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Isolation Theorems of the Bochner Curvature Type Tensors

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Abstract. An isolation theorem of the Bochner curvature tensor of a Kähler-Einstein manifold is given, when its $L^{n/2}$-norm is small. Similarly an isolation theorem of the contact Bochner curvature tensor for a Sasakian manifold is obtained. Those theorems are derived from the Weitzenböck formula which gives non-linearity constraint on the Bochner curvature tensors.

1. Introduction

The Weyl conformal curvature tensor is a tensor which measures deviation from the conformal flatness so that it is significant in conformal geometry. As its complex analogue we have the Bochner curvature tensor on a Kähler manifold, and also as its contact analogue the contact Bochner curvature tensor on a Sasakian manifold.

Our purpose of this paper is to show that these Bochner curvature type tensors B obey the following isolation theorems under certain Einstein conditions.

THEOREM A. *Let (M, J, g) be a compact, connected Kähler-Einstein n-manifold, $n = 2m \geq 4$, with positive scalar curvature s and of Vol$(g) = 1$. Then, there exists a constant $C(n)$, depending only on n such that if $L^{n/2}$-norm $\|B\|_{L^{n/2}} < C(n)s$, then $B = 0$ so that (M, J, g) is biholomorphically homothetic to the complex projective space $\boldsymbol{C}P^m$ with the Fubini-Study metric.*

REMARK. It is known that complex surfaces $\boldsymbol{C}P^2\#k\overline{\boldsymbol{C}P^2}$ $(3 \leq k \leq 8)$ and $\boldsymbol{C}P^1 \times \boldsymbol{C}P^1$ admit Kähler-Einstein metric with positive scalar curvature [16]. Here, $\boldsymbol{C}P^2\#k\overline{\boldsymbol{C}P^2}$ is the surface obtained by blowing up $\boldsymbol{C}P^2$ at k generic points.

THEOREM B. *Let (M, ϕ, ξ, η, g) be a compact, connected Sasakian η-Einstein n-manifold, $n = 2m + 1 \geq 5$, with scalar curvature $s > -(n-1)$ and of Vol$(g) = 1$. Then, there exists a constant $C(n)$, depending only on n such that if $L^{n/2}$-norm $\|B\|_{L^{n/2}} <$*

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$C(n)n(s+n-1)/(n+1)$, then $B=0$ so that M is D-homothetic to finite quotient of the standard n-sphere.

These theorems are analogous to the following isolation theorem of the Weyl conformal curvature tensor W.

THEOREM ([8]). *Let (M, g) be a compact, connected oriented Einstein n-manifold, $n \geq 4$, with positive scalar curvature s and of $Vol(g)=1$. Then, there exists a constant $C(n)$, depending only on n such that if $L^{n/2}$-norm $\|W\|_{L^{n/2}} < C(n)s$, then $W=0$ so that (M, g) is a finite isometric quotient of the standard n-sphere of unit volume.*

As a complex analogue of the Weyl conformal curvature tensor, S. Bochner [2] introduced the so-called Bochner curvature tensor using a complex local coordinate;

$$B_{\alpha\bar{\beta}\gamma\bar{\delta}} = R_{\alpha\bar{\beta}\gamma\bar{\delta}} - \frac{1}{m+2}(R_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + R_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}} + g_{\alpha\bar{\beta}}R_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}}R_{\alpha\bar{\delta}}) + \frac{s}{(m+1)(m+2)}(g_{\alpha\bar{\beta}}g_{\gamma\bar{\delta}} + g_{\gamma\bar{\beta}}g_{\alpha\bar{\delta}}) .$$

Y. Kamishima classified completely compact Kähler manifolds with vanishing Bochner curvature tensor in [10].

THEOREM ([10]). *Let M be a compact Kähler $2m(\geq 4)$-manifold having the vanishing Bochner curvature tensor B. Then M is holomorphically isometric to*

(1) *the complex projective space $\boldsymbol{C}P^m$,*
(2) *a complex Euclidean space form $T^m_{\boldsymbol{C}}/F$, $F \subset U(m)$,*
(3) *a complex hyperbolic space form $H^m_{\boldsymbol{C}}/\Gamma$, $\Gamma \subset PU(m,1)$,*
(4) *the fiber space $H^k_{\boldsymbol{C}} \times \boldsymbol{C}P^{m-k}/\Gamma$ where*

$$\Gamma \subset PU(k,1) \times PU(m-k+1) , \quad k = 1, 2, \cdots , m-1 .$$

Here, F is a finite group and Γ is a discrete cocompact subgroup, both acting properly discontinuously.

For odd dimensional Sasakian manifolds, M. Matsumoto and G. Chûman [13] defined the contact Bochner curvature tensor. A Sasakian manifold with vanishing contact Bochner curvature tensor is studied by M. Matsumoto and G. Chûman [13], T. Ikawa and M. Kon [4]. The isolation theorem of contact Bochner curvature tensor, namely Theorem B, is obtained by arguments based on those studies together with the result of [8].

2. Isolation of the Bochner curvature tensor

2.1. The Weitzenböck formula for the Bochner tensor. In this section, we establish the Weitzenböck formula on the left-exterior derivative d_L applied to the Bochner curvature tensor in a Kähler-Einstein manifold which plays an essential role in the proof of Theorem A.

Let (M, g) be a Riemannian n-manifold, and let Λ^p denote the bundle of exterior p-forms. The operator $d_L : \Gamma(\Lambda^p \otimes \Lambda^q) \to \Gamma(\Lambda^{p+1} \otimes \Lambda^q)$ exploited by the first author in [7] is given by

$$(d_L\psi)_{i_0 i_1\cdots i_p j_1\cdots j_q} = \sum_{k=0}^{p}(-1)^k \nabla_{i_k}\psi_{i_0 i_1\cdots \hat{i_k}\cdots i_p j_1\cdots j_q} \, . \tag{1}$$

The bundle $\Lambda^p \otimes \Lambda^q$ carries the inner product inherited from the metric g. Then, with respect to this inner product the operator d_L has the formal adjoint $\delta_L : \Gamma(\Lambda^{p+1}\otimes\Lambda^q) \to \Gamma(\Lambda^p \otimes \Lambda^q)$, given by

$$(\delta_L\psi)_{i_1\cdots i_p j_1\cdots j_q} = -\nabla^a \psi_{a i_1\cdots i_p j_1\cdots j_q} \, . \tag{2}$$

Similarly, the right-exterior derivative $d_R : \Gamma(\Lambda^p \otimes \Lambda^q) \to \Gamma(\Lambda^p \otimes \Lambda^{q+1})$ and the formal adjoint δ_R are also defined.

REMARK. We may consider the Riemannian curvature tensor R as a section $\Lambda^2 \otimes \Lambda^2$ and the Ricci tensor Ric as a section $\Lambda^1 \otimes \Lambda^1$. The following identities are well known

$$d_L R = 0 \, , \quad \delta_L R = -d_R Ric \, , \quad \delta_R Ric = \delta_L Ric = -\frac{1}{2} ds \, , \tag{3}$$

where s is the scalar curvature. The first one reads the second Bianchi identity.

By direct calculation, we have the Weitzenböck formula on the d_L as follows:

$$\begin{aligned}(\Delta_L\psi)_{ijst} &= (d_L\delta_L\psi + \delta_L d_L\psi)_{ijst} \\ &= \nabla^*\nabla\psi_{ijst} + R_i{}^a\psi_{ajst} + R_j{}^a\psi_{iast} - \{R, \psi\}_{ijst} \, ,\end{aligned} \tag{4}$$

for any $\psi \in \Gamma(\Lambda^2 \otimes \Lambda^2)$. Here, $\{\ ,\ \}$ is given by

$$\{S, T\}_{ijst} = S_{ij}{}^{ab}T_{abst} + S^a{}_{is}{}^b T_{ajtb} + S^a{}_{jt}{}^b T_{aisb} - S^a{}_{js}{}^b T_{aitb} - S^a{}_{it}{}^b T_{ajsb} \, , \quad S,\ T \in \Gamma(\Lambda^2 \otimes \Lambda^2) \, .$$

The Bochner curvature tensor B has also a real coordinate expression due to S. Tachibana [14]. We adopt in this paper his real coordinate formulation. Namely, let (M, J, g) be a Kähler n-manifold, $n = 2m \geq 4$. Then the Bochner curvature tensor is defined by

$$\begin{aligned}B_{ijst} = R_{ijst} &- \frac{1}{n+4}[R_{is}g_{jt} + R_{jt}g_{is} - R_{it}g_{js} - R_{js}g_{it} + J_i{}^r R_{rs}J_{jt} \\ &+ J_j{}^r R_{rt}J_{is} - J_i{}^r R_{rt}J_{js} - J_j{}^r R_{rs}J_{it} + 2J_i{}^r R_{rj}J_{st} + 2J_{ij}J_s{}^r R_{rt}] \\ &+ \frac{s}{(n+2)(n+4)}[g_{is}g_{jt} - g_{it}g_{js} + J_{is}J_{jt} - J_{it}J_{js} + 2J_{ij}J_{st}] \, ,\end{aligned} \tag{5}$$

where $J_{ij} = J_i{}^r g_{rj}$.

The following identities are obtained by the straightforward computation;

$$B_{ijst} = -B_{jist} = -B_{ijts} \, , \tag{6}$$

$$B_{ijst} + B_{jsit} + B_{sijt} = 0 \, , \tag{7}$$

$$B_{ijst} = B_{stij}\,, \tag{8}$$

$$B_{ijst} = J_i{}^p J_j{}^q B_{pqst}\,, \tag{9}$$

$$B^p{}_{jpt} = 0\,, \tag{10}$$

from which one sees that the Bochner curvature tensor in a Kähler manifold plays a role of the Weyl conformal curvature tensor in a Riemannian manifold. The identity (9) means that B is J-invariant, i.e., $B(JX, JY, Z, W) = B(X, Y, Z, W)$ for tangent vectors X, Y, X, W.

The identities $J_i{}^p B_{pjst} = -J_j{}^p B_{ipst}$ and $J^{pq} B_{pqst} = 0$ are derived from these identities.

Set the tensors G and Φ

$$G_{ijst} = g_{is} g_{jt} - g_{it} g_{js}\,, \quad \Phi_{ijst} = J_{is} J_{jt} - J_{it} J_{js} + 2J_{ij} J_{st}\,.$$

Then, from the above identities of the B, we have

$$\{G, B\} = \{\Phi, B\} = 0\,. \tag{11}$$

Now, we consider the Kähler-Einstein case. The Bochner curvature tensor has the following form if and only if M is Kähler-Einstein;

$$B = R - \frac{s}{n(n+2)}(G + \Phi)\,. \tag{12}$$

Moreover, when M is Kähler-Einstein, by applying (3) and $\nabla B = \nabla R$, we have

$$d_L B = 0\,, \quad \text{and} \quad \delta_L B = 0\,. \tag{13}$$

So that, from the Weitzenböck formula for the B, we have the following Lemma.

LEMMA. *Let (M, J, g) be a compact Kähler-Einstein manifold. Then the Bochner curvature tensor B fulfills*

$$0 = \Delta_L B = \nabla^* \nabla B + \frac{2s}{n} B - \{B, B\}\,. \tag{14}$$

2.2. Proof of Theorem A. The proof is quite similar to the proof in [8]. So we follow their argument.

Let (M, J, g) be a Kähler-Einstein manifold, $n = 2m \geq 4$, with positive scalar curvature s. We assume that the Bochner curvature B does not vanish identically and consider its norm $\|B\|_{L^{n/2}}$.

We apply the Sobolev inequality of a compact Riemannian n-manifold, $n \geq 3$, which is described in terms of Yamabe metrics. We take a Yamabe metric in the conformal class $[g]$, represented by g and then obtain the Sobolev inequality

$$4\frac{n-1}{n-2}\|\nabla f\|^2_{L^2} \geq s\mathrm{Vol}(g)^{(2/n)}\{\|f\|^2_{L^p} - \mathrm{Vol}(g)^{-(2/n)}\|f\|^2_{L^2}\}\,, \quad f \in H^2_1(M) \tag{15}$$

where $p = (2n/n-2)$. The inequality (15) still holds when f is replaced by any tensor T because of the Kato's inequality

$$|\nabla |T|| \leq |\nabla T| . \tag{16}$$

Remark that there is an improved Kato's inequality, for example, as given in [3] as

$$|\nabla |B|| \leq \gamma(n)|\nabla B|$$

for a certain positive constant $\gamma(n) < 1$, since B satisfies the elliptic equations (14). We can make use of this inequality to our argument. However, the improved Kato's constant $\gamma(n)$ is not essential in our argument. So we ignore here this constant.

It is known that any Einstein metric must be Yamabe, provided it is not conformal to the standard n-sphere ([6, 11]). Since the metric is Kähler-Einstein, g is indeed a Yamabe metric in the conformal class $[g]$. We normalize g by constant rescaling so that $\mathrm{Vol}(g) = 1$.

So we get

$$4\frac{n-1}{n-2}\|\nabla B\|_{L^2}^2 \geq s\{\|B\|_{L^p}^2 - \|B\|_{L^2}^2\} . \tag{17}$$

Next, we have the following inequality from (14);

$$\|\nabla B\|_{L^2}^2 + \frac{2s}{n}\|B\|_{L^2}^2 \leq C_n^{-1}\|B\|_{L^3}^3 \leq C_n^{-1}\|B\|_{L^{n/2}}\|B\|_{L^p}^2 . \tag{18}$$

Here, we used the Hölder inequality together with the pointwise inequality

$$\langle\{B, B\}, B\rangle \leq C_n^{-1}|B|^3 \tag{19}$$

for a constant $C_n > 0$, depending only on n. Applying the Sobolev inequality (17) yields

$$C_n^{-1}\|B\|_{L^{n/2}}\|B\|_{L^p}^2 \geq \frac{2s}{n}\|B\|_{L^2}^2 + \frac{n-2}{4(n-1)}s(\|B\|_{L^p}^2 - \|B\|_{L^2}^2) . \tag{20}$$

Assume that $4 \leq n \leq 9$. Then $(2/n) - ((n-2)/4(n-1)) > 0$ so that

$$C_n^{-1}\|B\|_{L^{n/2}} \geq \frac{n-2}{4(n-1)}s . \tag{21}$$

If, contrarily, $n \geq 10$, it holds $(2/n)-((n-2)/4(n-1)) < 0$. However $\|B\|_{L^2}^2 \leq \|B\|_{L^p}^2$, since $p > 2$. So,

$$\left(\frac{2}{n} - \frac{n-2}{4(n-1)}\right)s\|B\|_{L^2}^2 \geq \left(\frac{2}{n} - \frac{n-2}{4(n-1)}\right)s\|B\|_{L^p}^2 . \tag{22}$$

We have thus

$$C_n^{-1}\|B\|_{L^{n/2}}\|B\|_{L^p}^2 \geq \left\{\left(\frac{2}{n} - \frac{n-2}{4(n-1)}\right)s + \frac{n-2}{4(n-1)}s\right\}\|B\|_{L^p}^2 = \frac{2}{n}s\|B\|_{L^p}^2 \tag{23}$$

giving rise to $\|B\|_{L^{n/2}} \geq (2/n)C_n s$.

Therefore, if we put $C(n)$ as

$$
\begin{aligned}
C(n) &= \frac{n-2}{4(n-1)} C_n s\,, \quad 4 \le n \le 9\,, \\
&= \frac{2}{n} C_n s\,, \qquad\quad 10 \le n\,,
\end{aligned}
\tag{24}
$$

then we get a contradiction giving the complete proof.

REMARK. If $n = 4$, the Bochner curvature tensor B of a Kähler metric g is the anti-self-dual part of the Weyl curvature tensor W of g [5, 17]. That is, it holds in this case $B^+ = 0$ and $B^- = W^-$, where $B^\pm$ and $W^\pm$ are the restriction of B and W to $\Lambda^\pm$, respectively. Here the bundle Λ^2 splits as the Whitney sum $\Lambda^2 = \Lambda^+ \oplus \Lambda^-$, $\Lambda^\pm$ being the eigenspace bundles of the Hodge star operator $* \in \mathrm{End}(\Lambda^2)$. Then the W and the B leave $\Lambda^\pm$ invariant. As was discussed in [9], for a Kähler surface, we have

$$|\langle\{B, B\}, B\rangle| \le \sqrt{6}|B|^3\,.$$

The equality is achieved at a given point if and only if the curvature operator $B = W^- \in \mathrm{End}(\Lambda^-)$ has distinct eigenvalues at most two.

2.3. The optimal value of the estimate constant $C(4)$. We do not know in general the optimal value of $C(n)$ in Theorem A. However, when M is real 4-dimensional, we shall see that the constant $C(4) = \sqrt{\frac{1}{24}}$ is optimal in our theorem.

Let (M, J, g) be a compact, connected Kähler-Einstein 4-manifold with positive scalar curvature s. Then, since $|W^+|^2 = \frac{1}{24}s^2$, we have the Hirzebruch index theorem and the Gauss-Bonnet theorem (cf. [12])

$$\|W^+\|^2_{L^2} = \frac{1}{24}\int_M s^2 dV_g = \frac{4}{3}\pi^2(2\chi(M) + 3\tau(M))\,, \tag{25}$$

$$\|B\|^2_{L^2} = \|W^-\|^2_{L^2} = \frac{1}{24}\int_M s^2 dV_g - 12\pi^2\tau(M) = \frac{8}{3}\pi^2(\chi(M) - 3\tau(M))\,, \tag{26}$$

where $\chi(M)$ is the Euler-Poincáre characteristic and $\tau(M)$ is the signature of (M, g).

Moreover, since M is a Kähler-Einstein 4-manifold with positive scalar curvature, we see that the Betti numbers satisfy $b_1(M) = 0$ and $b_2^+(M) = 1$. So, we have

$$
\begin{aligned}
\tau(M) &= b_2^+(M) - b_2^-(M) = 1 - b_2^-(M)\,, \\
\chi(M) &= \sum_{k=0}^{4} (-1)^k b_k(M) = 3 + b_2^-(M)\,.
\end{aligned}
\tag{27}
$$

The identities (25), (26) and (27) imply

$$\|W^+\|^2_{L^2} = \frac{1}{24}\int_M s^2 dV_g = \frac{4}{3}\pi^2(9 - b_2^-(M))\,, \tag{28}$$

$$\begin{aligned}\|B\|_{L^2}^2 &= \frac{1}{24}\int_M s^2 dV_g - 12\pi^2\tau(M) = \frac{8}{3}\pi^2(\chi(M) - 3\tau(M)) \\ &= \frac{32}{3}\pi^2 b_2^-(M) = \frac{32}{3}\pi^2(1 - \tau(M)) = \frac{32}{3}\pi^2(\chi(M) - 3)\,.\end{aligned} \tag{29}$$

From (27) and (29), we easily obtain the following proposition.

PROPOSITION. *Let (M, J, g) be a compact, connected Kähler-Einstein 4-manifold, with positive scalar curvature s and of Vol$(g) = 1$. Then, the following are equivalent each other*;

i). *The Bochner curvature tensor does not vanish identically*,

ii). $\tau(M) \le 0$, iii). $\chi(M) \ge 4$, iv). $b_2^-(M) \ge 1$,

v). $\|B\|_{L^2}^2 \ge \frac{1}{24}s^2$, vi). $\|B\|_{L^2}^2 \ge \frac{8}{3}\pi^2\chi(M)$, vii). $\|B\|_{L^2}^2 \ge \frac{32}{3}\pi^2$.

For example, when M is homothetic to $\boldsymbol{C}P^1 \times \boldsymbol{C}P^1$ with the standard product metric, all the equalities hold in the above proposition. Thus we see that the value of the estimate constant $C(4) = \sqrt{\frac{1}{24}}$ is optimal in our theorem. Furthermore, in this case, we obtain the isolation theorem of the Bochner curvature tensor even though the constant $C(4)s$ is replaced by $\sqrt{\frac{8}{3}\pi^2\chi(M)}$ or $\sqrt{\frac{32}{3}\pi^2}$. That is,

THEOREM. *Let (M, J, g) be a compact, connected Kähler-Einstein 4-manifold with positive scalar curvature and set a positive constant $\varepsilon = \sqrt{\frac{8}{3}\pi^2\chi(M)}$ or $\sqrt{\frac{32}{3}\pi^2}$. If L^2-norm $\|B\|_{L^2} < \varepsilon$, then $B = 0$.*

REMARK. The L^2-norm of Bochner curvature tensor of a Kähler-Einstein 4-manifold with positive scalar curvature takes a discrete value represented by the topological invariant such that

$$\|B\|_{L^2}^2 = \frac{32}{3}\pi^2 b_2^-(M)\,, \quad (0 \le b_2^-(M) \le 8)\,.$$

Here, $0 \le b_2^-(M) \le 8$ is obtained from (28). For example, the Kähler-Einstein manifold $\boldsymbol{C}P^2\#k\overline{\boldsymbol{C}P^2}$ $(3 \le k \le 8)$ fulfills $b_2^- = k$ and $\|B\|_{L^2}^2 = \frac{32}{3}\pi^2 k$.

3. The contact Bochner curvature tensor

3.1. Curvature tensors of Sasakian manifolds. Let $(M, (\phi, \xi, \eta, g))$ be a Sasakian n-manifold, $n = 2m + 1 \ge 5$. Then g, η, ξ and ϕ are a Riemannian metric, a 1-form, a unit Killing vector field and a tensor field of type (1, 1), respectively, such that

$$\begin{aligned}\eta(X) = g(\xi, X)\,, &\quad (\nabla_X\eta)(Y) = g(X, \phi Y)\,, \\ \phi^2 X = -X + \eta(X)\xi\,, &\quad (\nabla_X\phi)(Y) = g(X, Y)\xi - \eta(Y)X\,,\end{aligned} \tag{30}$$

for any tangent vectors X, Y. From the above, the following identities are derived ([1, 15]).

$$\begin{gathered}\phi\xi = 0\,, \quad \eta(\phi X) = 0\,, \\ \phi X = -\nabla_X \xi\,, \quad d\eta(X, Y) = 2g(X, \phi Y)\,. \end{gathered} \tag{31}$$

It is well known that the Riemannian curvature tensor and the Ricci tensor of a Sasakian manifold satisfy

$$\begin{gathered}\xi^p R_{pjst} = g_{js}\eta_t - g_{jt}\eta_s\,, \\ \phi_i{}^p \phi_j{}^q R_{pqst} = R_{ijst} - G_{ijst} + \phi_i{}^p \phi_j{}^q G_{pqst}\,, \\ \phi_i{}^p \phi_j{}^q R_{pq} = R_{ij} - (n-1)\eta_i \eta_j\,, \end{gathered} \tag{32}$$

where $G_{ijst} = g_{is}g_{jt} - g_{it}g_{js}$.

We call $(M, (\phi, \xi, \eta, g))$ *η-Einstein*, if the Ricci tensor has the form $R_{ij} = a g_{ij} + b\eta_i\eta_j$, where $a = s/(n-1) - 1$ and $b = -s/(n-1) + n$. The scalar curvature of an η-Einstein is constant.

The *contact Bochner curvature tensor* is defined on M by (cf. [13])

$$\begin{aligned} B_{ijst} &= R_{ijst} \\ &- \frac{1}{n+3}[R_{is}g_{jt} + R_{jt}g_{is} - R_{it}g_{js} - R_{js}g_{it} + R_{ir}\phi_s{}^r\phi_{jt} + R_{jr}\phi_t{}^r\phi_{is} - R_{ir}\phi_t{}^r\phi_{js} \\ &- R_{jr}\phi_s{}^r\phi_{it} + 2R_{ir}\phi_j{}^r\phi_{st} + 2\phi_{ij}R_{sr}\phi_t{}^r - R_{is}\eta_j\eta_t - R_{jt}\eta_i\eta_s + R_{it}\eta_j\eta_s + R_{js}\eta_i\eta_t] \\ &+ \frac{k+n-1}{n+3}[\phi_{is}\phi_{jt} - \phi_{it}\phi_{js} + 2\phi_{ij}\phi_{st}] + \frac{k-4}{n+3}[g_{is}g_{jt} - g_{it}g_{js}] \\ &- \frac{k}{n+3}[g_{is}\eta_j\eta_t + g_{jt}\eta_i\eta_s - g_{it}\eta_j\eta_s - g_{js}\eta_i\eta_t]\,, \end{aligned} \tag{33}$$

where $k = (s+n-1)/(n+1)$ and $\phi_{ij} = g_{ir}\phi_j{}^r$. If $(M, (\phi, \xi, \eta, g))$ is a Boothby-Wang fibering over a Hodge manifold, then the contact Bochner curvature tensor coincides with the pull-back of the Bochner curvature tensor of the base Kähler manifold.

The following identities are obtained similarly to the ones for the Bochner curvature tensor;

$$B_{ijsl} = -B_{jist} = -B_{ijts}\,, \tag{34}$$

$$B_{ijst} + B_{jsit} + B_{sijt} = 0\,, \tag{35}$$

$$B_{ijst} = B_{stij}\,, \tag{36}$$

$$\xi^p B_{pjst} = 0\,, \tag{37}$$

$$B_{ijst} = \phi_i{}^p \phi_j{}^q B_{pqst}\,, \tag{38}$$

$$B^p{}_{jpt} = 0\,. \tag{39}$$

The contact Bochner curvature tensor in a Sasakian manifold plays a same role of Bochner curvature tensor in a Kähler manifold. (37) means $B(\xi, X, Y, Z) = 0$ for all tangent vectors

X, Y, Z. The identities $\phi_i{}^p B_{pjst} = -\phi_j{}^p B_{ipst}$ and $\phi^{pq} B_{pqst} = 0$ are derived from these identities.

A D-homothetic deformation $(\phi, \xi, \eta, g) \mapsto (\phi_c, \xi_c, \eta_c, g_c)$ is defined by

$$\phi_c = \phi\,, \quad \xi_c = c^{-1}\xi\,, \quad \eta_c = c\eta\,, \quad g_c = cg + c(c-1)\eta \otimes \eta\,,$$

for a positive constant c, where D means the distribution orthogonal to a contact form η. If (ϕ, ξ, η, g) is a Sasakian structure, then $(\phi_c, \xi_c, \eta_c, g_c)$ is also a Sasakian structure. By direct calculations, we have

$$R_{g_c} = cR_g + c(c^2-1)(\eta \otimes \eta) \bigcirc\!\!\!\!\wedge\, g - c(c-1)\Phi\,, \tag{40}$$

$$Ric_{g_c} = Ric_g - 2(c-1)g + (c-1)\{(n-1)c + n + 1\}\eta \otimes \eta\,, \tag{41}$$

$$s_{g_c} = c^{-1}s_g - c^{-1}(c-1)(n-1)\,, \tag{42}$$

where $\Phi_{ijst} = \phi_{is}\phi_{jt} - \phi_{it}\phi_{js} + 2\phi_{ij}\phi_{st}$. Here, $\bigcirc\!\!\!\!\wedge$ is the Nomizu-Kulkarni product of symmetric 2-tensors. Moreover, the volume form changes as $dV_{g_c} = c^{(n+1)/2} dV_g$.

When we emphasize that a tensor T is determined by the structure tensor (ϕ, ξ, η, g), we denote T by T_g.

LEMMA ([13]). *As a $(1,3)$-tensor the contact Bochner curvature tensor is invariant under any D-homothetic deformation.*

Now we shall introduce another important tensor U in M defined by

$$\begin{aligned} U_{ijst} &= R_{ijst} - (\rho+1)[g_{is}g_{jt} - g_{it}g_{js}] \\ &\quad - \rho[\phi_{is}\phi_{jt} - \phi_{it}\phi_{js} + 2\phi_{ij}\phi_{st} - g_{is}\eta_j\eta_t - g_{jt}\eta_i\eta_s + g_{it}\eta_j\eta_s + g_{js}\eta_i\eta_t] \\ &= R_{ijst} - (\rho+1)G_{ijst} - \rho(\Phi - (\eta\otimes\eta)\bigcirc\!\!\!\!\wedge\, g)_{ijst}, \end{aligned} \tag{43}$$

where $\rho + 1 = k/(n-1)$.

The contact Bochner curvature tensor coincides with U if and only if M is η-Einstein.

A Sasakian manifold M is called *Sasakian space form* if U vanishes identically. It is well known that a Sasakian space form is η-Einstein. So, the contact Bochner curvature tensor of a Sasakian space form vanishes identically.

3.2. Proof of Theorem B. First we show that a Sasakian η-Einstein structure can be D-homothetically deformed to a Sasakain Einstein structure whose contact Bochner curvature tensor coincides with the Weyl conformal curvature tensor. Namely,

PROPOSITION. *Let (M, ϕ, ξ, η, g) be a Sasakian η-Einstein $n(\geq 5)$-manifold with scalar curvature $s_g > -(n-1)$. Put the positive constant $\alpha = \frac{s_g+n-1}{(n-1)(n+1)}$ and consider the D-homothetically deformed structure $(\phi_\alpha, \xi_\alpha, \eta_\alpha, g_\alpha)$. Then the metric g_α is Einstein with $Ric_{g_\alpha} = (n-1)g_\alpha$ and $s_{g_\alpha} = n(n-1)$. Further the contact Bochner curvature tensor B_{g_α} coincides with the Weyl conformal curvature tensor*;

$$B_{g_\alpha} = W_{g_\alpha}\,.$$

PROOF. From (41) and (42) we see by putting $c = \alpha$ $Ric_{g_\alpha} = (n-1)g_\alpha$ and $s_{g_\alpha} = n(n-1)$. On the other hand

$$B_{g_\alpha} = R_{g_\alpha} - \frac{1}{2} g_\alpha \owedge g_\alpha = R_{g_\alpha} - \frac{s_{g_\alpha}}{2n(n-1)} g_\alpha \owedge g_\alpha = W_{g_\alpha} . \tag{44}$$

Now we will prove Theorem B. So, suppose that B_g vanishes. Since the contact Bochner curvature tensor is D-homothetic invariant, B_{g_α} also vanishes. From (44), (M, g_α) is a conformally flat, Einstein manifold with the scalar curvature $s_{g_\alpha} = n(n-1)$, so that (M, g_α) is a finite isometric quotient of the standard n-sphere.

We assume henceforth that B_g does not vanish identically and induces a contradiction. To show that, we put $c = \mathrm{Vol}(g_\alpha)^{-(2/n)}$, so that (M, cg_α) is compact, connected Einstein, with positive scalar curvature and of $\mathrm{Vol}(cg_\alpha) = 1$. As shown in [8], there exists a constant $C(n)$, depending only on n such that

$$\|W_{cg_\alpha}\|_{L^{n/2}, cg_\alpha} \geq C(n) s_{cg_\alpha} ,$$

Here the LHS and RHS are now, respectively

$$\|W_{cg_\alpha}\|_{L^{n/2}, cg_\alpha} = \|W_{g_\alpha}\|_{L^{n/2}, g_\alpha} = \|B_{g_\alpha}\|_{L^{n/2}, g_\alpha} = \alpha^{(1/n)} \|B_g\|_{L^{n/2}, g} ,$$

and

$$C(n) s_{cg_\alpha} = C(n) n(n-1) \mathrm{Vol}(g_\alpha)^{(2/n)} = C(n) n(n-1) \alpha^{(n+1)/n} \mathrm{Vol}(g)^{(2/n)} .$$

Hence, we have the inequality

$$\|B_g\|_{L^{n/2}, g} \geq C(n) n(n-1) \alpha \mathrm{Vol}(g)^{(2/n)} = C(n) \frac{n(s_g + n - 1)}{n+1} \mathrm{Vol}(g)^{(2/n)} . \tag{45}$$

The inequality (45) is invariant under D-homothetic deformation, while the $L^{n/2}$-norm of the contact Bochner curvature tensor is not an invariant under D-homothetic deformation. Normalizing the volume by D-homothetic deformation, we get a contradiction to the assumption $\|B\|_{L^{n/2}} < C(n) \frac{n(s+n-1)}{n+1}$ giving the complete proof.

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