j]} \in ex\tilde{\mathcal{A}}^2$. Now for $\gamma(t) = \gamma^{[0]}(t)$ and any $g \in \tilde{\mathcal{A}}^2$, since $\gamma g \in \tilde{\mathcal{A}}^2$ and $\gamma g = \sum_{i=0}^{\infty} \gamma_{4ni} g$, applying Lemma 4 implies

$$\sum_{i=0}^{\infty} b_{4ni}^2 \le \sum_{i=0}^{\infty} M(\gamma_{4ni} g)^2 \le \sum_{i=0}^{\infty} \|\gamma_{4ni} g\|_{ex}^2$$
$$\le \sum_{i=0}^{\infty} \|\gamma_{4ni} g\|^2 \le k_1 \|\gamma g\|^2 \le k_1 \|\gamma\|_{ex}^2 \|g\|^2\,.$$

As for $\gamma^{[j]} g$ $(j = 1, 2, \cdots, 4n-1)$, we have the same estimations, so that there is a constant k_2 such that

$$\sum_{m=0}^{\infty} b_m^2 = \sum_{j=0}^{4n-1} \left(\sum_{i=0}^{\infty} b_{4ni+j}^2 \right) \le {k_2}^2 \|g\|^2\,,$$

where

$$k_2 = \sqrt{k_1 \sum_{j=0}^{4n-1} \|\gamma^{[j]}\|_{ex}} \le \sqrt{4k_1 n} \|\gamma\|_{ex}\,. \tag{3.22}$$

LEMMA 6. *For any sequence $\{a_m\}_{m=0}^{\infty}$ of non-negative numbers with a finite square sum there exists a constant k_3 and $g \in \tilde{\mathcal{A}}^2$ such that*

$$\min_{m\le|t|\le m+1} |g(t)| \ge a_m \quad (m = 0, 1, 2, \cdots) \tag{3.23}$$

and

$$\|g\| \le k_3 \sqrt{\sum_{m=0}^{\infty} a_m^2}\,. \tag{3.24}$$

PROOF. Using $\gamma_m(t)$ in Lemma 5, we write

$$g_j(t) = \sum_{i=0}^{\infty} a_{4ni+j} \gamma_{4ni+j}(t) \quad (j = 0, 1, 2, \cdots, 4n-1)\,.$$

By the same lines as the proof of Lemma 4 we have

$$\eta^2(\alpha, g_j) \leq \sum_{i=0}^{\infty} a_{4ni+j}^2 \eta^2(\alpha, \gamma_{4ni+j}),$$

and so

$$A_1(g_j) \leq C' \sqrt{\sum_{i=0}^{\infty} a_{4ni+j}^2} \quad (j = 0, 1, 2, \cdots, 4n-1),$$

where a constant C' is determined by the inequalities

$$A_1(\gamma_{4ni+j}) \leq C' \quad \begin{pmatrix} i = 0, 1, 2, \cdots; \\ j = 0, 1, 2, \cdots, 4n-1 \end{pmatrix}. \tag{3.25}$$

Similarly, we have

$$\|g_j\|_{L^2} \leq C'' \sqrt{\sum_{i=0}^{\infty} a_{4ni+j}^2},$$

where a constant C'' is determined by the inequalities

$$\|\gamma_{4ni+j}\|_{L^2} \leq C'' \quad \begin{pmatrix} i = 0, 1, 2, \cdots; \\ j = 0, 1, 2, \cdots, 4n-1 \end{pmatrix}. \tag{3.26}$$

By Theorem III we have $g_j \in \tilde{\mathcal{A}}^2$ $(j = 0, 1, 2, \cdots, 4n-1)$ and the following inequalities hold:

$$\begin{aligned} \|g_j\| &\leq \frac{1}{c_n} A_1(g_j) + \left(1 + \frac{e_n}{c_n}\right) \frac{1}{(\sqrt{2\pi})^2} \|g\|_{L^2} \\ &\leq \left\{ \frac{C'}{c_n} + \frac{C''}{(\sqrt{2\pi})^n} \left(1 + \frac{e_n}{c_n}\right) \right\} \sqrt{\sum_{i=0}^{\infty} a_{4ni+j}^2} \quad (j = 0, 1, 2, \cdots, 4n-1). \end{aligned}$$

Then if we write $g = \sum_{j=0}^{4n-1} g_j$, then we have $g \in \tilde{\mathcal{A}}^2$ and the inequlities

$$\|g\| \leq \sum_{j=0}^{4n-1} \|g_j\| \leq k_3 \sqrt{\sum_{m=0}^{\infty} a_m^2}$$

with

$$k_3 \leq 4n \left\{ \frac{C'}{c_n} + \frac{C''}{(\sqrt{2\pi})^n} \left(1 + \frac{e_n}{c_n}\right) \right\}. \tag{3.27}$$

PROOF OF THEOREM 2. For any $g \in \tilde{\mathcal{A}}^2$ and a bounded continuous function h on $\mathbf{R}^n$ satisfying (3.13). Let

$$\xi(\alpha, g, h) = \sqrt{\frac{1}{(2\pi)^n} \int |g(t)|^2 |\Delta_\alpha^n h(t)|^2 dt}$$

and

$$b_m = \max_{m \le |t| \le m+1} |g(t)|, \quad a_m = \min_{m \le |t| \le m+1} |g(t)| \quad (m = 0, 1, 2, \cdots).$$

Then we have the inequalities

$$\text{(3.28)} \qquad \int_{|\alpha| \le 1} \sqrt{\sum_{m=0}^{\infty} a_m^2 \eta_m^2} \frac{d\alpha}{|\alpha|^{3n/2}} \le \int_{|\alpha| \le 1} \xi(\alpha, g, h) \frac{d\alpha}{|\alpha|^{3n/2}}$$
$$\le \int_{|\alpha| \le 1} \sqrt{\sum_{m=0}^{\infty} b_m^2 \eta_m^2} \frac{d\alpha}{|\alpha|^{3n/2}}.$$

Firstly let $h \in ex\tilde{\mathcal{A}}^2$ and let $\psi \in \mathcal{C}$ such that $\psi(\alpha) = \sum_{m=0}^{\infty} \tau_m \eta_m^2(\alpha, h)$, $\tau_m \ge 0$, $\sum_{m=0}^{\infty} \tau_m \le 1$. Hence by writing $\tau_m = a_m^2$ and applying Lemma 6 to the sequence $\{a_m/k_3\}$, there exists $g \in \tilde{\mathcal{A}}^2$ such that

$$\min_{m \le |t| \le m+1} |g(t)| \ge \frac{a_m}{k_3} \quad (m = 0, 1, 2, \cdots)$$

and

$$\|g\| \le k_3 \sqrt{\sum_{m=0}^{\infty} \left(\frac{a_m}{k_3}\right)^2} = \sqrt{\sum_{m=0}^{\infty} \tau_m} \le 1.$$

Then applying the first inequality of (3.28) yields

$$\int_{|\alpha| \le 1} \sqrt{\psi(\alpha)} \frac{d\alpha}{|\alpha|^{3n/2}} = \int_{|\alpha| \le 1} \sqrt{\sum_{m=0}^{\infty} a_m^2 \eta_m^2} \frac{d\alpha}{|\alpha|^{3n/2}}$$
$$\le \int_{|\alpha| \le 1} \xi(\alpha, g, h) \frac{d\alpha}{|\alpha|^{3n/2}},$$

therefore

$$K(h) = \sup_{\psi \in \mathcal{C}} \int_{|\alpha| \le 1} \sqrt{\psi(\alpha)} \frac{d\alpha}{|\alpha|^{3n/2}} \le \xi(h) < \infty$$

by Lemma 3.

Secondly suppose that $K(h) < \infty$. For any $g \in \tilde{\mathcal{A}}^2$ with $\|g\| \leq 1$, let us write $\max_{m \leq |t| \leq m+1} |g(t)| = b_m$ and $\tau_m = (b_m/k_2)^2$. Then by Lemma 5 we have $\tau_m \geq 0$, $\sum_{m=0}^{\infty} \tau_m \leq \|g\|^2 \leq 1$, $\psi(\alpha) = \sum_{m=0}^{\infty} \tau_m \eta_m^2 \in \mathcal{C}$ and

$$\sqrt{\sum_{m=0}^{\infty} b_m^2 \eta_m^2} = k_2 \sqrt{\sum_{m=0}^{\infty} \tau_m \eta_m^2} = k_2 \sqrt{\psi(\alpha)}\,.$$

Now applying the second inequality of (3.28), we have

$$\begin{aligned}\int_{|\alpha|\leq 1} \xi(\alpha, g, h) \frac{d\alpha}{|\alpha|^{3n/2}} &\leq \int_{|\alpha|\leq 1} \sqrt{\sum_{m=0}^{\infty} b_m^2 \eta_m^2}\, \frac{d\alpha}{|\alpha|^{3n/2}} \\ &\leq k_2 \int_{|\alpha|\leq 1} \sqrt{\psi(\alpha)} \frac{d\alpha}{|\alpha|^{3n/2}}\,,\end{aligned}$$

and therefore

$$\xi(h) = \sup_{\|g\|\leq 1} \int_{|\alpha|\leq 1} \xi(\alpha, g, h) \frac{d\alpha}{|\alpha|^{3n/2}} \leq k_2 K(h) < \infty\,.$$

From this with the help of Lemma 3, we can conclude that $h \in ex\tilde{\mathcal{A}}^2$. Thus the theorem has completely proved.

THE DIVISOR PROBLEM

Finally, we show the inequality

$$\|h^{-1}\|_{ex} \leq \frac{k_4}{m^{n+1}} \|h\|_{ex}^n$$

which is expected for a function $h \in ex\tilde{\mathcal{A}}^2$ with $|h(t)| > m > 0$.

We need some additional conditions.

THEOREM 3. *Suppose that $h \in ex\tilde{\mathcal{A}}^2$ satisfies the conditions*

(3.29) $$|h(t)| > m > 0 \quad (\forall t \in \mathbf{R}^n)$$

and

(3.30) $$|\Delta_\alpha^k h(t)| \leq C_k |\alpha|^k M(h) \quad (|\alpha| < 1, k = 1, 2, \cdots, n)\,.$$

Then $h^{-1} \in ex\tilde{\mathcal{A}}^2$ and

(3.31) $$\|h^{-1}\|_{ex} \leq \frac{k_4}{m^{n+1}} \|h\|_{ex}^n\,,$$

where

$$k_4 = \frac{C}{c_n}2^n + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\}\left(\frac{1}{\sqrt{2\pi}}\right)^n \tag{3.32}$$

and C *is a constant depending only on* n.

We need the following estimations.

LEMMA 7. *Suppose that* h *is a bounded continuous function which satisfies conditions* (3.29) *and* (3.30). *Then we have*

$$|\Delta_\alpha^k h^{-1}(t)| \le C_k' |\alpha|^k \frac{M(h)^k}{m^{k+1}} \quad (|\alpha| \le 1, k = 1, 2, \cdots, n)\,. \tag{3.33}$$

PROOF. These estimations are obtained by elementary calculations. For example, in the case of $k = 1$, since

$$\Delta_\alpha^1 h^{-1}(t) = \frac{1}{h(t+\alpha)} - \frac{1}{h(t)} = -\frac{\Delta_\alpha^1 h(t)}{h(t)h(t+\alpha)}$$

we obtain

$$|\Delta_\alpha^1 h^{-1}(t)| \le \frac{|\Delta_\alpha^1 h(t)|}{m^2} \le C_1|\alpha|\frac{M(h)}{m^2} \quad (|\alpha| \le 1)\,.$$

In the case of $k = 2$, we have

$$\begin{aligned}\Delta_\alpha^2 h^{-1}(t) &= \Delta_\alpha^1(\Delta_\alpha^1 h^{-1}(t)) \\ &= -\frac{\Delta_\alpha^1 h(t+\alpha)}{h(t+\alpha)h(t+2\alpha)} + \frac{\Delta_\alpha^1 h(t)}{h(t)h(t+\alpha)} \\ &= -\frac{h(t)\{\Delta_\alpha^1 h(t+\alpha) - \Delta_\alpha^1 h(t)\}}{h(t)h(t+\alpha)h(t+2\alpha)} + \frac{\Delta_\alpha^1 h(t)\{h(t+2\alpha) - h(t)\}}{h(t)h(t+\alpha)h(t+2\alpha)} \\ &= \frac{-h(t)\Delta_\alpha^2 h(t) + \Delta_\alpha^1 h(t)\Delta_{2\alpha}^1 h(t)}{h(t)h(t+\alpha)h(t+2\alpha)}\,,\end{aligned}$$

hence

$$|\Delta_\alpha^2 h^{-1}(t)| \le C_2|\alpha|^2\frac{M(h)^2}{m^3} \quad (|\alpha| \le 1)\,.$$

Finally we have

$$\begin{aligned}\Delta_\alpha^n h^{-1}(t) &= \Delta_\alpha^1(\Delta_\alpha^{n-1} h^{-1}(t)) \\ &= \frac{\Sigma_n}{h(t)h(t+\alpha)\cdots h(t+n\alpha)}\end{aligned}$$

with

$$\begin{aligned}\Sigma_n = & -h(t)h(t+\alpha)\cdots h(t+(n-2)\alpha)\Delta_\alpha^n h(t) + \\ & - h(t)h(t+\alpha)\cdots h(t+(n-3)\alpha)\Delta_\alpha^1 h(t+\alpha)\Delta_{2\alpha}^{n-1}h(t) \\ & + \cdots \\ & + \Delta_\alpha^1 h(t)\Delta_{2\alpha}^1 h(t)\cdots \Delta_{n\alpha}^1 h(t) .\end{aligned}$$

Here it should be remarked that the numerator consists of the sum of terms, each of which is the product of several kinds of differences and sum of their degrees are always just n and the denominator is estimated from below by $|h(t)|^{n+1} > m^{n+1} > 0$ ($\forall t \in \mathbf{R}^n$). Therefore we have

$$|\Delta_\alpha^n h^{-1}(t)| \le C_n |\alpha|^n \frac{M(h)^n}{m^{n+1}} \quad (|\alpha| \le 1) .$$

PROOF OF THEOREM 3. For any $g \in \tilde{\mathcal{A}}^2$ we consider

$$A_1(gh^{-1}) = \int_{|\alpha|\le 1} \frac{d\alpha}{|\alpha|^{3n/2}} \sqrt{\frac{1}{(2\pi)^n}\int |\Delta_\alpha^n(gh^{-1})|^2 dt} , \tag{3.34}$$

where

$$\Delta_\alpha^n(gh^{-1}) = \sum_{k=0}^n \binom{n}{k} \Delta_\alpha^k g\, \Delta_\alpha^{n-k} T_{k\alpha} h^{-1} .$$

By the use of estimations of Lemma 7 we have

$$|\Delta_\alpha^{n-k} T_{k\alpha} h^{-1}(t)| \le C'_{n-k} |\alpha|^{n-k} \frac{M(h)^{n-k}}{m^{n-k+1}} ,$$

where $|\alpha| \le 1$, $k = 0, 1, 2, \cdots, n$. Applying the Minkowski inequality, we have

$$A_1(gh^{-1}) \le \sum_{k=0}^n \binom{n}{k} \frac{C'_{n-k}}{m^{n-k+1}} M(h)^{n-k} A_{1,k}(g) ,$$
$$A_{1,k}(g) \le C_{n,k}\|g\| \quad (k = 0, 1, 2, \cdots, n) ,$$

where $C_{n,k}$ are constants depending only on k and n.

Putting $C = \max_{0\le k\le n} C_{n,k} C'_{n-k}$, we have

$$A_1(gh^{-1}) \le C 2^n \frac{M(h)^n}{m^{n+1}} \|g\| .$$

Taking the supremum with respect to $g \in \tilde{\mathcal{A}}^2$ with $\|g\| \leq 1$, we have

$$\|h^{-1}\|_{ex} \leq \frac{1}{c_n} \sup_{\|g\|\leq 1} A_1(gh^{-1}) + \left\{1 + \left(\frac{e_n}{c_n}\right)\right\} \frac{1}{\left(\sqrt{2\pi}\right)^n} M(h)$$

$$= k_4 \frac{M(h)^n}{m^{n+1}},$$

where k_4 is the constant defined in (3.32).

Since $M(h) \leq \|h\|_{ex}$, we have

$$\|h^{-1}\|_{ex} \leq \frac{k_4}{m^{n+1}} \|h\|_{ex}^n .$$

Thus Theorem 3 is proved.

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