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Upper Complete Intersection Dimension Relative to a Local Homomorphism

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Abstract. In this note, we introduce a homological invariant for finitely generated modules over commutative noetherian local rings by slightly modifying the definition of complete intersection dimension defined by Avramov, Gasharov, and Peeva [4], and observe it from a relative point of view.

1. Introduction

Throughout this note, we assume that all rings are commutative noetherian rings, and all modules are finitely generated.

Projective dimension and Gorenstein dimension (abbr. G-dimension) have played important roles in the classification of modules and rings. Recently, complete intersection dimension (abbr. CI-dimension) and Cohen-Macaulay dimension (abbr. CM-dimension) were introduced by Avramov, Gasharov, and Peeva [4] and Gerko [6], respectively. The former is defined by using projective dimension and the idea of quasi-deformation, and the latter is defined by using G-dimension and the idea of G-quasideformation.

These dimensions are homological invariants for modules, and share many properties with each other. For example, they satisfy the Auslander-Buchsbaum-type equalities. Every module over a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) local ring is of finite projective (resp. CI-, G-, CM-) dimension, and a local ring is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring if the projective (resp. CI-, G-, CM-) dimension of its residue class field is finite. Moreover, among these dimensions, there are inequalities which yield the well-known implications for a local ring R: R is regular $\Rightarrow$ R is a complete intersection $\Rightarrow$ R is Gorenstein $\Rightarrow$ R is Cohen-Macaulay.

In this note, we are interested in CI-dimension. Gulliksen [7] showed that every module over a complete intersection has finite complexity, that is, the Betti numbers are eventually bounded by a polynomial. As a result extending this, Avramov, Gasharov, and Peeva [4] proved that any module of finite CI-dimension has finite complexity. Hence, free resolutions of modules of finite CI-dimension are eventually well-behaved. However, there are a lot of

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unsolved problems on CI-dimension. For instance, it is unknown whether a module of finite complexity is always of finite CI-dimension. Though we do not discuss these problems in this note, it is important to consider CI-dimension.

Here we recall the definition of the CI-dimension of a module over a local ring R. It is similar to that of virtual projective dimension introduced by Avramov [2]:

(1) A local homomorphism $\phi : S \to R$ of local rings is called a *deformation* if ϕ is surjective and the kernel of ϕ is generated by an S-regular sequence.

(2) A diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings is called a *quasi-deformation* of R if α is faithfully flat and ϕ is a deformation.

(3) For an R-module M, the *complete intersection dimension* of M is defined as follows:

$$\text{CI-dim}_R M = \inf \left\{ \begin{matrix} \text{pd}_S(M \otimes_R R') \\ -\text{pd}_S R' \end{matrix} \;\middle|\; \begin{matrix} S \to R' \leftarrow R \text{ is a} \\ \text{quasi-deformation of } R \end{matrix} \right\}.$$

Now, slightly modifying the definition of CI-dimension, we define a homological invariant for a module over a local ring as follows.

DEFINITION 1.1. (1) We call a diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings an *upper quasi-deformation* of R if α is faithfully flat, the closed fiber of α is regular, and ϕ is a deformation.

(2) For an R-module M, we define the *upper complete intersection dimension* (abbr. CI*-dimension) of M as follows:

$$\text{CI}^*\text{-dim}_R M = \inf \left\{ \begin{matrix} \text{pd}_S(M \otimes_R R') \\ -\text{pd}_S R' \end{matrix} \;\middle|\; \begin{matrix} S \to R' \leftarrow R \text{ is an} \\ \text{upper quasi-deformation of } R \end{matrix} \right\}.$$

Here we itemize several properties of CI*-dimension, which are analogous to those of CI-dimension. We omit their proofs because we can prove them in the same way as the proofs of the corresponding results of CI-dimension given in [4]. Let R be a local ring with residue field k, $M \neq 0$ an R-module, and $\mathbf{x} = x_1, x_2, \cdots, x_n$ a sequence in R. We denote by $\Omega_R^r M$ the rth syzygy module of M.

(1) The following conditions are equivalent.
  i) R is a complete intersection.
  ii) $\text{CI}^*\text{-dim}_R X < \infty$ for any R-module X.
  iii) $\text{CI}^*\text{-dim}_R k < \infty$.

(2) If $\text{CI}^*\text{-dim}_R M < \infty$, then $\text{CI}^*\text{-dim}_R M = \text{depth } R - \text{depth}_R M$.

(3) $\text{CI}^*\text{-dim}_R \Omega_R^r M = \sup\{\text{CI}^*\text{-dim}_R M - r, 0\}$.

(4) $\text{CI}^*\text{-dim}_R M/\mathbf{x}M = \text{CI}^*\text{-dim}_R M + n$ if $\mathbf{x}$ is M-regular.

(5) $\text{CI}^*\text{-dim}_{R/(\mathbf{x})} M/\mathbf{x}M \leq \text{CI}^*\text{-dim}_R M$ if $\mathbf{x}$ is R-regular and M-regular.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.

(6) $\text{CI}^*\text{-dim}_{R/(\mathbf{x})} M \leq \text{CI}^*\text{-dim}_R M - n$ if $\mathbf{x}$ is R-regular and $\mathbf{x}M = 0$.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.

(7) $\text{CI-dim}_R M \leq \text{CI}^*\text{-dim}_R M \leq \text{pd}_R M$.
If any one of these dimensions is finite, then it is equal to those to its left.

Araya, Takahashi, and Yoshino [1], modifying the definition of CM-dimension, define a homological invariant for modules as a relative version of the modified CM-dimension. This invariant has a lot of properties similar to projective dimension, CI-dimension, G-dimension, and CM-dimension.

Let $\phi : S \to R$ be a local homomorphism of local rings. The main purpose of this note is to define a new homological invariant for an R-module M as a relative version of CI*-dimension over R, and to study its properties. We will call this the *upper complete intersection dimension* of M *relative* to ϕ, and denote it by $\text{CI}^*\text{-dim}_\phi M$. We shall observe that this invariant has many properties similar to those of the invariant defined by Araya, Takahashi, and Yoshino. For example, we will prove the following. Let k denote the residue class field of R.

THEOREM 2.10. *Let M be a non-zero R-module. If* $\text{CI}^*\text{-dim}_\phi M < \infty$, *then* $\text{CI}^*\text{-dim}_\phi M = \text{depth}\, R - \text{depth}_R M$.

THEOREM 2.14. *Suppose that $S = R$ and ϕ is the identity map on R. Then* $\text{CI}^*\text{-dim}_\phi M = \text{pd}_R M$ *for every R-module M.*

THEOREM 2.15. *The following conditions are equivalent.*
i) *R is a complete intersection and S is a regular ring.*
ii) $\text{CI}^*\text{-dim}_\phi M < \infty$ *for any R-module M.*
iii) $\text{CI}^*\text{-dim}_\phi k < \infty$.

2. Relative CI*-dimension

Throughout the section, $\phi : (S, \mathfrak{n}, l) \to (R, \mathfrak{m}, k)$ always denotes a local homomorphism of local rings.

In this section, we shall make the precise definition of the upper complete intersection dimension of an R-module relative to ϕ to observe CI*-dimension from a relative point of view. To do this, we need the notion of P-factorization, instead of that of upper quasi-deformation used in the definition of (absolute) CI*-dimension.

DEFINITION 2.1. Let

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R\,, \end{array}$$

be a commutative diagram of local homomorphisms of local rings. We call this diagram a *P-factorization* of ϕ if α and β are faithfully flat, the closed fiber of α is regular, and ϕ' is a deformation.

Note that this is an imitation of a G-factorization defined in [1]. The existence of a P-factorization of ϕ transmits several properties of R to S:

PROPOSITION 2.2. *Suppose that there exists a P-factorization of ϕ. Then, if R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring, so is S.*

PROOF. Let $S \overset{\beta}{\to} S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$ be a P-factorization of ϕ. Suppose that R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring. Since α is a faithfully flat homomorphism with regular closed fiber, R' is also a regular (resp. ...) ring. Since ϕ' is a deformation, we easily see that S' is also a regular (resp. ...) ring, and so is S by the flatness of β. □

From now on, we consider the existence of a P-factorization of ϕ. First of all, the above proposition yields the following example which says that ϕ may not have a P-factorization.

EXAMPLE 2.3. Suppose that $R = l$ is the residue class field of S and ϕ is the natural surjection from S to l. Then ϕ has no P-factorization unless S is regular.

Although there does not necessarily exist a P-factorization of ϕ in general, a P-factorization of ϕ seems to exist whenever the ring S is regular. We are able to show it if in addition we assume the condition that S contains a field:

THEOREM 2.4. *Suppose that S is a regular local ring containing a field. Then every local homomorphism $\phi : S \to R$ of local rings has a P-factorization.*

This theorem is essentially proved in [1]. But we shall give here a whole proof of it for this note to be as self-contained as possible. We need the following two lemmas:

LEMMA 2.5. [3, Theorem 1.1] *Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism of local rings, and α be the natural embedding from R into its $\mathfrak{m}$-adic completion $\hat{R}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & \hat{R} \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R \end{array}$$

of local homomorphisms of local rings such that β is faithfully flat, the closed fiber of β is regular, and ϕ' is surjective. (Such a diagram is called a Cohen factorization *of ϕ.)*

LEMMA 2.6. *Let $\phi : S \to R$ be a local homomorphism of complete local rings that admit the common coefficient field k. Put $S' = S \hat{\otimes}_k R$. Let $\lambda : S \to S'$ be the injective homomorphism mapping $b \in S$ to $b \hat{\otimes} 1 \in S'$, and $\varepsilon : S' \to R$ be the surjective homomorphism mapping $b \hat{\otimes} a \in S'$ to $\phi(b)a \in R$. Suppose that S is regular. Then $S \overset{\lambda}{\to} S' \overset{\varepsilon}{\to} R \overset{\mathrm{id}}{\leftarrow} R$ is a P-factorization of ϕ.*

PROOF. Let $y_1, y_2, \cdots, y_s$ be a minimal system of generators of the unique maximal ideal of S. Put $J = \operatorname{Ker}\varepsilon$ and $dy_i = y_i \hat{\otimes} 1 - 1 \hat{\otimes} \phi(y_i) \in S'$ for each $1 \leqq i \leqq s$.

CLAIM 1. *The ideal J of S' is generated by $dy_1, dy_2, \cdots, dy_s$.*

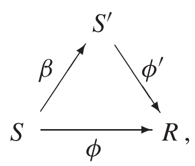
Indeed, put $J_0 = (dy_1, dy_2, \cdots, dy_s)S'$. Let $z = b \hat{\otimes} a$ be an element in J, and let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have $b \hat{\otimes} 1 = \sum b_{i_1 i_2 \cdots i_s} (y_1 \hat{\otimes} 1)^{i_1} (y_2 \hat{\otimes} 1)^{i_2} \cdots (y_s \hat{\otimes} 1)^{i_s} \equiv \sum b_{i_1 i_2 \cdots i_s} (1 \hat{\otimes} \phi(y_1))^{i_1} (1 \hat{\otimes} \phi(y_2))^{i_2} \cdots (1 \hat{\otimes} \phi(y_s))^{i_s} = 1 \hat{\otimes} \phi(b)$ modulo J_0. It follows that $z \equiv 1 \hat{\otimes} \phi(b)a$ modulo J_0. Since $\phi(b)a = \varepsilon(b \hat{\otimes} a) = 0$, we have $z \equiv 0$ modulo J_0, that is, the element $z \in J$ belongs to J_0. Thus, we see that $J = J_0$.

CLAIM 2. *The sequence $dy_1, dy_2, \cdots, dy_s$ is an S'-regular sequence.*

Indeed, since S is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i) \in S'$ for each $1 \leqq i \leqq s$. Note that the endomorphism on S' which sends Y_i to dy_i is an automorphism. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is S'-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form an S'-regular sequence.

These claims prove that the homomorphism ε is a deformation. On the other hand, it is easy to see that λ is faithfully flat. Thus, the lemma is proved. □

PROOF OF THEOREM 2.4. We may assume that R (resp. S) is complete in its $\mathfrak{m}$-adic (resp. $\mathfrak{n}$-adic) topology. Hence Lemma 2.5 implies that ϕ has a Cohen factorization



where β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is a surjective homomorphism. Hence S' is also a regular local ring containing a field. Therefore, replacing S with S', we may assume that ϕ is a surjection. In particular R and S have the common coefficient field, hence Lemma 2.6 implies that ϕ has a P-factorization, as desired. □

CONJECTURE 2.7. Whenever S is regular, the local homomorphism $\phi : S \to R$ would have a P-factorization.

Now, by using the idea of P-factorization, we define the CI*-dimension of a module in a relative sense.

DEFINITION 2.8. For an R-module M, we put

$$\text{CI}^*\text{-dim}_\phi M = \inf \left\{ \begin{matrix} \text{pd}_{S'}(M \otimes_R R') \\ -\text{pd}_{S'} R' \end{matrix} \;\middle|\; \begin{matrix} S \to S' \to R' \leftarrow R \\ \text{is a P-factorization of } \phi \end{matrix} \right\}$$

and call it the *upper complete intersection dimension* of M *relative* to ϕ.

By definition, $\mathrm{CI}^*\text{-}\dim_\phi M = \infty$ for an R-module M if ϕ has no P-factorization. Supppose that ϕ has at least one P-factorization $S \to S' \to R' \leftarrow R$. Then we have $\mathrm{pd}_{S'}(F \otimes_R R') = \mathrm{pd}_{S'} R'$ $(< \infty)$ for any free R-module F. Therefore Theorem 2.4 yields the following result:

PROPOSITION 2.9. *If S is a regular local ring that contains a field, then*

$$\mathrm{CI}^*\text{-}\dim_\phi F = 0\ (< \infty)$$

for any free R-module F.

In the rest of this section, we observe the properties of relative CI*-dimension $\mathrm{CI}^*\text{-}\dim_\phi$. We begin by proving that relative CI*-dimension also satisfies the Auslander-Buchsbaum-type equality:

THEOREM 2.10. *Let M be a non-zero R-module. If* $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$, *then*

$$\mathrm{CI}^*\text{-}\dim_\phi M = \operatorname{depth} R - \operatorname{depth}_R M\,.$$

PROOF. Since $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$, there exists a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of ϕ such that $\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{pd}_{S'}(M \otimes_R R') - \mathrm{pd}_{S'} R' < \infty$. Hence we see that

$$\begin{aligned}\mathrm{CI}^*\text{-}\dim_\phi M &= \mathrm{pd}_{S'}(M \otimes_R R') - \mathrm{pd}_{S'} R' \\ &= (\operatorname{depth} S' - \operatorname{depth}_{S'}(M \otimes_R R')) - (\operatorname{depth} S' - \operatorname{depth}_{S'} R') \\ &= \operatorname{depth}_{S'} R' - \operatorname{depth}_{S'}(M \otimes_R R')\,.\end{aligned}$$

Note that ϕ' is surjective. Since α and β are faithfully flat, we obtain

$$\begin{cases}\operatorname{depth}_{S'} R' = \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R', \\ \operatorname{depth}_{S'}(M \otimes_R R') = \operatorname{depth}_R M + \operatorname{depth} R'/\mathfrak{m}R'\,.\end{cases}$$

It follows that $\mathrm{CI}^*\text{-}\dim_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$. □

In view of this theorem, we notice that the value of the relative CI*-dimension of an R-module is given independently of the ring S if it is finite.

PROPOSITION 2.11. *Let M be an R-module. Then*

(1) $\mathrm{CI}^*\text{-}\dim_\phi M \geqq \mathrm{CI}^*\text{-}\dim_R M$.
The equality holds if $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$.

(2) $\mathrm{CI}^*\text{-}\dim_\phi M \leqq \mathrm{pd}_R M$ *if ϕ is faithfully flat.*
The equality holds if in addition $\mathrm{pd}_R M < \infty$.

PROOF. (1) Since the inequality holds if $\mathrm{CI}^*\text{-}\dim_\phi M = \infty$, assume that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a P-factorization of ϕ such that $\mathrm{pd}_{S'}(M \otimes_R R') - \mathrm{pd}_{S'} R' < \infty$. Then by definition $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a quasi-deformation of

R, which shows that $\mathrm{CI}^*\text{-}\dim_R M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum-type equality for CI*-dimension.

(2) Suppose that ϕ is faithfully flat. Since the inequality holds if $\mathrm{pd}_R M = \infty$, assume that $\mathrm{pd}_R M < \infty$. We easily see that the diagram $S \overset{\phi}{\to} R \overset{\mathrm{id}}{\to} R \overset{\mathrm{id}}{\leftarrow} R$ is a P-factorization of ϕ. Therefore we have $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$. Hence the assertion follows from Theorem 2.10 and the Auslander-Buchsbaum formula for projective dimension. □

The inequality in the second assertion of the above proposition may not hold without the faithful flatness of ϕ; see Remark 2.17 below.

Now, recall that

$$\mathrm{CI}^*\text{-}\dim_R M \leqq \mathrm{pd}_R M$$

for any R-module M. Hence the above proposition says that relative CI*-dimension is inserted between absolute CI*-dimension and projective dimension if ϕ is faithfully flat.

It is natural to ask when relative CI*-dimension $\mathrm{CI}^*\text{-}\dim_\phi$ coincides with absolute one $\mathrm{CI}^*\text{-}\dim_R$ as an invariant for R-modules. It seems to happen if S is the prime field of R.

Let us consider the case that the characteristic $\mathrm{char}\, k$ of k is zero. Then we easily see that $\mathrm{char}\, R = 0$. It follows that R has the prime field $\mathbf{Q}$. Let $S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$ be a quasi-deformation of R. Since α is injective and ϕ' is surjective, the residue class field of R' is of characteristic zero, and so is that of S'. Hence we see that $\mathrm{char}\, S' = 0$, and there exists a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ \mathbf{Q} & \xrightarrow[\phi]{} & R\,, \end{array}$$

where ϕ and β denote the natural embeddings. Note that β is faithfully flat because $\mathbf{Q}$ is a field. Therefore this diagram is a P-factorization of ϕ. Thus, Proposition 2.11(1) yields the following:

PROPOSITION 2.12. *Suppose that k is of characteristic zero. If S is the prime field of R, then*

$$\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{CI}^*\text{-}\dim_R M$$

for any R-module M.

CONJECTURE 2.13. If S is the prime field of R, then it would always hold that $\mathrm{CI}^*\text{-}\dim_\phi M = \mathrm{CI}^*\text{-}\dim_R M$ for any R-module M.

As we have observed in Proposition 2.11, the relative CI*-dimension $\mathrm{CI}^*\text{-}\dim_\phi M$ of an R-module M is always less than or equal to its projective dimension $\mathrm{pd}_R M$, as long as ϕ is faithfully flat. The next theorem gives a sufficient condition for these dimensions to coincide with each other as invariants for R-modules.

THEOREM 2.14. *Suppose that $S = R$ and ϕ is the identity map of R. Then*

$$\text{CI}^*\text{-dim}_\phi M = \text{pd}_R M$$

for every R-module M.

PROOF. The assumption in the theorem in particular implies that ϕ is faithfully flat. Hence Proposition 2.11(2) yields one inequality relation in the theorem. Thus we have only to prove the other inequality relation $\text{CI}^*\text{-dim}_\phi M \geqq \text{pd}_R M$. There is nothing to show if $\text{CI}^*\text{-dim}_\phi M = \infty$. Hence assume that $\text{CI}^*\text{-dim}_\phi M < \infty$. Then the identity map ϕ on R has a P-factorization $R \overset{\beta}{\to} S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$ such that $\text{pd}_{S'}(M \otimes_R R') < \infty$. Let l' denote the residue class field of S'. Taking an S'-sequence $\mathbf{x} = x_1, x_2, \cdots, x_r$ generating the kernel of ϕ', we have $\mathbf{R}\text{Hom}_{S'}(R', l') \cong \text{Hom}_{S'}(\text{K}_\bullet(\mathbf{x}), l') \cong \bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i]$, where $\text{K}_\bullet(\mathbf{x})$ is the Koszul complex of $\mathbf{x}$ over S'. Noting that both α and β are faithfully flat, we see that

$$\begin{aligned} \mathbf{R}\text{Hom}_{S'}(M \otimes_R R', l') &\cong \mathbf{R}\text{Hom}_{S'}((M \otimes_R^{\mathbf{L}} S') \otimes_{S'}^{\mathbf{L}} R', l') \\ &\cong \mathbf{R}\text{Hom}_{S'}(M \otimes_R^{\mathbf{L}} S', \mathbf{R}\text{Hom}_{S'}(R', l')) \\ &\cong \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', \textstyle\bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i]) \\ &\cong \textstyle\bigoplus_{i=0}^r \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i]\,. \end{aligned}$$

It follows from this that

$$\begin{aligned} \text{Ext}_{S'}^j(M \otimes_R R', l') &\cong \text{H}^j(\mathbf{R}\text{Hom}_{S'}(M \otimes_R R', l')) \\ &\cong \text{H}^j(\textstyle\bigoplus_{i=0}^r \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i]) \\ &\cong \textstyle\bigoplus_{i=0}^r \text{Ext}_{S'}^{j-i}(M \otimes_R S', l')^{\binom{r}{i}}\,. \end{aligned}$$

Note that $\text{Ext}_{S'}^j(M \otimes_R R', l') = 0$ for any $j \gg 0$ because $\text{pd}_{S'}(M \otimes_R R') < \infty$. Hence we obtain $\text{Ext}_{S'}^j(M \otimes_R S', l') = 0$ for any $j \gg 0$, which implies that $\text{pd}_{S'}(M \otimes_R S') < \infty$. Thus we get $\text{pd}_R M < \infty$. Then the Auslander-Buchsbaum-type equalities for projective dimension and CI*-dimension yield that $\text{CI}^*\text{-dim}_\phi M = \text{pd}_R M = \text{depth}\, R - \text{depth}_R M$. □

We know that $\text{CI}^*\text{-dim}_R M < \infty$ for any R-module M if R is a complete intersection and that R is a complete intersection if $\text{CI}^*\text{-dim}_R k < \infty$. We can prove the following result similar to this:

THEOREM 2.15. *The following conditions are equivalent.*

i) *R is a complete intersection and S is a regular ring.*

ii) $\text{CI}^*\text{-dim}_\phi M < \infty$ *for any R-module M.*

iii) $\text{CI}^*\text{-dim}_\phi k < \infty$.

PROOF. i) ⇒ ii): It follows from Lemma 2.5 that there is a Cohen factorization $S \overset{\beta}{\to} S' \overset{\phi'}{\to} \hat{R} \overset{\alpha}{\leftarrow} R$ of ϕ. Since both the ring S and the closed fiber of β are regular, so is S' by the faithful flatness of β. On the other hand, since R is a complete intersection, so is its $\mathfrak{m}$-adic completion $\hat{R}$. Hence the homomorphism ϕ' is a deformation. (A surjective homomorphism from a regular local ring to a local complete intersection must be a deformation; see [5, Theorem 2.3.3].) Thus, we see that the factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \hat{R} \xleftarrow{\alpha} R$ is a P-factorization of ϕ. The regularity of the ring S' implies that every S'-module is of finite projective dimension over S', from which the condition ii) follows.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): The condition iii) says that ϕ has a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\mathrm{pd}_{S'}(k \otimes_R R') < \infty$. Put $A = k \otimes_R R'$. Note that A is a regular local ring because it is the closed fiber of α. Let $\mathbf{a} = a_1, a_2, \cdots, a_t$ be a regular system of parameters of A. Since $\mathbf{a}$ is an A-regular sequence, we have $\mathrm{pd}_{S'} A/(\mathbf{a}) = \mathrm{pd}_{S'} A + t < \infty$. Since ϕ' is surjective, we see that the quotient ring $A/(\mathbf{a})$ is isomorphic to the residue class field l' of S'. Hence we obtain $\mathrm{pd}_{S'} l' < \infty$, which implies that S' is regular, and so is S. On the other hand, it follows from Theorem 2.11(1) that R is a complete intersection. □

Suppose that R is regular. Then, by Proposition 2.2, S is also regular if ϕ has at least one P-factorization. Thus the above theorem implies the following corollary:

COROLLARY 2.16. *Suppose that R is regular. If* $\mathrm{CI}^*\text{-}\dim_\phi N < \infty$ *for some R-module N, then* $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ *for every R-module M.*

REMARK 2.17. Relating to the second assertion of Proposition 2.11, there is no inequality relation between relative CI*-dimension and projective dimension in a general setting. In fact, the following results immediately follow from Theorem 2.15:

(1) $\mathrm{CI}^*\text{-}\dim_\phi k < \mathrm{pd}_R k$ if R is a complete intersection which is not regular and S is a regular ring.

(2) $\mathrm{CI}^*\text{-}\dim_\phi k > \mathrm{pd}_R k$ if R is regular and S is not regular.

We can calculate the relative CI*-dimension of each of the syzygy modules of an R-module M by using the relative CI*-dimension of M:

PROPOSITION 2.18. *For an R-module M and an integer $n \geqq 0$,*

$$\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^n M = \sup\{\mathrm{CI}^*\text{-}\dim_\phi M - n, 0\}.$$

PROOF. We claim that $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ if and only if $\mathrm{CI}^*\text{-}\dim_\phi \Omega_R^1 M < \infty$. Indeed, let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. There is a short exact sequence

$$0 \to \Omega_R^1 M \to R^m \to M \to 0$$

with some integer m. Since R' is flat over R, we obtain

$$0 \to \Omega_R^1 M \otimes_R R' \to R'^m \to M \otimes_R R' \to 0.$$

Note that $\mathrm{pd}_{S'} R' < \infty$. Hence we see that $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$ if and only if $\mathrm{pd}_{S'}(\Omega_R^1 M \otimes_R R') < \infty$. This implies the claim.

It follows from the claim that CI*-dim$_\phi M < \infty$ if and only if CI*-dim$_\phi \Omega_R^n M < \infty$. Thus, in order to prove the proposition, we may assume that CI*-dim$_\phi M < \infty$ and CI*-dim$_\phi \Omega_R^n M < \infty$. In particular, we have CI*-dim$_R M < \infty$ by Proposition 2.11(1), hence we also have CI-dim$_R M < \infty$. Therefore [4, (1.9)] gives us the equality

$$\operatorname{depth}_R \Omega_R^n M = \min\{\operatorname{depth}_R M + n, \operatorname{depth} R\}\,.$$

Consequently we obtain

$$\begin{aligned}\text{CI}^*\text{-dim}_\phi \Omega_R^n M &= \operatorname{depth} R - \operatorname{depth}_R \Omega_R^n M \\ &= \max\{\operatorname{depth} R - \operatorname{depth}_R M - n, 0\} \\ &= \max\{\text{CI}^*\text{-dim}_\phi M - n, 0\}\,,\end{aligned}$$

as desired. □

As the last result of this note, we state the relationship between relative CI*-dimension and regular sequences.

PROPOSITION 2.19. *Let* $\mathbf{x} = x_1, x_2, \cdots, x_m$ *(resp.* $\mathbf{y} = y_1, y_2, \cdots, y_n$*) be a sequence in* R *(resp.* S*). Denote by* $\bar{\phi}$ *(resp.* $\tilde{\phi}$*) the local homomorphism* $S/(\mathbf{y}) \to R/\mathbf{y}R$ *(resp.* $S \to R/(\mathbf{x})$*) induced by* ϕ*. Then*

(1) CI*-dim$_\phi M/\mathbf{x}M =$ CI*-dim$_\phi M + m$ *if* $\mathbf{x}$ *is* M*-regular.*

(2) CI*-dim$_{\bar{\phi}} M/\mathbf{y}M \leqq$ CI*-dim$_\phi M$ *if* $\mathbf{y}$ *is* S*-regular,* R*-regular, and* M*-regular.* *The equality holds if* CI*-dim$_\phi M < \infty$.

(3) CI*-dim$_{\tilde{\phi}} M \leqq$ CI*-dim$_\phi M - m$ *if* $\mathbf{x}$ *is* R*-regular and* R*-regular and* $\mathbf{x}M = 0$. *The equality holds if* CI*-dim$_\phi M < \infty$.

PROOF. (1) By Theorem 2.10 we have only to show that CI*-dim$_\phi M/\mathbf{x}M < \infty$ if and only if CI*-dim$_\phi M < \infty$. Let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. Since R' is R-flat, the sequence $\mathbf{x}$ is also $(M \otimes_R R')$-regular. Hence we obtain $\operatorname{pd}_{S'}(M \otimes_R R')/\mathbf{x}(M \otimes_R R') = \operatorname{pd}_{S'}(M \otimes_R R') + m$. Note that $(M \otimes_R R')/\mathbf{x}(M \otimes_R R') \cong (M/\mathbf{x}M) \otimes_R R'$. Therefore we see that $\operatorname{pd}_{S'}(M/\mathbf{x}M) \otimes_R R' < \infty$ if and only if $\operatorname{pd}_{S'}(M \otimes_R R') < \infty$. Thus the desired result is proved.

(2) We may assume that CI*-dim$_\phi M < \infty$ because the assertion immediately follows if CI*-dim$_\phi M = \infty$. It suffices to prove that the left side of the inequality is also finite, because the equality is implied by Theorem 2.10. There exists a P-factorization $S \to S' \to R' \leftarrow R$ of ϕ such that $\operatorname{pd}_{S'}(M \otimes_R R') < \infty$. Since $\mathbf{y}$ is both S-regular and R-regular, it is easy to see that the induced diagram $S/(\mathbf{y}) \to S'/\mathbf{y}S' \to R'/\mathbf{y}R' \leftarrow R/\mathbf{y}R$ is a P-factorization of $\bar{\phi}$. As $\mathbf{y}$ is M-regular, it is also $(M \otimes_R R')$-regular, and we have $\operatorname{pd}_{S'/\mathbf{y}S'}(M/\mathbf{y}M) \otimes_R R' = \operatorname{pd}_{S'/\mathbf{y}S'}(M \otimes_R R')/\mathbf{y}(M \otimes_R R') = \operatorname{pd}_{S'}(M \otimes_R R') < \infty$. Hence we have CI*-dim$_{\bar{\phi}} M/\mathbf{y}M < \infty$.

(3) Suppose that CI*-dim$_\phi M < \infty$. It is enough to prove that CI*-dim$_{\tilde{\phi}} M < \infty$ by Theorem 2.10. Let $S \to S' \to R' \leftarrow R$ of ϕ be a P-factorization of ϕ with $\operatorname{pd}_{S'}(M \otimes_R R') <$

∞. Then we easily see that the induced diagram $S \to S' \to R'/\mathbf{x}R' \leftarrow R/(\mathbf{x})$ is a P-factorization of $\tilde{\phi}$. Since $M \otimes_{R/(\mathbf{x})} R'/\mathbf{x}R' \cong M \otimes_R R'$ has finite projective dimension over S', we have CI*-dim$_{\tilde{\phi}} M < \infty$, as desired. □

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References

[1] T. ARAYA, R. TAKAHASHI and Y. YOSHINO, Upper Cohen-Macaulay dimension, to appear in Math. J. Okayama Univ.

[2] L. L. AVRAMOV, Modules of finite virtual projective dimension, Invent. Math. **96** (1989), 71–101.

[3] L. L. AVRAMOV, H. -B. FOXBY and B. HERZOG, Structure of local homomorphisms, J. Algebra **164** (1994), 124–145.

[4] L. L. AVRAMOV, V. N. GASHAROV and I. V. PEEVA, Complete intersection dimension, Inst. Hautes Etudes Sci. Publ. Math. **86** (1997), 67–114.

[5] W. BRUNS and J. HERZOG, *Cohen-Macaulay rings, revised edition*, Cambridge University Press (1998).

[6] A. A. GERKO, On homological dimensions, Mat. Sb. **192** (2001), 79–94; translation in Sb. Math. **192** (2001), 1165–1179.

[7] T. H. GULLIKSEN, On the deviations of a local ring, Math. Scand. **47** (1980), 5–20.

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