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On Two Step Tensor Modules of the Maximal Compact Subgroups of Inner Type Noncompact Real Simple Lie Groups

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1. Introduction

Let $\mathbf{C}$ (resp. $\mathbf{R}$) be the complex (resp. real) number field. We consider a connected simply connected complex simple Lie group $G_{\mathbf{C}}$ and its connected noncompact simple real form G. In this article we shall always fix a maximal compact subgroup K of G, and assume that rank G = rank K. This assumption is equivalent to G is inner. Let $\mathfrak{g}$ and $\mathfrak{k}$ be respectively the Lie algebras of G and K. Let θ be the Cartan involution which stabilizes K. Then $\mathfrak{g}$ is decompsed by $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{p}$ is the eigenspace of θ in $\mathfrak{g}$ with the eigenvalue -1. Let $\mathfrak{g}_{\mathbf{C}}$ be the Lie algebra of $G_{\mathbf{C}}$. We shall denote, for each subspace $\mathfrak{v}$ of $\mathfrak{g}$, by $\mathfrak{v}_{\mathbf{C}}$ the complexification of $\mathfrak{v}$ in $\mathfrak{g}_{\mathbf{C}}$. $\mathfrak{p}_{\mathbf{C}}$ is a K-module by the adjoint action of K. Let B be a maximal abelian subgroup of K. Then B is also a maximal abelian subgroup (Cartan subgroup) of G. Let $\mathfrak{b}$ be the Lie algebra of B. Then the root system Σ of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ is decomposed by $\Sigma = \Sigma_K \cup \Sigma_n$, where Σ_K (resp. Σ_n) is the set of all compact (resp. noncompact) roots in Σ. Then Σ_K is also the root system of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. We choose a positive root system P_K, and always fix it.

Let us state our purpose of this article. Let μ be a P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ and (π_μ, V_μ) a simple K-module with highest weight μ. We consider a simple Harish-Chandra $(\mathfrak{g}, K)$-module $U(\mathfrak{g}_{\mathbf{C}})V_\mu$ which contains (π_μ, V_μ) with multiplicity one, where $U(\mathfrak{g}_{\mathbf{C}})$ is the universal enveloping algebra of $\mathfrak{g}_{\mathbf{C}}$. Let $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ be the tensor K-module. Canonically this space has a unitary K-module structure. We define a K-linear homomorphism ϖ of $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ to $U(\mathfrak{g}_{\mathbf{C}})V_\mu$ by $\varpi(X \otimes Y \otimes v) = XYv$ for $X, Y \in \mathfrak{p}_{\mathbf{C}}$, $v \in V_\mu$. Let V be a finite K-module. We define a projection operator P_μ on V by

$$P_\mu(v) = \deg \pi_\mu \int_K kv\,\overline{trace\pi_\mu(k)}dk \quad \text{for } v \in V\,, \tag{1.1}$$

where $\deg \pi_\mu = \dim V_\mu$ and dk is the Haar measure on K normalized as $\int_K dk = 1$. Since $P_\mu \varpi = \varpi P_\mu$, ϖ induces a K-module linear homomorphism of $M(\mu) = P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ to $V_\mu \subset U(\mathfrak{g}_{\mathbf{C}})V_\mu$. Let $m = m(\mu)$ be the multiplicity of V_μ in $M(\mu)$. $M(\mu)$ is decomposed by $M(\mu) = \bigoplus_{j=1}^{m} U(\mathfrak{k}_{\mathbf{C}})v_j$, where v_j is the highest weight vector of the simple K-module $U(\mathfrak{k}_{\mathbf{C}})v_j$ and $U(\mathfrak{k}_{\mathbf{C}})$ is the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$. Let $v(\mu)$ be the highest weight

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vector of V_μ. Since ϖ is a K-module linear homomorphism of $M(\mu)$ to V_μ, there exsists a complex number x_i such that $\varpi(v_i) = x_i v(\mu)$, $1 \le i \le m$. We choose the root vectors $X_\alpha, \alpha \in \Sigma$ normalized as $\phi(X_\alpha, X_{-\alpha}) = 1$, where ϕ is the Killing form on $\mathfrak{g}_{\mathbf{C}}$. Then we have $H_\alpha = ad(X_\alpha)X_{-\alpha} \in \mathfrak{b}_{\mathbf{C}}$. Let X_ω be a root vector corresponding to a noncompact root ω. We have $(H - \mu(H)1)P_\mu(X_\omega \otimes X_{-\omega} \otimes v(\mu)) = 0$, $H \in \mathfrak{b}$, where 1 is the identity in $U(\mathfrak{k}_{\mathbf{C}})$. Since μ is the highest weight of V_μ, there exist the complex constants $c_{\omega,j}$ such that

$$P_\mu(X_\omega \otimes X_{-\omega} \otimes v(\mu)) - P_\mu(X_{-\omega} \otimes X_\omega \otimes v(\mu)) = \sum_{j=1}^{m} c_{\omega,j} v_j \,.$$

Let P be a positive root system of Σ containing P_K and P_n the set all noncompact roots in P. We put $P_n = \{\omega_1, \omega_2, \cdots, \omega_N\}$, $\mathbf{x_0} = {}^t(x_1, x_2, \cdots, x_m)$, $\mathbf{b} = {}^t(\mu(H_{\omega_1}), \mu(H_{\omega_2}), \cdots, \mu(H_{\omega_N}))$ and $A = (c_{\omega_i,j})$. Then $\mathbf{x_0}$ is a solution of the system of the linear equations;

$$A\mathbf{x} = \mathbf{b}\,. \tag{1.2}$$

We note that all entries in A are given by the Clebsch-Gordan coefficients of the tensor K-module $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ (see Corollary 4.7). This indicates that the action of X_ω on $V_\mu \subset U(\mathfrak{g}_{\mathbf{C}})V_\mu$ is controlled by the Clebsch-Gordan coefficients of $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ (cf. also [1]). Our motivation is to study the equation (1.2).

Let us state the first result after the following preparations. Let H_μ be the element in $\mathfrak{b}_{\mathbf{C}}$ satisfying $\phi(H_\mu, H) = \mu(H)$ for all $H \in \mathfrak{b}_{\mathbf{C}}$. Then the centralizer $K(\mu)$ of H_μ in K is reductive, and contains B. Let $\Sigma_{K(\mu)}$ be the root system of the pair $(\mathfrak{k}(\mu)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$, where $\mathfrak{k}(\mu)$ is the Lie algebra of $K(\mu)$. We put $P_{K(\mu)} = P_K \cap \Sigma_{K(\mu)}$. $P_{K(\mu)}$ is a positive root system of $\Sigma_{K(\mu)}$. A noncompact root $\omega \in \Sigma_n$ is said to be $P_{K(\mu)}$-highest if $\omega + \alpha \notin \Sigma$ for all α in $P_{K(\mu)}$. When ω in Σ_n is $P_{K(\mu)}$-highest, ω is actually the highest weight of a simple $K(\mu)$-submodule of $\mathfrak{p}_{\mathbf{C}}$. The set of all P_K-dominant integral form on $\mathfrak{b}_{\mathbf{C}}$ will be denoted by Γ_K. In §5 we shall prove the following theorem.

THEOREM I. *Let $\mu \in \Gamma_K$ and assume that μ is admissible* (*see Definition* 5.2). *Then the multiplicity $m(\mu)$ of V_μ in the K-module $M(\mu)$ is given by*

$$m(\mu) = \sharp\{\omega \in \Sigma_n : \omega \text{ is } P_{K(\mu)}\text{-highest}\}\,,$$

where $\sharp S$ is the number of the elements in a set S.

We shall state our second result. Let P be a positive root system containing P_K. For a subset Θ in the simple root system Ψ of P, we denote by $P(\Theta)$ the set of all positive roots in P generated by Θ over the ring of integers. The dual space of the real vector space $\sqrt{-1}\mathfrak{b}$ will be denoted by $(\sqrt{-1}\mathfrak{b})^*$. Let C^* be the positive Weyl chamber of $(\sqrt{-1}\mathfrak{b})^*$ corresponding to P. We define a subset $C(\Theta)^*$ in the topological closure $cl(C^*)$ of C^* by

$$C(\Theta)^* = \{\eta \in cl(C^*) : (\alpha, \eta) = 0 \text{ for } \alpha \in P(\Theta) \text{ and } (\alpha, \eta) > 0 \text{ for } \alpha \in P\backslash P(\Theta)\}\,,$$

where (α, η) is the inner product on $(\sqrt{-1}\mathfrak{b})^*$ induced from the Killing form ϕ on $\mathfrak{g}_{\mathbf{C}}$. Let η be an element in $C(\Theta)^*$ and H_η the element in $\sqrt{-1}\mathfrak{b}$ determined by $\phi(H_\eta, H) = \eta(H)$, $H \in$

$\sqrt{-1}\mathfrak{b}$. Consider the centralizer $K(\eta)$ of H_η in K. Then $K(\eta)$ contains B, and is uniquely determined by $C(\Theta)^*$. We put $K(\Theta) = K(\eta)$. Let $\mathfrak{p}^+$ be the subspace of $\mathfrak{p}_{\mathbf{C}}$ generated by the set of all root vectors corresponding to $P \cap \Sigma_n$. Let τ be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the compact real form $\mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$. A simple $K(\Theta)$-submodule $\mathfrak{q}$ of $\mathfrak{p}_{\mathbf{C}}$ is said to be the first (resp. the second) kind if $\tau(\mathfrak{q}) = \mathfrak{q}$ (resp. $\mathfrak{q} \subset \mathfrak{p}^+$ or $\tau(\mathfrak{q}) \subset \mathfrak{p}^+$). A noncompact root ω in Σ_n is said to be the first (resp. the second) kind if ω is a weight of a simple $K(\Theta)$-submodule of $\mathfrak{p}_{\mathbf{C}}$ of the first (resp. the second) kind. The triple $(P_K, P(\Theta), P)$ is standard if each simple $K(\Theta)$-submodule $\mathfrak{q}$ of $\mathfrak{p}_{\mathbf{C}}$ is either the first kind or the second kind. The following theorem will be proved in §7.

THEOREM II. *Let $\mu \in \Gamma_K$. Then there exists a standard triple $(P_K, P(\Theta), P)$ such that $\mu \in C(\Theta)^*$. Moreover, we have $K(\Theta) = K(\mu)$.*

Let $(P_K, P(\Theta), P)$ be a standard triple. We consider an element μ in $C(\Theta)^* \cap \Gamma_K$ and a noncompact root ω satisfying $\mu+\omega \in \Gamma_K$. We define a projection operator $P_{\mu+\omega}$ on $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by the same as in (1.1). We put

$$P_+ = \sum_{\omega \in \Sigma_n \cap P, \mu+\omega \in \Gamma_K} P_{\mu+\omega} .$$

Let us define a K-submodule $N(\mu)$ of $M(\mu)$ by $N(\mu)$ = the K-module generated by the set

$$N = \{P_\mu(X \otimes P_+(Y \otimes v) - Y \otimes P_+(X \otimes v)) : X, Y \in \mathfrak{p}_{\mathbf{C}}, v \in V_\mu\} .$$

THEOREM III. *Let $(P_K, P(\Theta), P)$ be a standard triple and $\mu \in C(\Theta)^* \cap \Gamma_K$. Suppose that μ is sufficiently $P_K \backslash P_{K(\Theta)}$-regular. Then μ is admissible. Furthermore, we have*

$$n(\mu) = \sharp\{\omega \in P \cap \Sigma_n : \omega \text{ is } P_{K(\Theta)}\text{-highest and of the second kind}\} ,$$

where $n(\mu)$ is the multiplicity of V_μ in $N(\mu)$.

In §8 we shall prove this theorem by using the asymptotic behaviour of the Clebsch-Gordan coefficients of $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$.

2. Preliminaries

Let Σ be the root system of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. We put, for $\alpha \in \Sigma$,

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}_{\mathbf{C}} : ad(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{b}_{\mathbf{C}}\} .$$

Then we have $\mathfrak{g}_{\mathbf{C}} = \mathfrak{b}_{\mathbf{C}} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha)$. Let $\mathfrak{g}_u = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{p}$ be the compact real form of $\mathfrak{g}_{\mathbf{C}}$. We choose a canonical Weyl basis $X_\alpha \in \mathfrak{g}_\alpha, \alpha \in \Sigma$ satisfying the followings (cf. the proof of Theorem 6.3 in [2]):

$$(2.1) \qquad X_\alpha - X_{-\alpha} , \quad \sqrt{-1}(X_\alpha + X_{-\alpha}) \in \mathfrak{g}_u \quad \text{and} \quad \phi(X_\alpha, X_{-\alpha}) = 1 ,$$

where $\phi(X, Y) = trace(ad(X)ad(Y))$ is the Killing form on $\mathfrak{g}_{\mathbf{C}}$. We put $H_\alpha = ad(X_\alpha)X_{-\alpha}$. Then we have $\phi(H_\alpha, H) = \alpha(H)$ for all H in $\mathfrak{b}_{\mathbf{C}}$. Let μ be a linear form on the real vector space $\sqrt{-1}\mathfrak{b}$. Then there exists a unique H_μ in $\sqrt{-1}\mathfrak{b}$ such that $\phi(H_\mu, H) = \mu(H)$ for all H in $\sqrt{-1}\mathfrak{b}$. Let $(\sqrt{-1}\mathfrak{b})^*$ be the dual space of $\sqrt{-1}\mathfrak{b}$. We define a positive definite bilinear form (μ, λ) by $(\mu, \lambda) = \phi(H_\mu, H_\lambda)$ for $\mu, \lambda \in (\sqrt{-1}\mathfrak{b})^*$. We put, for each pair of α and β in Σ, a complex number $< \alpha, \beta >$ by

$$(2.2) \qquad \langle \alpha, \beta \rangle = \begin{cases} \phi(ad(X_\alpha)X_\beta, X_{-\alpha-\beta}) & \text{if } \alpha+\beta \in \Sigma\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then $\langle \alpha, \beta \rangle$ is a pure imaginary number. Let p and q be two nonnegative integers satisfying $j\alpha + \beta \in \Sigma$ iff $-q \leq j \leq p$. Then we have (cf. Lemma 4.3.8 and Corollary 4.3.12 in [4])

$$(2.3) \qquad 2(\beta, \alpha)|\alpha|^{-2} = q - p\,, \quad p + q \leq 3\,.$$

Furthermore, we have (cf. Lemma 4.3.22 in [4])

$$(2.4) \qquad |\langle \alpha, \beta \rangle|^2 = p(q+1)\frac{|\alpha|^2}{2}\,.$$

A root α in Σ is compact (resp. noncompact) if $X_\alpha \in \mathfrak{k}_{\mathbf{C}}$ (resp. $X_\alpha \in \mathfrak{p}_{\mathbf{C}}$). Since $\mathfrak{k}_{\mathbf{C}}$ and $\mathfrak{p}_{\mathbf{C}}$ are invariant under $ad(\mathfrak{b})$, Σ is given by the disjoint union of the set of all compact roots Σ_K and the set of all noncompact roots Σ_n. Σ_K is also the root system of the pair $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Let σ (resp. τ) be the conjugation of $\mathfrak{g}_{\mathbf{C}}$ with respect to the real form $\mathfrak{g}$ (resp. $\mathfrak{g}_u$). By our choice for the Weyl basis of $\mathfrak{g}_{\mathbf{C}}$ we have

$$(2.5) \qquad \sigma(X_\alpha) = -X_\alpha \quad \text{for} \quad \alpha \in \Sigma_K\,, \quad \sigma(X_\alpha) = X_{-\alpha} \quad \text{for} \quad \alpha \in \Sigma_n\,,$$

$$(2.6) \qquad \tau(X_\alpha) = -X_{-\alpha} \quad \text{for} \quad \alpha \in \Sigma\,.$$

3. Two step tensor K-module

The adjoint action $Ad(k)$ $(k \in K)$ on $\mathfrak{p}_{\mathbf{C}}$ will be denoted by kX for X in $\mathfrak{p}_{\mathbf{C}}$. We define a hermitian structure (X, Y) of $\mathfrak{p}_{\mathbf{C}}$ by $(X, Y) = -\phi(X, \tau(Y))$, $X, Y \in \mathfrak{p}_{\mathbf{C}}$. Thereby $\mathfrak{p}_{\mathbf{C}}$ is a unitary K-module. Fix $\mu \in \Gamma_K$, and consider a unitary simple K-module (π_μ, V_μ) with highest weight μ. For the simplicity of our notations we shall denote the action $\pi(k)$ $(k \in K)$ on V_μ by kv for $v \in V_\mu$. Let dk be the Haar measure on K normalized as $\int_K dk = 1$. We define a character χ_μ of the K-module (π_μ, V_μ) by

$$(3.1) \qquad \chi_\mu(k) = \deg \pi_\mu trace \pi_\mu(k)\,,$$

where $k \in K$ and $\deg \pi_\mu = \dim V_\mu$. Then we have

$$(3.2) \qquad \int_K \chi_\mu(k^{-1}k')\chi_\mu(k)dk = \chi_\mu(k')\,.$$

Let V be a finite dimensional K-module. We define a projection operator P_μ on V by

$$(3.3) \qquad P_\mu(v) = \int_K kv\overline{\chi_\mu(k)}dk \quad \text{for} \quad v \in V\,,$$

where $\overline{\chi_\mu(k)}$ is the complex conjugate of $\chi_\mu(k)$. By (3.2) we have

$$(P_\mu)^2 = P_\mu . \tag{3.4}$$

Furthermore, we have

$$kP_\mu = P_\mu k \quad \text{for all } k \in K . \tag{3.5}$$

A unitary K-module structure on the two step tensor space $\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ is defined by

$$k(X \otimes Y \otimes v) = (kX \otimes kY \otimes kv) \quad \text{for} \quad X, Y \in \mathfrak{p}_{\mathbf{C}}, \ \ v \in V_\mu \ \text{ and } \ k \in K , \tag{3.6}$$

$$(X \otimes Y \otimes v, X' \otimes Y' \otimes v') = (X, X')(Y, Y')(v, v') \tag{3.7}$$

for $X, Y, X', Y' \in \mathfrak{p}_{\mathbf{C}}$ and $v, v' \in V_\mu$. The K-module $M(\mu) = P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$ is decomposed into a finite number of the simple modules which are K-isomorphic to V_μ. Therefore

$$M(\mu) \cong m(\mu)V_\mu , \tag{3.8}$$

where $m(\mu)$ is the multiplicity of V_μ in $M(\mu)$.

LEMMA 3.1. *We put*

$$W(\mu) = \{Z \in M(\mu) : HZ = \mu(H)Z \textit{ for all } H \in \mathfrak{b}\} .$$

Then we have $m(\mu) = \dim W(\mu)$.

PROOF. Let $M(\mu) = \bigoplus_{i=1}^{m(\mu)} V_i$ be the decomposition of $M(\mu)$ by the simple K-modules V_i. Then we have

$$W(\mu) = \bigoplus_{i=1}^{m(\mu)} W(\mu) \cap V_i .$$

Since V_i is a simple K-module, we have $\dim W(\mu) \cap V_i = 1$ for all i, $1 \le i \le m(\mu)$. This implies that $\dim W(\mu) = m(\mu)$.

DEFINITION 3.2. Let p be a nonnegative integer and $\tilde{\phi}$ a symbol. We define Π_p by $\Pi_0 = \{\tilde{\phi}\}$, $\Pi_p = \{(\alpha_1, \alpha_2, \cdots, \alpha_p) : \alpha_i \in P_K\}$ for $p > 0$, and put $\Pi = \bigcup_{p=0}^{\infty} \Pi_p$. Then Π is a semigroup by the $\star$-operation with the identity $\tilde{\phi}$, where $\star$ is defined by

$$I \star J = (\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q), \quad I = (\alpha_1, \cdots, \alpha_p), \quad J = (\beta_1, \cdots, \beta_q) \in \Pi .$$

DEFINITION 3.3. Let $U(\mathfrak{k}_{\mathbf{C}})$ be the universal enveloping algebra of $\mathfrak{k}_{\mathbf{C}}$. We define a semigroup homomorphism of Π to $U(\mathfrak{k}_{\mathbf{C}})$ by

$$Q(\tilde{\phi}) = 1 \quad \text{and} \quad Q(I) = X_{-\alpha_1} X_{-\alpha_2} \cdots X_{-\alpha_p} \quad \text{for } I = (\alpha_1, \alpha_2, \cdots, \alpha_p) .$$

DEFINITION 3.4. Let $I = (\alpha_1, \alpha_2, \cdots, \alpha_p) \in \Pi$ and $J \in \Pi$. When J is of the form $J = (\alpha_{i_1}, \alpha_{i_2}, \cdots, \alpha_{i_q})$, $1 \le i_1 < i_2 < \cdots < i_q \le p$ or $J = \tilde{\phi}$ we denote by $J \preccurlyeq I$. We also define $I\backslash J \in \Pi$ by

$$I\backslash J = (\alpha_{j_1}, \alpha_{j_2}, \cdots, \alpha_{j_{p-q}}), \quad \text{where } \{j_1, j_2, \cdots, j_{p-q}\} = \{1, 2, \cdots, p\}\backslash\{i_1, \cdots, i_q\}$$

satisfying $j_1 < j_2 < \cdots < j_{p-q}$.

We note that $I\backslash(I\backslash J) = J$ and $I\backslash J \preccurlyeq I$.

Let ψ be the mapping of Π defined by $\psi(I) = (\alpha_p, \alpha_{p-1}, \cdots, \alpha_1)$, $I = (\alpha_1, \alpha_2, \cdots, \alpha_p)$ $\in \Pi$. Since ψ^2 is the identity on Π, ψ is a bijection. Let $J \in \Pi$ and $\alpha \in P_K$. Then we have

$$Q(\psi(J))X_{-\alpha} = Q(\psi(\alpha \star J)). \tag{3.9}$$

For $I = (\alpha_1, \alpha_2, \cdots, \alpha_p)$, we put $\sharp I = p$ and $\langle I \rangle = \sum_{i=1}^{p} \alpha_i$.

LEMMA 3.5. *Let $\gamma, \delta \in \Sigma_n$ and $I \in \Pi$. Assume that $\gamma + \delta = \langle I \rangle$. Then we have*

$$P_\mu(X_\gamma \otimes X_\delta \otimes Q(I)v(\mu)) = \sum_{J \preccurlyeq I, J \in \Pi} (-1)^{\sharp I} P_\mu(Q(\psi(J))X_\gamma \otimes Q(\psi(I\backslash J))X_\delta \otimes v(\mu)),$$

where $v(\mu)$ is the highest weight vector of V_μ normalized as $|v(\mu)| = 1$.

Proof by an induction on $\sharp I$. When $\sharp I = 0$, our assertion is obvious. Assume that the identity is true for all L in Π and $\xi, \eta \in \Sigma_n$ satisfying $\sharp L < \sharp I$ and $\xi + \eta = \langle L \rangle$. We have $\alpha \star L = I$ for $\alpha \in P_K$ and $L \in \Pi$. Bearing in mind $-\langle L \rangle + \gamma + \delta + \mu > \mu$ and μ is the highest weight of $M(\mu)$ we have $P_\mu(X_\gamma \otimes X_\delta \otimes Q(L)v(\mu)) = 0$. Since $Q(I) = X_{-\alpha}Q(L)$, we have

$$\begin{aligned} P_\mu(X_\gamma \otimes X_\delta \otimes Q(I)v(\mu)) &= P_\mu(X_\gamma \otimes X_{-\alpha}(X_\delta \otimes Q(L)v(\mu))) \\ &\quad - P_\mu(X_\gamma \otimes ad(X_{-\alpha})X_\delta \otimes Q(L)v(\mu)) \\ &= -P_\mu(ad(X_{-\alpha})X_\gamma \otimes X_\delta \otimes Q(L)v(\mu)) \\ &\quad - P_\mu(X_\gamma \otimes ad(X_{-\alpha})X_\delta \otimes Q(L)v(\mu)). \end{aligned}$$

Applying the inductive hypothesis to $L \in \Pi$ and $\gamma, \delta - \alpha$ (resp. $\gamma - \alpha, \delta$) we have

$$\begin{aligned} &P_\mu(X_\gamma \otimes X_\delta \otimes Q(I)v(\mu)) \\ &= (-1)^{\sharp I} \sum_{J \preccurlyeq L} \{P_\mu(Q(\psi(\alpha \star J))X_\gamma \otimes Q(\psi(L\backslash J))X_\delta \otimes v(\mu)) \\ &\quad + P_\mu(Q(\psi(J))X_\gamma \otimes Q(\psi(\alpha \star (L\backslash J)))X_\delta \otimes v(\mu))\}. \end{aligned} \tag{3.10}$$

Here we used (3.9). Since $\alpha \star (L\backslash J) = I\backslash J$, $L\backslash J = I\backslash \alpha \star J$ for $J \preccurlyeq L$ and

$$\{J : J \preccurlyeq I, J \in \Pi\} = \{J : J \preccurlyeq L, J \in \Pi\} \cup \{\alpha \star J : J \preccurlyeq L, J \in \Pi\},$$

(3.10) implies the identity of this lemma.

LEMMA 3.6. *Let S be the set of all vectors $P_\mu(X_{-\gamma} \otimes X_\gamma \otimes v(\mu))$, $\gamma \in \Sigma_n$. Then we have $W(\mu) = [S]$, where $[S]$ is the linear span of the set S.*

PROOF. Since V_μ is a simple K-module, V_μ is generated by the set $\{Q(I)v(\mu) : I \in \Pi\}$. By (3.5) we have $HP_\mu = P_\mu H$, $H \in \mathfrak{b}$. This implies that $W(\mu)$ is generated by the set

$$S' \equiv \{P_\mu(X_\gamma \otimes X_\delta \otimes Q(I)v(\mu))) : \gamma, \delta \in \Sigma_n, I \in \Pi, \gamma + \delta = \langle I \rangle\}.$$

Let us prove that $S' \subset [S]$. Let $Z = P_\mu(X_\gamma \otimes X_\delta \otimes Q(I)v(\mu))$ be each element in S'. By Lemma 3.5 we have

$$\begin{aligned} Z &= (-1)^{\sharp I} \sum_{J \preccurlyeq I} P_\mu(Q(\psi(J))X_\gamma \otimes Q(\psi(I\backslash J))X_\delta \otimes v(\mu)) \\ &= (-1)^{\sharp I} \sum_{J \preccurlyeq I} c_{\gamma,J} c_{\delta,I\backslash J} P_\mu(X_{\gamma-\langle J\rangle} \otimes X_{\delta-\langle I\backslash J\rangle} \otimes v(\mu)), \end{aligned}$$

where $c_{\gamma,J} = \phi(Q(\psi(J))X_\gamma, X_{-\gamma+\langle J\rangle})$. Since $\langle J\rangle + \langle I\backslash J\rangle = \langle I\rangle$ and $\gamma+\delta = \langle I\rangle$, we have $S' \subset [S]$. Moreover since $W(\mu) = [S'] \subset [S] \subset W(\mu)$, we have $W(\mu) = [S]$.

4. Weight subspace $W(\mu)$ of $M(\mu)$

First we restate the following three lemmas in [3].

LEMMA 4.1. *Let (π_μ, V_μ) be a simple K-module with highest weight μ. Then we have*

$$\mathfrak{p}_{\mathbf{C}} \otimes V_\mu = \bigoplus_{\omega\in\Sigma_n, \mu+\omega\in\Gamma_K} P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu),$$

where $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) = \{0\}$ or is a simple K-module.

For a proof cf. Lemma 3.4 in [3].

The following two lemmas are also proved respectively by Corollary 3.5 and Lemma 3.6 in [3].

LEMMA 4.2. *Let ω be a noncompact root in Σ. Assume that $\mu \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then we have $|P_{\mu+\omega}(X_\omega \otimes v(\mu))| \neq 0$, where $v(\mu)$ is the highest weight vector in V_μ.*

LEMMA 4.3. *Let $\mu \in \Gamma_K$, $\omega \in \Sigma_n$, and assume that $\mu+\omega \in \Gamma_K$, $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then, for each $\gamma \in \Sigma_n$, we have*

$$(|\lambda+\omega|^2 - |\lambda+\gamma|^2)|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \sum_{\alpha\in P_K} 2|\langle\alpha,\gamma\rangle|^2 |P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu))|^2,$$

where $\lambda = \mu + \rho_K$ and ρ_K is one half the sum of all roots in P_K.

LEMMA 4.4. *Let $\mu \in \Gamma_K$ and $\gamma, \omega \in \Sigma_n$. Assume that $\mu+\omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then we have*

$$\begin{aligned} &(P_\mu(X_{-\gamma} \otimes P_{\mu+\omega}(X_\gamma \otimes v(\mu))), P_\mu(X_{-\omega} \otimes P_{\mu+\omega}(X_\omega \otimes v(\mu)))) \\ &\quad = c(\mu;\omega)^2 |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2, \end{aligned}$$

where $c(\mu;\omega)=\sqrt{\frac{\deg\pi_\mu}{\deg\pi_{\mu+\omega}}}$.

PROOF. We note that $(kX_{-\omega}, X_{-\omega})=\overline{(kX_\omega, X_\omega)}$. By (3.6) and (3.7) we have

$$(4.1)\qquad (P_\mu(X\otimes Y\otimes v), P_\mu(X'\otimes Y'\otimes v'))=\int_K (kX, X')(kY, Y')(kv, v')\overline{\chi_\mu(k)}dk\,.$$

Let $\{v_i\}\,(1\le i\le \deg\pi_\mu, v_1=v(\mu))$ be an orthonormal basis of V_μ. Since $\chi_\mu(k)=\deg\pi_\mu\sum_i(kv_i, v_i)$, we have

$$\begin{aligned}
&(P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu)))\,,\,P_\mu(X_{-\omega}\otimes P_{\mu+\omega}(X_\omega\otimes v(\mu))))\\
&\quad=\deg\pi_\mu\sum_i\int_K\overline{(k(X_\gamma\otimes v_i)\,,\,X_\omega\otimes v_i)}(kP_{\mu+\omega}(X_\gamma\otimes v(\mu)), P_{\mu+\omega}(X_\omega\otimes v(\mu)))dk\\
&\quad=\deg\pi_\mu\sum_i\int_K\overline{(kP_{\mu+\omega}(X_\gamma\otimes v_i)\,,\,P_{\mu+\omega}(X_\omega\otimes v_i))}\\
&\qquad\times(kP_{\mu+\omega}(X_\gamma\otimes v(\mu)), P_{\mu+\omega}(X_\omega\otimes v(\mu)))dk\\
&\quad=\deg\pi_\mu(\deg\pi_{\mu+\omega})^{-1}\sum_i(P_{\mu+\omega}(X_\gamma\otimes v(\mu))\,,\,P_{\mu+\omega}(X_\gamma\otimes v_i))\\
&\qquad\times\overline{(P_{\mu+\omega}(X_\omega\otimes v(\mu)), P_{\mu+\omega}(X_\omega\otimes v_i))}\\
&\quad=c(\mu;\omega)^2|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\,.
\end{aligned}$$

Here we used the orthogonality relation on K. Hence the lemma follows.

COROLLARY 4.5. *Assume that* $\mu, \mu+\omega\in\Gamma_K$ *and* $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\ne\{0\}$. *Then* $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))$ *is a simple* K*-module with highest weight* μ. *Let* $v_\omega(\mu)$ *be the highest weight vector of the simple* K*-module* $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))$ *determined by*

$$P_\mu(X_{-\omega}\otimes P_{\mu+\omega}(X_\omega\otimes v(\mu)))=c(\mu;\omega)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2v_\omega(\mu)\,.$$

Then we have $|v_\omega(\mu)|=1$ *and*

$$P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu)))=c(\mu;\omega)|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2v_\omega(\mu)\quad \textit{for all}\ \ \gamma\in\Sigma_n\,,$$

where $c(\mu;\omega)=\sqrt{\frac{\deg\pi_\mu}{\deg\pi_{\mu+\omega}}}$.

PROOF. By Lemma 4.2 and Lemma 4.4, we have

$$|P_\mu(X_{-\omega}\otimes P_{\mu+\omega}(X_\omega\otimes v_\mu))|=c(\mu;\omega)|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2\ne 0\,.$$

Therefore $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))\ne\{0\}$ and $|v_\omega(\mu)|=1$. Replacing V_μ with the simple K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)$ in Lemma 4.1, we have $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))$ is simple. We put

$$P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu)))=c(\gamma)c(\mu;\omega)|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2v_\omega(\mu)\,,$$

where $c(\gamma)$ is a complex number. By Lemma 4.4 we have

$$\begin{aligned} c(\gamma)c(\mu;\omega)^2|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2 \\ = (P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu))), P_\mu(X_{-\omega}\otimes P_{\mu+\omega}(X_\omega\otimes v(\mu)))) \\ = c(\mu;\omega)^2|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2 . \end{aligned}$$

This implies that $c(\gamma)=1$, and hence we have the formula.

THEOREM 4.6. *Let $\mu\in\Gamma_K$ and $W(\mu)$ the weight subspace of the K-module $M(\mu)$. Then we have*

$$\dim W(\mu) = \sharp\Sigma_{W(\mu)} , \tag{4.2}$$

where $\Sigma_{W(\mu)} = \{\omega\in\Sigma_n : \mu+\omega\in\Gamma_K,\ P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)\neq\{0\}\}$.

PROOF. We put $A=\{P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu))) : \gamma,\omega\in\Sigma_n,\ \mu+\omega\in\Gamma_K\}$. First we shall prove that $W(\mu)=[A]$. Let $Z=P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu)))$ be an element in A. Since the action of K commutes with P_μ and $P_{\mu+\omega}$ (see (3.5)), we have $HZ=\mu(H)Z$ for all H in $\mathfrak{b}_{\mathbf{C}}$. This implies that $A\subset W(\mu)$. Conversely let Z be an element in $W(\mu)$. By Lemma 3.6 we have

$$Z=\sum_{\gamma\in\Sigma_n} c_\gamma P_\mu(X_{-\gamma}\otimes X_\gamma\otimes v(\mu)) ,$$

where c_γ is a complex constant. Then by Lemma 4.1 we have

$$Z=\sum_{\gamma\in\Sigma_n}\ \sum_{\omega\in\Sigma_n,\mu+\omega\in\Gamma_K} c_\gamma P_\mu(X_{-\gamma}\otimes P_{\mu+\omega}(X_\gamma\otimes v(\mu))) .$$

Thus $W(\mu)=[A]$ as claimed. Let us now prove this theorem. By Corollary 4.5 we have

$$W(\mu)=[A]=[\{|P_{\mu+\omega}(X_\omega\otimes v(\mu))|^2 v_\omega(\mu) : \omega\in\Sigma_n,\ \mu+\omega\in\Gamma_K\}] . \tag{4.3}$$

Let $\omega,\gamma\in\Sigma_n,\omega\neq\gamma$. Assume that $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)$ and $P_{\mu+\gamma}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu)$ are nontrivial. Since these spaces are orthogonal, $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))$ and $P_\mu(\mathfrak{p}_{\mathbf{C}}\otimes P_{\mu+\gamma}(\mathfrak{p}_{\mathbf{C}}\otimes V_\mu))$ are also orthogonal (see (4.1)). Hence (4.3) and Lemma 4.2 imply (4.2).

In view the proof of the above theorem we have the following.

COROLLARY 4.7. *Let $\omega,\gamma\in\Sigma_n,\omega\neq\gamma$. Consider two highest weight vectors $v_\omega(\mu)$ and $v_\gamma(\mu)$ as in Corollary 4.5. Then $v_\omega(\mu)$ and $v_\gamma(\mu)$ are orthogonal. Moreover, we have*

$$P_\mu(X_{-\gamma}\otimes X_\gamma\otimes v(\mu)) = \sum_{\omega\in\Sigma_{W(\mu)}} c(\mu;\omega)|P_{\mu+\omega}(X_\gamma\otimes v(\mu))|^2 v_\omega(\mu) .$$

5. Admissible dominant integral form

In this section we shall determine the multiplicity $m(\mu)$ of V_μ in the K-module $M(\mu)$ for an admissible integral form μ in Γ_K (for the definition, see below). Let $\mathfrak{z}(H_\mu)$ be the centralizer of H_μ in $\mathfrak{g}_{\mathbf{C}}$. Since one dimensional algebra $\mathbf{C}H_\mu$ is σ and τ invariant, $\mathfrak{z}(H_\mu)$ is also invariant under these anti-automorphisms of $\mathfrak{g}_{\mathbf{C}}$. We now put $\mathfrak{l}(\mu) = \mathfrak{z}(H_\mu) \cap \mathfrak{g}$. Since $\theta = \sigma\tau$, $\mathfrak{l}(\mu)$ is a θ-stable reductive algebra with Cartan subalgebra $\mathfrak{b}$. Therefore $\mathfrak{l}(\mu)$ has the following Cartan decomposition.

$$\mathfrak{l}(\mu) = \mathfrak{k}(\mu) \oplus \mathfrak{p}(\mu), \quad \text{where} \quad \mathfrak{k}(\mu) = \mathfrak{k} \cap \mathfrak{l}(\mu) \quad \text{and} \quad \mathfrak{p}(\mu) = \mathfrak{p} \cap \mathfrak{l}(\mu). \tag{5.1}$$

Let $L(\mu)$ be the centralizer of H_μ in G. We put $K(\mu) = K \cap L(\mu)$. Then $K(\mu)$ is a maximal compact subgroup of $L(\mu)$. Furthermore, since $H_\mu \in \mathfrak{b}_{\mathbf{C}}$, B is a Cartan subgroup of $K(\mu)$ (resp. $L(\mu)$).

DEFINITION 5.1. Let $\mu \in \Gamma_K$ and $K(\mu)$ the centralizer of H_μ in K. For the root system $\Sigma_{K(\mu)}$ of the pair $(\mathfrak{k}(\mu)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ we put $P_{K(\mu)} = P_K \cap \Sigma_{K(\mu)}$.

DEFINITION 5.2. An element $\mu \in \Gamma_K$ is admissible if μ has the following properties.
For $Sp(n, \mathbf{R}) and SO(2m, 2n+1)$, $(\mu, \alpha) \geq 2$ for all short roots $\alpha \in P_K \backslash P_{K(\mu)}$.
For the the type of G_2, $2(\mu, \alpha)|\alpha|^{-2} \geq 3$ for all short roots $\alpha \in P_K \backslash P_{K(\mu)}$.
If G satisfies that all noncompact roots have the same length, then μ is always admissible.

REMARK. The inner type noncompact real simple Lie groups are classified by $Sp(n, \mathbf{R})$, $SO(2m, 2n+1)$, the type G_2 and the groups which satisfy all noncompact roots have the same length (cf. Table II, p. 354 in [2]). When G is of the type G_2 then P_K has exactly one simple short (resp. long) root.

DEFINITION 5.3. A noncompact root ω in Σ is $P_{K(\mu)}$-highest if $\omega + \alpha \notin \Sigma$ for all $\alpha \in P_{K(\mu)}$.

Let ω be a noncompact root and m a nonnegative integer. We define five sets $\Delta(\omega)$, $\Delta_\pm(\omega)$, $\Delta_m(\omega)$ and $\Delta_m(\omega)^*$ by

$$\begin{aligned} &\Delta(\omega) = \{\alpha \in P_K : \omega + \alpha \in \Sigma\}, \\ &\Delta_\pm(\omega) = \{\alpha \in P_K : \pm(\alpha, \omega) > 0\}, \\ &\Delta_m(\omega) = \{\alpha \in \Delta(\omega) : 2(\omega, \alpha)|\alpha|^{-2} = m\}, \\ &\Delta_m(\omega)^* = \{\alpha \in \Delta_m(\omega) : \omega - \alpha \in \Sigma\}. \end{aligned} \tag{5.2}$$

We have the following lemma (see Lemma 6.1 in [3]).

LEMMA 5.4. *Let G be an inner type noncompact real simple Lie group and ω a noncompact root. Then we have the followings.*
(1) $\Delta(\omega) = \Delta_-(\omega) \cup \Delta_0(\omega) \cup \Delta_1(\omega)$, $\Delta_0(\omega) = \Delta_0(\omega)^*$ *and* $\Delta_1(\omega) = \Delta_1(\omega)^*$.
(2) *If* $\Delta_0(\omega) \neq \phi$, *then* G *is either* $Sp(n, \mathbf{R})$ *or* $SO(2m, 2n+1)$, *and* $\Delta(\omega) = \Delta_0(\omega)^* \cup \Delta_{-1}(\omega)$.
(3) *If* $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* \neq \phi$, *then G is of the type* G_2.

Let $\mu \in \Gamma_K$ and $\omega \in \Sigma_n$. Assume that $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Then there exists a rational function $f(\eta;\omega)$ in $\eta \in (\sqrt{-1}\mathfrak{b})^*$ (cf. Theorem 5.5 in [3]) such that

$$|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = f(\lambda+\omega;\omega)\,, \tag{5.3}$$

where $\lambda = \mu + \rho_K$. The function $f(\eta;\omega)$ has the following product formula (cf. Theorem 6.5 in [3]).

THEOREM 5.5. *Let ω be a noncompact root in Σ. Then $f(\eta+\omega;\omega)$ is given by the followings.*

(1) *If $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, then we have*

$$f(\eta+\omega;\omega) = \prod_{\alpha\in\Delta_-(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1}\,.$$

(2) *If $\Delta_0(\omega)^* \neq \phi$, then G is either $Sp(n,\mathbf{R})$ or $SO(2m,2n+1)$ and*

$$f(\eta+\omega;\omega) = \prod_{\alpha\in\Delta_{-1}(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1} \times \prod_{\alpha\in\Delta_0(\omega)^*} (2(\eta,\alpha)-|\alpha|^2)(2(\eta,\alpha)+|\alpha|^2)^{-1}\,.$$

(3) *If $\Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^* \neq \phi$, then G is of the type G_2 and*

$$f(\eta+\omega;\omega) = \prod_{\alpha\in\Delta_-(\omega)} (\eta+\omega,\alpha)(\eta,\alpha)^{-1} \times \prod_{\alpha\in\Delta_1(\omega)^*} (2(\eta,\alpha)-|\alpha|^2)(2((\eta,\alpha)+|\alpha|^2))^{-1} \times \prod_{\alpha\in\Delta_{-1}(\omega)^*} 2((\eta,\alpha)-|\alpha|^2)(2(\eta,\alpha)+|\alpha|^2)^{-1}\,.$$

We also restate the following theorem (see Theorem 7.6 in [3]).

THEOREM 5.6. *Let $\mu \in \Gamma_K$ and $\omega \in \Sigma_n$. Assume that $\mu+\omega \in \Gamma_K$. Then the K-module $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ if and only if $f(\lambda+\omega;\omega) > 0$.*

LEMMA 5.7. *Let $\mu \in \Gamma_K$ and $\omega \in \Sigma_n$. Assume that $\mu+\omega \in \Gamma_K$ and $\Delta_0(\omega)^* \cap P_{K(\mu)} \neq \{\phi\}$. Then there exists a simple root $\alpha \in P_K$ such that $\alpha \in \Delta_0(\omega)^* \cap P_{K(\mu)}$.*

PROOF. Let α be the lowest root in $\Delta_0(\omega)^* \cap P_{K(\mu)}$. Assume that α is not simple in P_K. Then we can choose $\beta, \gamma \in P_K$ satisfying $\alpha = \beta + \gamma$. From $(\mu,\alpha) = 0$ and $\mu \in \Gamma_K$, it follows that $(\mu,\beta) = (\mu,\gamma) = 0$. Moreover, since $(\omega,\alpha) = 0$, we have either $(\omega,\beta) = (\omega,\gamma) = 0$ or $(\omega,\beta)(\omega,\gamma) < 0$. Consider the first case. Since $[X_\omega,[X_\beta,X_\gamma]] \neq 0$, Jacobi's identity implies $\omega+\beta \in \Sigma$ or $\omega+\gamma \in \Sigma$. There is no loss of generality assuming that $\omega+\beta \in \Sigma$. Since $\beta \in \Delta_0(\omega)^* \cap P_{K(\mu)}$ and $\alpha > \beta$, we have a contradiction to the choice of α. For the latter case we can assume $(\omega,\beta) < 0$. Therefore $(\mu+\omega,\beta) < 0$, $\beta \in P_K$. This is a contradiction to the assumption $\mu+\omega \in \Gamma_K$. Thus α is simple in P_K.

LEMMA 5.8. *Let $\mu \in \Gamma_K$ and $\omega \in \Sigma_n$. Assume that μ is admissible. Then we have that $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ if and only if ω is $P_{K(\mu)}$-highest.*

PROOF. Bearing in mind ω is $P_{K(\mu)}$-highest iff $\Delta(\omega) \cap P_{K(\mu)} = \phi$, it is sufficient to prove that $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$ iff $\Delta(\omega) \cap P_{K(\mu)} = \phi$ (see Theorem 5.6). First we assume that $\Delta(\omega) \cap P_{K(\mu)} = \phi$. We note that $(\mu, \alpha) > 0$ for $\alpha \in \Delta(\omega)$. Let us prove that $\mu+\omega \in \Gamma_K$ and $f(\lambda+\omega; \omega) > 0$. If $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* = \phi$, then by (1) in Lemma 5.4 we have $\Delta(\omega) = \Delta_-(\omega)$. By (2.3) we have $\Delta(\omega) = \Delta_{-1}(\omega) \cup \Delta_{-2}(\omega) \cup \Delta_{-3}(\omega)$. Let α be an element in $\Delta_{-1}(\omega)$. Since $(\mu, \alpha) > 0$, we have $2(\lambda + \omega, \alpha)|\alpha|^{-2} > 0$. If $\alpha \in \Delta_{-2}(\omega)$, then $\alpha \in \Delta_0(\omega+\alpha)^*$. By (2) in Lemma 5.4 we have G is one of $Sp(n, \mathbf{R})$ and $SO(2m, 2n+1)$. Since α is a short root, the admissibility of μ implies $2(\lambda + \omega, \alpha)|\alpha|^{-2} > 0$. If $\alpha \in \Delta_{-3}(\omega)$, then $\alpha \in \Delta_{-1}(\omega + \alpha)^*$. By (3) in Lemma 5.4 G is of the type G_2, and α is a short root. By the admissibility of μ we have also $(\lambda + \omega, \alpha) > 0$. Thus $(\lambda + \omega, \alpha) > 0$ for all $\alpha \in P_K$, and especially $\mu+\omega \in \Gamma_K$. Moreover, by (1) in Theorem 5.5 we have $f(\lambda+\omega; \omega) > 0$. Consider the case $\Delta_0(\omega)^* \neq \phi$. By (2) in Lemma 5.4 we have $\Delta(\omega) = \Delta_0(\omega)^* \cup \Delta_{-1}(\omega)$. By using the same arguments as above we can prove that $\mu + \omega \in \Gamma_K$ and $(\lambda + \omega, \alpha) > 0$ for $\alpha \in P_K$. Moreover, since $(\mu, \alpha) > 0$ for $\alpha \in \Delta_0(\omega)^*$, we have $2(\lambda, \alpha)|\alpha|^{-2} > 1$. Hence by (2) in Theorem 5.5 we have $f(\lambda + \omega; \omega) > 0$ for this case. Assume that $\Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^* \neq \phi$. Then G is of the type G_2. From Lemma 5.4 and (2.3) it follows that $\Delta(\omega) = \Delta_{-3}(\omega) \cup \Delta_{-1}(\omega) \cup \Delta_1(\omega)^*$. For $\alpha \in \Delta_{-3}(\omega)$ the admissibility of μ implies $(\mu + \omega, \alpha) \geq 0$. If $\alpha \in \Delta_{-1}(\omega)$, then by $(\mu, \alpha) > 0$ we have $(\mu + \omega, \alpha) \geq 0$. Let $\alpha \in \Delta_1(\omega)^* \cup \Delta_{-1}(\omega)^*$. Since α is a short root, the admissibility implies $2(\mu, \alpha)|\alpha|^2 \geq 3$. Therefore $\mu + \omega \in \Gamma_K$ and $2(\lambda, \alpha)|\alpha|^{-2} > 1$ (resp. $(\lambda, \alpha)|\alpha|^{-2} > 1$) for $\alpha \in \Delta_1(\omega)^*$ (resp. $\alpha \in \Delta_{-1}(\omega)^*$). By (3) in Theorem 5.5 we have $f(\lambda + \omega; \omega) > 0$. Conversely assume that $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$. Since $\Delta(\omega) = \Delta_-(\omega)$, for the case $\Delta_0(\omega)^* \cup \Delta_{-1}(\omega)^* \cup \Delta_1(\omega) = \phi$, the assumption $\mu+\omega \in \Gamma_K$ implies that $\Delta(\omega) \cap P_{K(\mu)} = \phi$. Suppose that $\Delta_0(\omega)^* \neq \phi$. Then we have $(\mu, \alpha) > 0$ for $\alpha \in \Delta_{-1}(\omega)$. Let $\alpha \in \Delta_0(\omega)^*$. We shall prove that $(\mu, \alpha) > 0$. Suppose that $(\mu, \alpha) = 0$. Since $\Delta_0(\omega)^* \cap P_{K(\mu)} \neq \phi$, Lemma 5.7 implies that there is a simple root β in P_K such that $\beta \in \Delta_0(\omega)^* \cap P_{K(\mu)}$. We have $2(\lambda, \beta)|\beta|^{-2} = 1$, $\beta \in \Delta_0(\omega)^*$, and hence by (2) in Theorem 5.5 we have $f(\lambda + \omega, \omega) = 0$. This is a contradiction to the assumption $f(\lambda + \omega; \omega) > 0$. Thus $(\mu, \alpha) > 0$ for $\alpha \in \Delta(\omega)$, and hence $\Delta(\omega) \cap P_{K(\mu)} = \phi$ for the case $\Delta_0(\omega)^* \neq \phi$. Finally assume that $\alpha \in \Delta_{-1}(\omega)^* \cup \Delta_1(\omega)^*$. We note that α is a simple short root in P_K. Since $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$, (3) in Theorem 5.5 implies $(\mu, \alpha) > 0$. Thus we can prove that if $\mu + \omega \in \Gamma_K$ and $f(\lambda + \omega; \omega) > 0$, then $\Delta(\omega) \cap P_{K(\mu)} = \phi$.

THEOREM 5.9. *Let $\mu \in \Gamma_K$ and V_μ a simple K-module with the highest weight μ. Consider the K-module $M(\mu) = P_\mu(\mathfrak{p}_{\mathbf{C}} \otimes \mathfrak{p}_{\mathbf{C}} \otimes V_\mu)$, and assume that μ is admissible. Then the multiplicity $m(\mu)$ of V_μ in $M(\mu)$ is given by*

$$m(\mu) = \sharp\{\omega \in \Sigma_n : \omega \text{ is } P_{K(\mu)}\text{-highest}\} .$$

PROOF. Let $\omega \in \Sigma_n$. Then by Lemma 5.8 $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ if and only if ω is $P_{K(\mu)}$-highest. Consequently by Theorem 4.6 we have our assertion.

6. Positive root system associated with a P_K-dominant integral form

In this section we shall give a good positive root system associated with $\mu \in \Gamma_K$ (see Lemma 6.5 below). An element H in $\sqrt{-1}\mathfrak{b}$ is said to be regular if $\alpha(H) \neq 0$ for all α in Σ. An element H in $\sqrt{-1}\mathfrak{b}$ is said to be singular unless H is regular. Let $(\sqrt{-1}\mathfrak{b})'$ denote the set of all regular elements in $\sqrt{-1}\mathfrak{b}$ and P a positive root system satisfying $P_K \subset P$. We define a subset C in $(\sqrt{-1}\mathfrak{b})'$ by

$$C = \{H \in \sqrt{-1}\mathfrak{b} : \alpha(H) > 0 \text{ for all } \alpha \in P\}.$$

Each topological connected component of $(\sqrt{-1}\mathfrak{b})'$ is said to be a Weyl chamber. Especially C is the positive Weyl chamber corresponding to P. Let W be the Weyl group of the pair $(\mathfrak{g}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. W acts simply transitively on the set of all Weyl chambers (cf. Theorem 4.3.18 in [4]). Moreover we have

$$(\sqrt{-1}\mathfrak{b})' = \bigcup_{s \in W} sC \text{ (disjoint union)}.$$

Let s be an element in W. Then sC is the positive Weyl chamber corresponding to the positive root system sP.

LEMMA 6.1. *The number of positive root systems containing P_K is $(W : W_K)$, where $(* : *)$ is the group index and W_K is the Weyl group of $(\mathfrak{k}_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$.*

PROOF. We denote the set of all positive root systems containing P_K by $\{s_i P : 1 \leq i \leq p, s_i \in W, s_1 = 1\}$. It is enough to prove that

$$W = \bigcup_{i=1}^{p} W_K s_i \text{ (disjoint union)}. \tag{6.1}$$

Let C_K be the positive Weyl chamber corresponding to P_K. First we shall prove $W = \bigcup_{i=1}^{p} W_K s_i$. Let s be an element in W. Since $sC \subset \bigcup_{t \in W_K} tC_K$, there is t in W_K such that $tC_K \cap sC \neq \phi$. We can choose $H \in C$ satisfying $t^{-1}sH \in C_K$. Since $\alpha(t^{-1}sH) > 0$ for all $\alpha \in P_K$, we have $P_K \subset t^{-1}sP$. We let $t^{-1}s = s_i$ for i, $1 \leq i \leq p$. Then $s \in W_K s_i$, and hence the identity in (6.1) follows. Next we shall prove that if $W_K s_i \cap W_K s_j \neq \phi$, then $i = j$. There is $t \in W_K$ such that $ts_i = s_j$. If $t \neq 1$, then we have $t\alpha < 0$ for $\alpha \in P_K$. Since $\alpha \in s_i P$, we have $\alpha = s_i \beta$ for $\beta \in P$. This implies that $ts_i\beta \in s_j P \cap (-P_K)$. Since $s_j P$ is a positive root system and $P_K \subset s_j P$, we have a contradiction. Thus $t = 1$ and $i = j$.

LEMMA 6.2. *Assume that the pair (G, K) is hermitian symmetric. Then for the positive root system P_K of Σ_K we can choose a positive root system P' satisfying the following properties:*

$$P_K \subset P', \quad \mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^- \text{ and } ad(\mathfrak{k}_{\mathbf{C}})\mathfrak{p}^{\pm} \subset \mathfrak{p}^{\pm},$$

where $\mathfrak{p}^{\pm}$ *is the subspace of* $\mathfrak{p}_{\mathbf{C}}$ *generated by the set of all root vectors corresponding to the noncompact roots in* P' (*resp.* $-P'$).

PROOF. Let H_0 be a nonzero element in the center of $\mathfrak{k}_{\mathbf{C}}$. We note that $\gamma(H_0) \neq 0$ for all $\gamma \in \Sigma_n$ (cf. Corollary 7.3 in [2]). We can assume that $H_0 \in \sqrt{-1}\mathfrak{b}$. Let $\mathfrak{b}_1$ be the orthogonal complement of H_0 in $\mathfrak{b}$. Then $\mathfrak{b}_1$ is a Cartan subalgebra of the semisimple Lie algebra $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$. Let K_1 be the analytic subgroup of G corresponding to $\mathfrak{k}_1$. Then $P_{K_1} = P_K$ is a positive root system of $((\mathfrak{k}_1)_{\mathbf{C}}, (\mathfrak{b}_1)_{\mathbf{C}})$. Let C_{K_1} be the positive Weyl chamber in $\sqrt{-1}\mathfrak{b}_1$ corresponding to P_{K_1}. We choose $H \in C_{K_1}$, and put $H_n = \frac{1}{n}H + H_0$ for all positive integers n. Since $\lim_{n\to+\infty} \gamma(H_n) = \gamma(H_0)$, there exists a sufficiently large number N such that $\gamma(H_N)$ and $\gamma(H_0)$ have the same signature for all $\gamma \in \Sigma_n$. We put

$$P' = \{\alpha \in \Sigma : \alpha(H_N) > 0\}.$$

Since H_N is regular, P' is a positive root system of Σ containing P_K. Then $\mathfrak{p}^{\pm}$ are $\mathfrak{k}_{\mathbf{C}}$-invariant and $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$.

LEMMA 6.3. *Let* $\mu \in \Gamma_K$ *and* $\mathfrak{l}(\mu)$ *the centralizer of* H_μ *in* $\mathfrak{g}$. *Then the inner type reductive Lie algebra* $\mathfrak{l}(\mu)$ *has the following decomposition by the ideals.*

$$\mathfrak{l}(\mu) = \mathfrak{l}_0 \oplus \mathfrak{l}_1 \oplus \mathfrak{l}_2,$$

where all $\mathfrak{l}_i$*'s are inner and* θ*-invariant*, $\mathfrak{l}_0 \subset \mathfrak{k}$, *and each simple ideal of* $\mathfrak{l}_1$ (*resp.* $\mathfrak{l}_2$) *is noncompact nonhermitian* (*resp. hermitian*).

PROOF. Let $\mathfrak{l}(\mu) = \bigoplus_{i=0}^{p} \mathfrak{q}_i$ be the decompostion by ideals of $\mathfrak{l}(\mu)$, where $\mathfrak{q}_0$ is the center of $\mathfrak{l}(\mu)$ and the other $\mathfrak{q}_i'$s are all simple. Since $\mathfrak{q}_0 \subset \mathfrak{b}$, it is enough to prove that $\mathfrak{q}_i$ $(1 \leq i \leq p)$ is an inner type θ-invariant simple Lie algebra. Let p_i be the projection of $\mathfrak{l}(\mu)$ to $\mathfrak{q}_i$. Then we have $[p_i(\mathfrak{b}), p_j(\mathfrak{b})] = \{0\}$ for $i, j, i \neq j$. This implies that $\{0\} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{i=1}^{q}[p_i(\mathfrak{b}), p_i(\mathfrak{b})]$, and hence $p_i(\mathfrak{b})$ is an abelian subalgebra of $\mathfrak{q}_i$. Since $[\mathfrak{b}, p_i(\mathfrak{b})] = \{0\}$ and $\mathfrak{b}$ is a maximal abelian subalgebra of $\mathfrak{l}(\mu)$, we have $p_i(\mathfrak{b}) \subset \mathfrak{b}$ and $p_i(\mathfrak{b})$ is maximal abelian in $\mathfrak{q}_i$. Thus $\mathfrak{q}_i$ is an inner type simple Lie algebra. Moreover since $p_i(\mathfrak{b}) \subset \mathfrak{q}_i \cap \theta(\mathfrak{q}_i) \subset \mathfrak{l}(\mu)$, we have $\mathfrak{q}_i = \theta(\mathfrak{q}_i)$.

DEFINITION 6.4. Let P be a positive root system of Σ containing P_K. We put $\mathfrak{p}^+ = \bigoplus_{\alpha\in\Sigma_n\cap P} \mathfrak{g}_\alpha$ and $\mathfrak{p}^- = \bigoplus_{\alpha\in\Sigma_n\cap P} \mathfrak{g}_{-\alpha}$. Let $\mathfrak{q}$ be a simple $K(\mu)$-submodule of $\mathfrak{p}_{\mathbf{C}}$. Then $\mathfrak{q}$ is said to be the first (resp. the second) kind with respect to P if $\tau(\mathfrak{q}) = \mathfrak{q}$ (resp. $\mathfrak{q} \subset \mathfrak{p}^+$ or $\mathfrak{q} \subset \mathfrak{p}^-$).

LEMMA 6.5. *Let* $\mu \in \Gamma_K$. *Then we can choose a positive root system* $P^\star$ *of* Σ *satisfying the following properties:* $P_K \subset P^\star$, *and each* $K(\mu)$*-simple submodule* $\mathfrak{q}$ *of* $\mathfrak{p}_{\mathbf{C}}$ *is either the first kind or the second kind with respect to* $P^\star$.

PROOF. Consider the decomposition of $\mathfrak{l}(\mu)$ as in Lemma 6.3. Let Σ_i $(0 \leq i \leq 2)$ be the root system of the pair $((\mathfrak{l}_i)_{\mathbf{C}}, (\mathfrak{l}_i \cap \mathfrak{b})_{\mathbf{C}})$. Since each $\alpha \in \Sigma_i$ can be extended to $\mathfrak{b}$, we have $\Sigma_i \subset \Sigma$. Furthermore since $\mathfrak{l}_i$ is θ-invariant, we have $P_{K(\mu)} = \bigcup_{i=0}^{2}(P_{K(\mu)} \cap \Sigma_i)$ and

$P_{K(\mu)} \cap \Sigma_i$ is a positive root system of $((\mathfrak{l}_i \cap \mathfrak{k})_{\mathbf{C}}, (\mathfrak{l}_i \cap \mathfrak{b})_{\mathbf{C}})$. We put $P_0 = P_{K(\mu)} \cap \Sigma_0$. For the algebra $\mathfrak{l}_1$ we choose a positive root system P_1 of Σ_1 satisfying $P_{K(\mu)} \cap \Sigma_1 \subset P_1$. For the hermitian case $\mathfrak{l}_2$, we choose a positive root system P_2 of Σ_2 satisfying $P_{K(\mu)} \cap \Sigma_2 \subset P_2$ as in Lemma 6.2. We now put

$$P(\mu) = \bigcup_{i=0}^{2} P_i \, . \tag{6.2}$$

Then $P(\mu)$ is a positive root system of $(\mathfrak{l}(\mu)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$, and $P_{K(\mu)} \subset P(\mu)$. Let us now choose a positive root system $P^\star$ of Σ as follows. Let $\mathfrak{l}_* = [\mathfrak{l}(\mu), \mathfrak{l}(\mu)]$ be the drived algebra of $\mathfrak{l}(\mu)$. We put $\mathfrak{b}_* = \mathfrak{b} \cap \mathfrak{l}_*$. Then $\mathfrak{b}_*$ is a Cartan subalgebra of the real semisimple Lie algebra $\mathfrak{l}_*$. Let $C_*(\mu)$ be the positive Weyl chamber of $\sqrt{-1}\mathfrak{b}_*$ corresponding to $P(\mu)$. We choose an element H_0 in $C_*(\mu)$ and put $H_n = \frac{1}{n} H_0 + H_\mu$ for all positive integers n. Then for a sufficiently large number N, $\alpha(H_\mu)$ and $\alpha(H_N)$ have the same signature for all $\alpha \in \Sigma \backslash (P(\mu) \cup -P(\mu))$. We now put

$$P^\star = \{\alpha \in \Sigma : \alpha(H_N) > 0\} \, . \tag{6.3}$$

Immediately we have $P(\mu) \subset P^\star$. Moreover by the choice of H_N we have $\alpha(H_N) > 0$ for $\alpha \in P_K \backslash P_{K(\mu)}$. This implies that $P_K \subset P^\star$. Finally we shall prove that each simple $K(\mu)$-submodule $\mathfrak{q}$ of $\mathfrak{p}_{\mathbf{C}}$ is the first kind or the second kind with respect to $P^\star$. Let $\mathfrak{l}(\mu) = \mathfrak{k}(\mu) \oplus \mathfrak{p}(\mu)$ be the Cartan decomposition of $\mathfrak{l}(\mu)$ as in (5.1) and $\mathfrak{r}$ the orthogonal complement of $\mathfrak{p}(\mu)$ in $\mathfrak{p}$. Then $\mathfrak{r}$ is $K(\mu)$-invariant and $\mathfrak{p}_{\mathbf{C}} = \mathfrak{p}(\mu)_{\mathbf{C}} \oplus \mathfrak{r}_{\mathbf{C}}$. Since $\mathfrak{q}$ is a simple $K(\mu)$-module, we have

$$\mathfrak{q} \subset \mathfrak{p}(\mu)_{\mathbf{C}} \quad \text{or} \quad \mathfrak{q} \subset \mathfrak{r}_{\mathbf{C}} \, . \tag{6.4}$$

In the first case in (6.4), we have (1) $\mathfrak{q} \subset (\mathfrak{l}_1)_{\mathbf{C}}$ or (2) $\mathfrak{q} \subset (\mathfrak{l}_2)_{\mathbf{C}}$. Since each simple ideal of $\mathfrak{l}_1$ is nonhermitian, $\mathfrak{q}$ is the first kind for the case (1). For the case (2) the choice of the positive root system P_2 implies that $\mathfrak{q}$ is the second kind. Let us consider the latter case in (6.4). Let X_ω be the $K(\mu)$-highest weight vector in $\mathfrak{q}$. Since $\omega \notin P(\mu)$, we have that $\omega(H_\mu) \neq 0$. Since each weight (noncompact root) δ of $\mathfrak{q}$ is of the form $\delta = \omega - \sum_{\alpha \in P_{K(\mu)}} m_\alpha \alpha$, m_α is an integer, we have $\delta(H_\mu) = \omega(H_\mu)$. This implies that $\delta(H_N)$ and $\omega(H_N)$ have the same signature. Hence $\mathfrak{q}$ is the second kind for this case. Thus each $K(\mu)$-simple submodule $\mathfrak{q}$ of $\mathfrak{p}_{\mathbf{C}}$ is the first kind or the second kind.

COROLLARY 6.6. *Let $P(\mu)$ be the positive root system of $\Sigma(\mathfrak{l}(\mu)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$ as in* (6.2) *and $\mathfrak{p}(\mu) = \mathfrak{p} \cap \mathfrak{l}(\mu)$. Then each simple $K(\mu)$-submodule of $\mathfrak{p}(\mu)_{\mathbf{C}}$ is the first kind or the second kind with respect to $P(\mu)$. Moreover each simple root in $P(\mu)$ is also simple in $P^\star$.*

PROOF. It is sufficient to prove that if α is simple in $P(\mu)$, then α is simple in $P^\star$. Suppose that α is not simple in $P^\star$. Then there exist β and γ in $P^\star$ such that $\alpha = \beta + \gamma$. Therefore $0 = \alpha(H_\mu) = \beta(H_\mu) + \gamma(H_\mu)$. By the choice of $P^\star$ in (6.3) we have $\beta(H_\mu) = 0$ and $\gamma(H_\mu) = 0$, and hence $\alpha = \beta + \gamma$, $\beta, \gamma \in P_{K(\mu)}$. This is a contradiction to α is simple in $P_{K(\mu)}$.

7. Standard triple of the positive root systems

Our purpose of this section is to prove Theorem 7.5.

LEMMA 7.1. *Let P be a positive root system of Σ containing P_K and Ψ the simple root system of P. For a subset Θ of Ψ we denote by $P(\Theta)$ the set of all roots in P generated by the set Θ over the ring of integers. Then there exists a reductive subalgebra $\mathfrak{l}(\Theta)$ of $\mathfrak{g}$ containing $\mathfrak{b}$ such that $P(\Theta)$ is a positive root system of the pair $(\mathfrak{l}(\Theta)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$.*

PROOF. Let C be the positive Weyl chamber of $\sqrt{-1}\mathfrak{b}$ corresponding to P. We put

$$C(\Theta) = \{H \in cl(C) : \alpha(H) = 0 \text{ for } \alpha \text{ in } \Theta \text{ and } \alpha(H) > 0 \text{ for } \alpha \text{ in } \Psi \backslash \Theta\}, \tag{7.1}$$

where $cl(C)$ is the topological closure of C in $\sqrt{-1}\mathfrak{b}$. It is sufficient to prove this lemma for the case $P(\Theta) \neq P$. Since $\Theta \neq \Psi$, we can choose $H \in C(\Theta)\backslash\{0\}$. The centralizer $\mathfrak{l}(H)$ of H in $\mathfrak{g}$ is reductive, and contains $\mathfrak{b}$. Let Σ_H be the root system of $(\mathfrak{l}(H)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$. Then we have $P(\Theta) = \Sigma_H \cap P$. Hence $P(\Theta)$ is a positive root system of the pair $(\mathfrak{l}(\Theta)_{\mathbf{C}}, \mathfrak{b}_{\mathbf{C}})$, where $\mathfrak{l}(\Theta) = \mathfrak{l}(H)$.

LEMMA 7.2. *Let Θ be a subset of Ψ, and define $C(\Theta)$ by (7.1). Let H be an element in $C(\Theta)$ and $K(\Theta)$ the centralizer of H in K. Then the group $K(\Theta)$ is determined independently by the choice of H in $C(\Theta)$.*

PROOF. Let $K(\Theta)^0$ be the analytic subgroup of G corresponding to $\mathfrak{l}(\Theta) \cap \mathfrak{k}$. In view of the proof of Lemma 7.1 $K(\Theta)^0$ is uniquely determined by Θ. Let k be an element in $K(\Theta)$. Then there exists k_0 in $K(\theta)^0$ such that $Ad(k) = Ad(k_0)$. We put $z = k^{-1}k_0$. Since z belongs to the center Z of K, we have $K(\Theta) \subset ZK(\Theta)^0$. On the other hand, since K is connected, B is a maximal abelian subgroup of K (cf. Corollary 2.7 in [2]). This implies that $Z \subset B$. Since $B \subset K(\Theta)$, we have $K(\Theta) = ZK(\Theta)^0$. Thus $K(\Theta)$ is determined independently by the choice of H.

DEFINITION 7.3. Let P be a positive root system of Σ containing P_K. For a subset Θ in the simple root system Ψ of P, we consider the positive root system $P(\Theta)$ as in Lemma 7.1. Then the triple $(P_K, P(\Theta), P)$ is standard if each simple $K(\Theta)$-submodule of $\mathfrak{p}_{\mathbf{C}}$ is either the first kind or the second kind with respect to P.

DEFINITION 7.4. Let $(P_K, P(\Theta), P)$ be a standard triple. A root γ in Σ_n is said to be the first (resp. the second) kind if the simple $K(\Theta)$-module $\mathfrak{q}_\gamma$ generated by X_γ is the first (resp. the second) kind.

REMARK. Let P be a positive root system containing P_K. For $\Theta = \phi$, we have $P(\Theta) = \phi$ and $K(\Theta) = B$. Moreover, (P_K, ϕ, P) is standard, and $C(\Theta)$ is the positive Weyl chamber.

THEOREM 7.5. *For $\mu \in \Gamma_K$, there exists a standard triple $(P_K, P(\Theta), P)$ such that $H_\mu \in C(\Theta)$. Moreover we have $K(\Theta) = K(\mu)$.*

PROOF. We first assume that H_μ is regular. Then we have $\mathfrak{k}(\mu) = \mathfrak{b}$. In this case we put $P = \{\alpha \in \Sigma : \alpha(H_\mu) > 0\}$. Then (P_K, ϕ, P) is standard, $H_\mu \in C(\phi)$, $K(\Theta) = K(\mu)$ and $\mathfrak{l}(\Theta) = \mathfrak{b}$. Let us now assume that H_μ is singular. Let $\mathfrak{l}(\mu)$ be the centralizer of H_μ in $\mathfrak{g}$. We choose the positive root systems $P^\star$ and $P(\mu)$ the same as in Lemma 6.5 and Corollary 6.6 respectively. Let $\Psi^\star$ be the simple root system of $P^\star$. By Corollary 6.6 the simple root system Θ of $P(\mu)$ is a subset of $\Psi^\star$. We put $P = P^\star$. Since $P(\Theta) = P(\mu)$, the triple $(P_K, P(\Theta), P)$ is standard, $H_\mu \in C(\Theta)$ and $K(\Theta) = K(\mu)$.

8. Principal weight space $PW(\mu)$

In this section we shall fix a standard triple $(P_K, P(\Theta), P)$, and consider the convex cone $C(\Theta)$ corresponding to this triple. We now put $C(\Theta)^* = \{\eta \in (\sqrt{-1}\mathfrak{b})^* : H_\eta \in C(\Theta)\}$. Let $\mu \in C(\Theta)^* \cap \Gamma_K$ and V_μ a unitary simple K-module with highest weight μ. We shall fix the highest weight vector $v(\mu)$ normalized as $|v(\mu)| = 1$.

DEFINITION 8.1. Let P_n be the set of all noncompact roots in P. We define a projection operator P_+ on the K-module $\mathfrak{p}_{\mathbf{C}} \otimes V_\mu$ by $P_+ = \sum_{\omega \in P_n, \mu+\omega \in \Gamma_K} P_{\mu+\omega}$.

DEFINITION 8.2. Let $W(\mu)$ be the weight subspace of $M(\mu)$ as in Lemma 3.1. We define a subspace $PW(\mu)$ of $W(\mu)$ by

$$PW(\mu) = [\{P_\mu(X_{-\gamma} \otimes P_+(X_\gamma \otimes v(\mu)) - X_\gamma \otimes P_+(X_{-\gamma} \otimes v(\mu))) : \gamma \in P_n\}] .$$

LEMMA 8.3. *Let $N(\mu)$ be the K-submodule of $M(\mu)$ generated by the set*

$$\{P_\mu(X \otimes P_+(Y \otimes v) - Y \otimes P_+(X \otimes v)) : X, Y \in \mathfrak{p}_{\mathbf{C}}, v \in V_\mu\} .$$

Then we have $N(\mu) \cap W(\mu) = PW(\mu)$. Especially $\dim PW(\mu)$ *is the multiplicity of V_μ in $N(\mu)$.*

PROOF. It is enough to prove that $N(\mu) \cap W(\mu) \subset PW(\mu)$. Let Z be an element in $N(\mu) \cap W(\mu)$. We can assume that

$$Z = P_\mu(X_\gamma \otimes P_+(X_\delta \otimes Q(I)v(\mu)) - X_\delta \otimes P_+(X_\gamma \otimes Q(I)v(\mu))) ,$$

where $\gamma, \delta \in \Sigma_n$, $I \in \Pi$, $\gamma + \delta = \langle I \rangle$. By Lemma 3.5 we have

$$\begin{aligned} Z = \sum_{J \preccurlyeq I} (-1)^{\sharp I} \{ & P_\mu(Q(\psi(J))X_\gamma \otimes P_+(Q(\psi(I \backslash J))X_\delta \otimes v(\mu))) \\ & - P_\mu(Q(\psi(J))X_\delta \otimes P_+(Q(\psi(I \backslash J))X_\gamma \otimes v(\mu)))\} . \end{aligned}$$

Since $(I \backslash J) \preccurlyeq I$ and $I \backslash (I \backslash J) = J$, we have

$$\begin{aligned} Z = (-1)^\sharp \Bigg\{ & \sum_{J \preccurlyeq I} P_\mu(Q(\psi(J))X_\gamma \otimes P_+(Q(\psi(I \backslash J))X_\delta \otimes v(\mu))) \\ & - \sum_{J \preccurlyeq I} P_\mu(Q(\psi(I \backslash J))X_\delta \otimes P_+(Q(\psi(J))X_\gamma \otimes v(\mu))) \Bigg\} . \end{aligned}$$

Since $\gamma + \delta = \langle J \rangle + \langle I \backslash J \rangle$, we have $Z \in PW(\mu)$.

LEMMA 8.4. *Let μ be an element in $C(\Theta)^* \cap \Gamma_K$ and V_μ the simple K-module with highest weight μ. Suppose that $U(\mathfrak{k}(\Theta)_{\mathbf{C}})X_\gamma \ni X_\delta$ for two noncompact roots γ, δ in Σ. Then, for each noncompact root ω satisfying $\mu + \omega \in \Gamma_K$, we have*

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = |P_{\mu+\omega}(X_\delta \otimes v(\mu))|^2 .$$

PROOF. We first prove that $X_\alpha v(\mu) = 0$ for all $\alpha \in \Sigma_{K(\Theta)}$. Since $v(\mu)$ is the highest weight vector of V_μ, it is sufficient to prove that $X_{-\alpha} v(\mu) = 0$ for all $\alpha \in P_{K(\Theta)}$. Since $ad(X_\alpha)X_{-\alpha}v(\mu) = \alpha(H_\mu)v(\mu) = 0$, we have $X_\alpha X_{-\alpha}v(\mu) = 0$. By the choice of X_α in (2.1), we have $0 = (X_\alpha X_{-\alpha}v(\mu), v(\mu)) = |X_{-\alpha}v(\mu)|^2$. This implies that $X_{-\alpha}v(\mu) = 0$. Let us now prove this lemma. By the asummption for γ and δ, there exist a nonzero complex number c and a finite number of roots $\alpha_1, \alpha_2, \cdots, \alpha_q \in \Sigma_{K(\Theta)}$ such that

$$ad(X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma = cX_\delta .$$

Then we have

$$\begin{aligned}
c|P_{\mu+\omega}(X_\delta \otimes v(\mu))|^2 &= (P_{\mu+\omega}(ad(X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma \otimes v(\mu)), P_{\mu+\omega}(X_\delta \otimes v(\mu))) \\
&= (X_{\alpha_1} P_{\mu+\omega}(ad(X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma \otimes v(\mu)), P_{\mu+\omega}(X_\delta \otimes v(\mu))) \\
&\quad - (P_{\mu+\omega}(ad(X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma \otimes X_{\alpha_1} v(\mu)), P_{\mu+\omega}(X_\delta \otimes v(\mu))) , \\
&= (P_{\mu+\omega}(ad(X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma \otimes v(\mu)), P_{\mu+\omega}(ad(X_{-\alpha_1})X_\delta \otimes v(\mu))) \\
&\cdots \\
&= (P_\mu(X_\gamma \otimes v(\mu)), P_\mu(ad(X_{-\alpha_q} \cdots X_{-\alpha_1})X_\delta \otimes v(\mu))) \\
&= c'|P_{\mu+\omega}(X_\gamma \otimes v(\mu)|^2 ,
\end{aligned}$$

where $c' = \phi(ad(X_{-\alpha_q} \cdots X_{-\alpha_1})X_\delta, X_{-\gamma})$. Since the Killing form ϕ is τ invariant, (2.6) implies that

$$c' = (-1)^q \phi(ad(X_{\alpha_q} \cdots X_{\alpha_1})X_{-\delta}, X_\gamma) = \phi(X_{-\delta}, ad(X_{\alpha_1} X_{\alpha_2} \cdots X_{\alpha_q})X_\gamma) = c .$$

Thus we have $|P_{\mu+\omega}(X_\gamma \otimes v(\mu))| = |P_{\mu+\omega}(X_\delta \otimes v(\mu))|$.

THEOREM 8.5. *Let $(P_K, P(\Theta), P)$ be a standard triple and $\mu \in C(\Theta)^* \cap \Gamma_K$. Assume that μ is admissible. Then we have*

$$PW(\mu) = [\{Z(\gamma) : \gamma \textit{ is a } P_{K(\Theta)}\textit{-highest root in } P_n \textit{ and of the second kind}\}] ,$$

where $Z(\gamma) = P_\mu(X_{-\gamma} \otimes P_+(X_\gamma \otimes v(\mu)) - X_\gamma \otimes P_+(X_{-\gamma} \otimes v(\mu)))$.

PROOF. Let γ be a noncompact root in Σ. By using Corollary 4.7 we have

$$(8.1)\quad Z(\gamma) = \sum_{\omega \in P_n \cap \Sigma_{W(\mu)}} c(\mu; \omega)(|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 - |P_{\mu+\omega}(X_{-\gamma} \otimes v(\mu))|^2)v_\omega(\mu) .$$

By Lemma 8.4 if two vectors X_γ and X_δ belong to the same simple $K(\Theta)$-submodule in $\mathfrak{p}_{\mathbf{C}}$, then we have

$$Z(\gamma) = Z(\delta)\,. \tag{8.2}$$

Especially if γ is of the first kind, then we have

$$Z(\gamma) = 0\,. \tag{8.3}$$

Hence by (8.1), (8.2) and (8.3) we have our assertion of this theorem.

DEFINITION 8.6. Let $(P_K, P(\Theta), P)$ be a standard triple and $\mu \in C(\Theta)^* \cap \Gamma_K$. We define $|\mu|_\Theta$ by

$$|\mu|_\Theta = \min\left\{\frac{2(\mu,\alpha)}{|\alpha|^2} : \alpha \in P_K \backslash P_{K(\Theta)}\right\}. \tag{8.4}$$

We note that if $|\mu|_\Theta \geq 3$, then μ is admissible. Hence by Lemma 5.8 we have $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$ for μ satisfying $|\mu|_\Theta \geq 3$ and a $P_{K(\Theta)}$-highest root $\omega \in P_n$.

LEMMA 8.7. *Let ω be a $P_{K(\Theta)}$-highest noncompact root in P and γ a noncompact root. Then there exists a positive integer N (≥ 3) such that*

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 \leq |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2$$

for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq N$.

PROOF. Let $\mathfrak{q}_\omega$ be the simple $K(\Theta)$-module generated by X_ω. Suppose that $X_\gamma \in \mathfrak{q}_\omega$. By Lemma 8.4 we have the inequality in this lemma for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq 3$. Let us consider the case $X_\gamma \notin \mathfrak{q}_\omega$. By Lemma 4.3 we have

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \sum_{\alpha\in P_K} \frac{2|\langle\alpha,\gamma\rangle|^2}{|\lambda+\omega|^2 - |\lambda+\gamma|^2} |P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu))|^2\,. \tag{8.5}$$

By (2.4) we have

$$2|\langle\alpha,\gamma\rangle| \leq 3|\alpha|^2\,. \tag{8.6}$$

By Lemma 3.8 in [3] if $P_{\mu+\omega}(X_\gamma \otimes v(\mu)) \neq 0$, then there exists $I = (\alpha_1, \alpha_2, \cdots, \alpha_q) \in \Pi$ such that $\omega - \gamma = \langle I\rangle$. Moreover, since $X_\gamma \notin \mathfrak{q}_\omega$, we have $\alpha_p \notin P_{K(\Theta)}$ for a root α_p in $\{\alpha_1, \alpha_2, \cdots, \alpha_q\}$. This implies that

$$\begin{aligned} |\lambda+\omega|^2 - |\lambda+\gamma|^2 &= \sum_{i=1}^{q} 2(\mu,\alpha_i) + |\omega|^2 - |\gamma|^2 \\ &\geq 2(\mu,\alpha_p) + |\omega|^2 - |\gamma|^2 \\ &\geq |\mu|_\Theta |\alpha_p|^2 + |\omega|^2 - |\gamma|^2\,. \end{aligned} \tag{8.7}$$

Hence by (8.5), (8.6) and (8.7) there exists a positive integer N_1 such that

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 \leq \max_{\alpha \in P_K} |P_{\mu+\omega}(X_{\gamma+\alpha} \otimes v(\mu))|^2$$

for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq N_1$. By using this argument successively we can prove this lemma.

COROLLARY 8.8. *Let $\omega, \gamma \in \Sigma_n$. Suppose that ω and γ are $P_{K(\Theta)}$-highest. Then we have*

$$\lim_{|\mu|_\Theta \to +\infty} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = \delta_{\omega,\gamma}\,,$$

where $\delta_{\omega,\gamma}$ is Kronecker's delta.

PROOF. Assume that $X_\gamma \notin \mathfrak{q}_\omega$. By Lemma 8.7 and (8.5), there exists a number N' such that

$$|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 \leq \sum_{\alpha \in P_K} \frac{2|\langle \alpha, \gamma \rangle|^2}{|\lambda+\omega|^2 - |\lambda+\gamma|^2} |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2$$

for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq N'$. This inequality and (8.7) imply

$$\lim_{|\mu|_\Theta \to +\infty} |P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 = 0\,.$$

Consider the case $\gamma = \omega$. We can assume that $|\mu|_\Theta$ of $\mu \in C(\Theta)^* \cap \Gamma_K$ is sufficiently large. Then $\mu + \omega \in \Gamma_K$ and $P_{\mu+\omega}(\mathfrak{p}_{\mathbf{C}} \otimes V_\mu) \neq \{0\}$. Since $|P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = f(\lambda+\omega : \omega)$, Theorem 5.5 implies that

$$\lim_{|\mu|_\Theta \to +\infty} |P_{\mu+\omega}(X_\omega \otimes v(\mu))|^2 = 1\,.$$

THEOREM 8.9. *Let $(P_K, P(\Theta), P)$ be a standard triple, and $C(\Theta)^* = \{\eta \in (\sqrt{-1}\mathfrak{b})^* : H_\eta \in C(\Theta)\}$. Then there exists a sufficiently large number N such that*

$$\dim PW(\mu) = \sharp\{\omega \in P_n : \omega \text{ is } P_{K(\Theta)}\text{-highest and of the second kind}\}$$

for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq N$.

PROOF. Let ω and γ be two $P_{K(\Theta)}$-highest roots in P_n. We put

$$a_{\omega,\gamma}(\mu) = c(\mu;\omega)(|P_{\mu+\omega}(X_\gamma \otimes v(\mu))|^2 - |P_{\mu+\omega}(X_{-\gamma} \otimes v(\mu))|^2)\,.$$

By Corollary 4.5 we have

$$\begin{aligned} &P_\mu(X_{-\gamma} \otimes P_+(X_\gamma \otimes v(\mu)) - X_\gamma \otimes P_+(X_{-\gamma} \otimes v(\mu))) \\ &\quad = \sum_{\omega \in P_n \cap \Sigma_{W(\mu)}} a_{\omega,\gamma}(\mu) v_\omega(\mu)\,. \end{aligned} \tag{8.8}$$

Since $\deg \pi_\mu = \prod_{\alpha \in P_K} (\lambda, \alpha)(\rho_K, \alpha)^{-1}$, we have $\lim_{|\mu|_\Theta \to +\infty} c(\mu; \omega) = d(\omega)$, where $d(\omega)$ is a positive constant. Hence by Corollary 8.8 we have

$$\lim_{|\mu|_\Theta \to +\infty} a_{\omega,\gamma}(\mu) = d(\omega)\delta_{\omega,\gamma} \quad \text{for } \omega, \gamma \in P_\Theta \,,$$

where

$$P_\Theta = \{\gamma \in P_n : \gamma \text{ is } P_{K(\Theta)}\text{-highest and of the second kind}\}\,.$$

In view of Theorem 8.5 and (8.8) we can prove there exists a number N such that $\dim PW(\mu) = \sharp P_\Theta$ for all $\mu \in C(\Theta)^* \cap \Gamma_K$ satisfying $|\mu|_\Theta \geq N$.

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