TOKYO J. MATH.
VOL. 27, NO. 1, 2004

On the Iwasawa Invariants of Z_2-Extensions of Certain Real Quadratic Fields

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(Communicated by Y. Yamada)

Abstract. For a real quadratic field k, we denote by $\lambda_2(k)$, $\mu_2(k)$ and $\nu_2(k)$ the Iwasawa λ-, μ- and ν-invariants of the cyclotomic $\mathbf{Z}_2$-extension of k, respectively. In this paper, we give certain families of real quadratic fields k such that $\lambda_2(k) = \mu_2(k) = 0$ and $\nu_2(k) = 2$, by using Kuroda's class number formula.

1. Introduction

Let k be a finite extension of the field $\boldsymbol{Q}$ of rational numbers. For a fixed prime number l, we denote by k_∞ the cyclotomic $\mathbf{Z}_l$-extension of k. Then the Galois group $\mathrm{Gal}(k_\infty/k)$ is isomorphic to the additive group $\mathbf{Z}_l$ of l-adic integers. For each integer $n \geq 0$, k_∞ has a unique subfield k_n which is a cyclic extension of degree l^n over k, it is called n-th layer. Let e_n be the highest power of l dividing the class number of n-th layer k_n. We denote by $\lambda_l(k)$, $\mu_l(k)$ and $\nu_l(k)$ the Iwasawa λ-, μ- and ν-invariants of k_∞, respectively, satisfying Iwasawa's class number formula: $e_n = \lambda_l(k)n + \mu_l(k)l^n + \nu_l(k)$ for all sufficiently large $n \geq 0$ (cf. [6] or [12]).

For each prime number l, it is conjectured (cf. [3]) that if k is a totally real number field then $\lambda_l(k) = \mu_l(k) = 0$, i.e., e_n is bounded as $n \to \infty$. This is often called Greenberg's conjecture. Since this conjecture presented, it has been studied by many authors. In these studies of Greenberg's conjecture, the case for real quadratic fields k and even prime $l = 2$ seems to be rather a special case because of the effects of genus theory. Ozaki and Taya (cf. [10]) proved the existence of infinitely many real quadratic fields k with $\lambda_2(k) = \mu_2(k) = 0$ in various situations. In this paper, we also deal with the cyclotomic $\mathbf{Z}_2$-extensions of certain real quadratic fields k. The main theorem is the following.

THEOREM 1. *Let p, q, r be prime numbers such that*

$$p \equiv q \equiv 5 \pmod 8, \quad r \equiv 3 \pmod 4, \quad \textit{and} \quad \left(\frac{pq}{r}\right) = -1,$$

Received May 29, 2003; revised June 24, 2003
2000 Mathematics Subject Classification. Primary 11R23, 11R11.
Key words and phrases. Iwasawa invariants, real quadratic fields.

where $\left(\frac{*}{*}\right)$ *is Legendre's symbol. Put* $k = \boldsymbol{Q}(\sqrt{pqr})$ *or* $\boldsymbol{Q}(\sqrt{2pqr})$. *Let* $\lambda_2(k)$, $\mu_2(k)$ *and* $\nu_2(k)$ *be the Iwasawa* λ-, μ- *and* ν-*invariants of the cyclotomic* $\boldsymbol{Z}_2$-*extension* k_∞ *of* k, *respectively. Then we have* $\lambda_2(k) = \mu_2(k) = 0$ *and* $\nu_2(k) = 2$.

Here we note that as for μ-invariants, it is well known that $\mu_l(k) = 0$ for any prime number l when k is an abelian extension of $\boldsymbol{Q}$, by the theorem of Ferrero and Washington (cf. [1] or [12]). However, we show $\mu_2(k) = 0$ here independently of it.

In the section 3, we prove Theorem 1 by using "Kuroda's class number formula" (cf. Proposition 1) and by the explicit description of the unit group of the first layer k_1 of the cyclotomic $\boldsymbol{Z}_2$-extension k_∞ of the real quadratic field $k = \boldsymbol{Q}(\sqrt{pqr})$.

2. Some known results

In this section, we mention some known results about Iwasawa invariants of the cyclotomic $\boldsymbol{Z}_2$-extensions of real quadratic fields.

For each integer $n \geq 0$, let $\zeta_n = \exp(2\pi\sqrt{-1}/2^{n+2})$, a primitive 2^{n+2}-th roots of unity in the complex number field. Then the n-th layer $\boldsymbol{Q}_n$ of the cyclotomic $\boldsymbol{Z}_2$-extension $\boldsymbol{Q}_\infty/\boldsymbol{Q}$ is the field $\boldsymbol{Q}(\zeta_n + \zeta_n^{-1})$. Especially, the first layer $\boldsymbol{Q}_1$ is the real quadratic field $\boldsymbol{Q}(\sqrt{2})$. It is proved by Weber (cf. [4], Satz 6, p. 29) that the class number of $\boldsymbol{Q}_n$ is odd for all $n \geq 0$, i.e., $\lambda_2(\boldsymbol{Q}) = \mu_2(\boldsymbol{Q}) = \nu_2(\boldsymbol{Q}) = 0$.

Let m be a positive square-free integer, and let $k = \boldsymbol{Q}(\sqrt{m})$, a real quadratic field. The cyclotomic $\boldsymbol{Z}_2$-extension k_∞ of k is given by $k\boldsymbol{Q}_\infty$. If $m = 2$, i.e., $k = \boldsymbol{Q}_1 = \boldsymbol{Q}(\sqrt{2})$ then $k_\infty = \boldsymbol{Q}_\infty$, so we already know that $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$. Therefore we consider the case $m > 2$. In this case, the first layer $k_1 = k\boldsymbol{Q}_1 = \boldsymbol{Q}(\sqrt{2}, \sqrt{m})$ has just three real quadratic fields $\boldsymbol{Q}_1 = \boldsymbol{Q}(\sqrt{2})$, $k = \boldsymbol{Q}(\sqrt{m})$, $k' = \boldsymbol{Q}(\sqrt{2m})$ as its subextensions. We note that k and k' have the same cyclotomic $\boldsymbol{Z}_2$-extension, i.e., $k_\infty = k'_\infty$, so the Iwasawa invariants are also the same.

In [5], Iwasawa proved the theorem which states that for each prime number l, if a Galois l-extension K/k of number fields has at most one (finite or infinite) ramified prime and the class number of k is not divisible by l, then the class number of K is also not divisible by l. By this theorem, if a real quadratic field k with odd class number has only one prime ideal above the prime number 2, then for each $n \geq 0$ the class number of the n-th layer k_n of the cyclotomic $\boldsymbol{Z}_2$-extension k_∞/k is also odd, i.e., $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$. Furthermore, by genus theory and the theorem of Rédei and Reichardt (cf. [11]), we can see that a real quadratic field k has odd class number and only one prime ideal above the prime number 2 if and only if $k = \boldsymbol{Q}(\sqrt{m})$ with positive square free integer m satisfies one of the following conditions.

$$m = \begin{cases} 2, \\ p, \ p \equiv 5 \pmod 8, \\ q, \ q \equiv 3 \pmod 4, \\ 2q, \ q \equiv 3 \pmod 4, \\ pq, \ p \equiv 3, \ q \equiv 7 \pmod 8, \end{cases}$$

where p and q denote prime numbers. Therefore the cyclotomic $\boldsymbol{Z}_2$-extension of $k = \boldsymbol{Q}(\sqrt{m})$ or $\boldsymbol{Q}(\sqrt{2m})$ with square free positive integer m satisfying the above condition has the Iwasawa invariants $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$. These cases are often called 'trivial' cases. In [10], Ozaki and Taya treated 'non-trivial' cases and proved the following theorem.

THEOREM 2 (Ozaki-Taya [10]). *Let $k = \boldsymbol{Q}(\sqrt{m})$ or $\boldsymbol{Q}(\sqrt{2m})$ and let $\lambda_2(k)$, $\mu_2(k)$ and $\nu_2(k)$ be the Iwasawa λ-, μ- and ν-invariants of the cyclotomic $\boldsymbol{Z}_2$-extension k_∞/k, respectively. Suppose that m is one of the following*:

(1) $m = p$, $\quad p \equiv 1 \pmod 8$ *and* $2^{\frac{p-1}{4}} \not\equiv (-1)^{\frac{p-1}{8}} \pmod p$,
(2) $m = pq$, $\quad p \equiv q \equiv 3 \pmod 8$,
(3) $m = pq$, $\quad p \equiv 3, \ q \equiv 5 \pmod 8$,
(4) $m = pq$, $\quad p \equiv 5, \ q \equiv 7 \pmod 8$,
(5) $m = pq$, $\quad p \equiv q \equiv 5 \pmod 8$,

where p and q are distinct prime numbers. Then we have $\lambda_2(k) = \mu_2(k) = \nu_2(k) = 0$ for (1) *and* (2), *and $\lambda_2(k) = \mu_2(k) = 0$, $\nu_2(k) > 0$ for* (3), (4) *and* (5).

On the other hands, Yamamoto [13] determined all real abelian 2-extensions $K/\boldsymbol{Q}$ with $\lambda_2(K) = \mu_2(K) = \nu_2(K) = 0$. As a corollary to the results of Yamamoto, we obtain the following.

THEOREM 3 (cf. Yamamoto [13]). *Let p, q, r be prime numbers such that*

$$p \equiv q \equiv 3, \quad r \equiv 7 \pmod 8, \quad \text{and} \quad \left(\frac{pq}{r}\right) = -1,$$

where $\left(\frac{*}{*}\right)$ is Legendre's symbol. Put $k = \boldsymbol{Q}(\sqrt{pqr})$ or $\boldsymbol{Q}(\sqrt{2pqr})$. Let $\lambda_2(k)$, $\mu_2(k)$ and $\nu_2(k)$ be the Iwasawa λ-, μ- and ν-invariants of the cyclotomic $\boldsymbol{Z}_2$-extension k_∞ of k, respectively. Then we have $\lambda_2(k) = \mu_2(k) = 0$ and $\nu_2(k) = 2$.

PROOF. As mentioned before, it is sufficient to prove the case of $k = \boldsymbol{Q}(\sqrt{pqr})$. By genus theory and the theorem of Rédei and Reichardt (cf. [11]), we can see that the Hilbert 2-class field of k is the field $K = \boldsymbol{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$. In [13], Yamamoto proved that $\lambda_2(K) = \mu_2(K) = \nu_2(K) = 0$ for the cyclotomic $\boldsymbol{Z}_2$-extension K_∞ of this field K. Then for each $n \geq 0$, the Hilbert 2-class field of n-th layer k_n is $K_n = K\boldsymbol{Q}_n$, the n-th layer of K_∞/K. By class field theory, the highest power e_n of 2 dividing the class number of k_n is 2, i.e., $e_n = 2$ for all $n \geq 0$. This complete the proof. □

REMARK. The statements of Theorem 1 and Theorem 3 are similar. In fact, we can prove Theorem 3 by the arguments similar to the proof of Theorem 1. But Theorem 1 is not obtained as a corollary to Yamamoto's results in [13].

In addition, Ozaki treated several cases different from each of the above theorems and proved $\lambda_2(k) = \mu_2(k) = 0$ for certain real quadratic fields k in his thesis [9]. The real quadratic fields, which were treated in Theorem 2, have the ideal class group of 2-rank smaller than 2. But in [9], Ozaki proved $\lambda_2(k) = \mu_2(k) = 0$ for certain infinitely many real quadratic fields k with the ideal class group of 2-rank 2. Similarly, the real quadratic fields $k = \boldsymbol{Q}(\sqrt{pqr})$ in Theorem 1 and Theorem 3 have the ideal class group of 2-rank 2.

3. Proof of Theorem 1

To prove Theorem 1, we need several propositions. The equation in the following proposition is often called "Kuroda's class number formula".

PROPOSITION 1 (cf. Kuroda [8], Kubota [7]). *Let K be a real bicyclic biquadratic extension of $\boldsymbol{Q}$ with the unit group $E(K)$. The field K has three real quadratic subextensions $F_i/\boldsymbol{Q}$ $(i = 1, 2, 3)$. Let ε_i (> 0) be the fundamental unit of F_i $(i = 1, 2, 3)$, and $h(K)$, $h(F_i)$ the class numbers of K, F_i, respectively. Put the group index $Q(K) = [E(K) : \langle -1, \varepsilon_1, \varepsilon_2, \varepsilon_3 \rangle]$. Then we have the equation*

$$h(K) = \frac{1}{4} \cdot Q(K) \cdot h(F_1) \cdot h(F_2) \cdot h(F_3) .$$

Furthermore, we have $Q(K) = 1$, 2 or 4, and a system of the fundamental units of K is one of the following types:

$$\begin{array}{rll}
\text{i)} & \varepsilon_1, \varepsilon_2, \varepsilon_3 & \\
\text{ii)} & \sqrt{\varepsilon_1}, \varepsilon_2, \varepsilon_3 & (N\varepsilon_1 = 1) \\
\text{iii)} & \sqrt{\varepsilon_1}, \sqrt{\varepsilon_2}, \varepsilon_3 & \multirow{2}{*}{$\Big\}\ (N\varepsilon_1 = N\varepsilon_2 = 1)$} \\
\text{iv)} & \sqrt{\varepsilon_1\varepsilon_2}, \varepsilon_2, \varepsilon_3 & \\
\text{v)} & \sqrt{\varepsilon_1\varepsilon_3}, \sqrt{\varepsilon_2}, \varepsilon_3 & \multirow{2}{*}{$\Big\}\ (N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = 1)$} \\
\text{vi)} & \sqrt{\varepsilon_1\varepsilon_2}, \sqrt{\varepsilon_2\varepsilon_3}, \sqrt{\varepsilon_3\varepsilon_1} & \\
\text{vii)} & \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}, \varepsilon_2, \varepsilon_3 & (N\varepsilon_1 = N\varepsilon_2 = N\varepsilon_3 = \pm 1)
\end{array}$$

where $N\varepsilon_i$ is the abbreviation of the absolute norm $N_{F_i/\boldsymbol{Q}}(\varepsilon_i)$ $(i = 1, 2, 3)$.

PROPOSITION 2 (Fukuda [2]). *Let k_∞/k be any $\mathbf{Z}_l$-extension of number fields such that any prime of k_∞ which is ramified in k_∞/k is totally ramified. For each integer $n \geq 0$, we denote by $A(k_n)$ the l-Sylow subgroup of the ideal class group of k_n, the n-th layer of the $\mathbf{Z}_l$-extension k_∞/k. If $|A(k_1)| = |A(k)|$, then $|A(k_n)| = |A(k)|$ for all $n \geq 0$, where $| * |$ means the order of the group.*

Now, we prove Theorem 1 by using the above propositions.

PROOF OF THEOREM 1. As already mentioned, it is enough to show only the case of $k = \boldsymbol{Q}(\sqrt{pqr})$. We may assume that prime numbers p, q, r satisfy the condition

$$p \equiv q \equiv 5 \pmod 8, \quad r \equiv 3 \pmod 4, \quad \text{and} \quad \left(\frac{p}{r}\right) = +1, \quad \left(\frac{q}{r}\right) = -1. \qquad (\dagger)$$

The first layer of the cyclotomic $\boldsymbol{Z}_2$-extension k_∞ of the real quadratic field $k = \boldsymbol{Q}(\sqrt{pqr})$ is the real bicyclic biquadratic field $k_1 = \boldsymbol{Q}(\sqrt{2}, \sqrt{pqr})$. The field k_1 contains just three real quadratic fields: $\boldsymbol{Q}(\sqrt{2})$, k and $k' = \boldsymbol{Q}(\sqrt{2pqr})$.

We denote by $A(k)$, $A(k')$ and $A(k_1)$ the 2-Sylow subgroups of the ideal class groups of k, k' and k_1, respectively. Let L and L' be the Hilbert 2-class fields of k and k', respectively. By genus theory and the theorem of Rédei and Reichardt (cf. [11]), we can see that both $A(k)$ and $A(k')$ are the abelian 2-group of type $(2, 2)$, and we have $L = \boldsymbol{Q}(\sqrt{p}, \sqrt{q}, \sqrt{r})$ and $L' = \boldsymbol{Q}(\sqrt{p}, \sqrt{q}, \sqrt{2r})$. Especially, we have $|A(k)| = |A(k')| = 4$.

Let ε and ε' be the fundamental units of the real quadratic fields k and k', respectively. By genus theory, we can also see that each of the real quadratic fields k and k' has the narrow class number different from the class number in wider sense. Therefore we have $N\varepsilon = N\varepsilon' = 1$, where N means the absolute norm. Now, we have the following lemma.

LEMMA. *$\sqrt{\varepsilon}$, $\sqrt{\varepsilon'}$ and $\sqrt{\varepsilon\varepsilon'}$ are not contained in the first layer k_1.*

PROOF. (I) First, we assume that $\left(\frac{q}{p}\right) = +1$. Let $\mathfrak{p}$ be a prime ideal of k above the prime number p, which is ramified in k. Since $\mathfrak{p}^2 = (p)$, the ideal class containing $\mathfrak{p}$ is an element of $A(k)$. By the assumption and the condition $(\dagger)$, we can see that the prime $\mathfrak{p}$ splits completely in L, so that $\mathfrak{p}$ is a principal ideal of k. Let $\alpha \in k^\times$ be a generator of the prime ideal $\mathfrak{p}$: $\mathfrak{p} = (\alpha)$. Since $(p) = \mathfrak{p}^2 = (\alpha^2)$ and α is real, we have $p = \varepsilon^z \alpha^2$ for some integer z. If z is even, $\sqrt{p} = \pm\alpha\,\varepsilon^{z/2} \in k^\times$, which is a contradiction. Therefore z must be odd, and there is an element $\beta \in k^\times$ such that $p = \varepsilon\,\beta^2$. Since $\sqrt{p} = \pm\beta\sqrt{\varepsilon}$, we know that $k(\sqrt{\varepsilon}) = k(\sqrt{p})$ and $\sqrt{\varepsilon}$ is not contained in the first layer $k_1 = k(\sqrt{2})$.

Let $\mathfrak{q}'$ be a prime ideal of k' above the prime number q, which is ramified in k'. By the similar arguments, we can see that the prime $\mathfrak{q}'$ is a principal ideal of k', and there is an element $\beta' \in k'^\times$ such that $\sqrt{q} = \pm\beta'\sqrt{\varepsilon'}$. Then we know that $k'(\sqrt{\varepsilon'}) = k'(\sqrt{q})$ and $\sqrt{\varepsilon'}$ is also not contained in the first layer $k_1 = k'(\sqrt{2})$.

We have $k_1(\sqrt{\varepsilon}) = k_1(\sqrt{p}) \neq k_1(\sqrt{q}) = k_1(\sqrt{\varepsilon'})$, so that $\sqrt{\varepsilon\varepsilon'}$ must not be contained in k_1.

(II) Secondly, we assume that $\left(\frac{q}{p}\right) = -1$. Let $\mathfrak{p}$ and $\mathfrak{l}$ be the prime ideals of k above the prime numbers p and 2, respectively, which are ramified in k. We note that both of the ideal classes containing $\mathfrak{p}$ or $\mathfrak{l}$ are elements of $A(k)$. By the assumption and the condition $(\dagger)$, we can see that $\mathfrak{p}$ and $\mathfrak{l}$ have the same decomposition field $k(\sqrt{r})$ with respect to the extension L/k. This means that

$$\left(\frac{L/k}{\mathfrak{p}}\right) = \left(\frac{L/k}{\mathfrak{l}}\right),$$

where $\left(\frac{L/k}{*}\right)$ is the Artin symbol. Therefore the ideal classes containing $\mathfrak{p}$ or $\mathfrak{l}$ are the same element of the ideal class group of k, and there is an element $\alpha \in k^{\times}$ such that $\mathfrak{l} = (\alpha)\mathfrak{p}$. Since $(2) = \mathfrak{l}^2 = (\alpha)^2\mathfrak{p}^2 = (\alpha^2 p)$ and α is real, $2 = \varepsilon^z\alpha^2 p$ for some integer z. If z is even, $\sqrt{2} = \pm\alpha\,\varepsilon^{z/2}\sqrt{p}$, so that $k_1 = k(\sqrt{p})$, which is a contradiction. Then z must be odd, and $2 = \varepsilon\,\beta^2 p$ for some $\beta \in k^{\times}$. Since $\sqrt{2} = \pm\beta\sqrt{\varepsilon}\sqrt{p}$, we know that $k_1(\sqrt{\varepsilon}) = k_1(\sqrt{p})$ and $\sqrt{\varepsilon}$ is not contained in k_1.

Let $\mathfrak{q}'$ and $\mathfrak{l}'$ be the prime ideals of k' above the prime number q and 2, respectively, which are ramified in k'. By the assumption and the condition (†), we can see that $\mathfrak{q}'$ and $\mathfrak{l}'$ have the same decomposition field $k'(\sqrt{2r})$ with respect to the extension L'/k', so that the ideal classes containing $\mathfrak{q}'$ or $\mathfrak{l}'$ are the same element of the ideal class group of k'. By the similar arguments, we know that $\sqrt{2} = \pm\beta'\sqrt{\varepsilon'}\sqrt{q}$ for some $\beta' \in k'^{\times}$, and $k_1(\sqrt{\varepsilon'}) = k_1(\sqrt{q})$. Therefore $\sqrt{\varepsilon'}$ is also not contained in k_1.

We have $k_1(\sqrt{\varepsilon}) = k_1(\sqrt{p}) \neq k_1(\sqrt{q}) = k_1(\sqrt{\varepsilon'})$, so that $\sqrt{\varepsilon\varepsilon'}$ must not be contained in k_1. Now, we complete the proof of the lemma. □

We note that the real quadratic field $\boldsymbol{Q}(\sqrt{2})$ has the class number 1 and the fundamental unit $1+\sqrt{2}$ with the absolute norm $N(1+\sqrt{2}) = -1$. By the above lemma and Proposition 1, a system of the fundamental units of k_1 must be $\{1+\sqrt{2},\ \varepsilon,\ \varepsilon'\}$. Therefore the group index $Q(k_1) = [E(k_1) : \langle -1,\ 1+\sqrt{2},\ \varepsilon,\ \varepsilon'\rangle] = 1$, where $E(k_1)$ is the group of the units of k_1. By the Kuroda's class number formula in Proposition 1, we have

$$|A(k_1)| = \frac{1}{4}\cdot Q(k_1)\cdot|A(k)|\cdot|A(k')| = \frac{1}{4}\cdot 1\cdot 4\cdot 4 = 4\,.$$

Then we know that $|A(k_1)| = |A(k)| = 4$. Note that any prime of k_∞ which is ramified in k_∞/k is totally ramified. By Proposition 2, $|A(k_n)| = |A(k)| = 4$ for all $n \geq 0$, so that the Iwasawa invariants satisfy $\lambda_2(k) = \mu_2(k) = 0$ and $\nu_2(k) = 2$. This complete the proof of Theorem 1. □

ACKNOWLEDGEMENT. The author expresses his hearty thanks to Professor Keiichi Komatsu for many valuable discussions and advice. He also expresses his thanks to referee for helpful advice on revising this paper.

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