TOKYO J. MATH.
VOL. 27, NO. 1, 2004

Bianchi Surfaces with Constant Chebyshev Angle

Atsushi FUJIOKA

Hitotsubashi University

(Communicated by Y. Ohnita)

Dedicated to Professor Takushiro Ochiai on his sixtieth birthday

Abstract. We consider Bianchi surfaces parametrized by a generalized Chebyshev net and show that such a surface with constant Chebyshev angle is a piece of a right helicoid.

1. Introduction

Bianchi surfaces are a class of surfaces with negative Gaussian curvature discovered by generalizing Bäcklund transformation for surfaces with constant negative Gaussian curvature [1, 2]. On the other hand Bobenko [3] introduced a new class of surfaces which are called harmonic inverse mean curvature (HIMC for short) surfaces as a generalization of surfaces with constant mean curvature. Nowadays both surfaces are studied from the viewpoint of theory of integrable systems [4, 7, 8, 10, 11, 12, 13, 15]. In particular they admit a one-parameter family of deformations preserving the ratio of the principal curvatures and if the ambient space is $\mathbf{R}^3$ any isothermic HIMC surface is a dual to a Bonnet surface. Moreover isothermic HIMC surfaces in $\mathbf{R}^3$ with constant ratio of the principal curvatures can be expressed more concretely than general case as well as Bonnet surfaces with constant curvature [5, 6, 9, 14, 16].

In this paper we introduce the notion of generalized Chebyshev nets which is a natural generalization of Chebyshev nets for surfaces with constant negative Gaussian curvature and show that the following:

THEOREM 1.1. *A Bianchi surface with constant Chebyshev angle parametrized by a generalized Chebyshev net is a piece of a right helicoid.*

In the following we consider only surfaces with negative Gaussian curvature in the Euclidean 3-space $\mathbf{R}^3$. Since such a surface has two directions, called the asymptotic directions, in which the curvature vanishes, we can parametrize the surface locally by asymptotic line coordinates (x, y):

$$F : D \to \mathbf{R}^3 ,$$

Received April 25, 2003; revised June 10, 2003
Partially supported by Grant-in-Aid for Scientific Research No. 14740038, Ministry of Education, Culture, Sports, Science and Technology, Japan.

where $D \subset \mathbf{R}^2$. If the Gaussian curvature is $-\frac{1}{\rho^2}$ for a positive function ρ on D then the fundamental forms become as follows:

$$\mathrm{I} = A^2dx^2 + 2AB\cos\phi dxdy + B^2dy^2, \quad \mathrm{II} = \frac{2AB\sin\phi}{\rho}dxdy\,,$$

where $A = |F_x|$, $B = |F_y|$ and ϕ is the angle between the asymptotic lines, called the Chebyshev angle. Changing the coordinates if necessary, we may assume that $0 < \phi < \pi$. If we put $a = \frac{A}{\rho}$, $b = \frac{B}{\rho}$ then the Gauss-Codazzi equations have the following form [3]:

$$\phi_{xy} + \left(\frac{\rho_x b}{2\rho a}\sin\phi\right)_x + \left(\frac{\rho_y a}{2\rho b}\sin\phi\right)_y - ab\sin\phi = 0\,, \tag{1.1}$$

$$\begin{cases} a_y + \dfrac{\rho_y}{2\rho}a - \dfrac{\rho_x}{2\rho}b\cos\phi = 0\,, \\ b_x + \dfrac{\rho_x}{2\rho}b - \dfrac{\rho_y}{2\rho}a\cos\phi = 0\,. \end{cases} \tag{1.2}$$

DEFINITION 1.1. A surface is called a Bianchi surface if $\rho_{xy} = 0$, i.e., $\rho = f(x) + g(y)$, where f and g are functions of x and y only respectively.

DEFINITION 1.2. A parametrization of a surface is called a generalized Chebyshev net if $A = B$, i.e., $a = b$.

EXAMPLE 1.1. For a surface with constant negative Gaussian curvature, (1.2) implies that a and b are functions of x and y only respectively. Changing the coordinates conformally if necessary, we may assume that $a = b = C$ for a positive constant C. Hence the surface is parametrized by a so-called Chebyshev net.

EXAMPLE 1.2. A surface with negative Gaussian curvature is minimal if and only if $\phi = \frac{\pi}{2}$ since the mean curvature is $\frac{1}{\rho}\cot\phi$. In this case we can solve (1.2) directly:

$$a = \alpha(x)\rho^{-\frac{1}{2}}\,, \quad b = \beta(y)\rho^{-\frac{1}{2}}\,,$$

where α and β are positive functions of x and y only respectively. Similar to Example 1.1 the surface is parametrized by a generalized Chebyshev net.

2. Proof of Theorem 1.1

For a Bianchi surface with constant Chebyshev angle parametrized by a generalized Chebyshev net, (1.1) and (1.2) become

$$\left(\frac{\rho_x}{2\rho}\right)_x + \left(\frac{\rho_y}{2\rho}\right)_y - a^2 = 0\,, \tag{2.1}$$

$$\begin{cases} \dfrac{a_y}{a} + \dfrac{\rho_y}{2\rho} - \dfrac{\rho_x}{2\rho}\cos\phi = 0\,, \\ \dfrac{a_x}{a} + \dfrac{\rho_x}{2\rho} - \dfrac{\rho_y}{2\rho}\cos\phi = 0\,. \end{cases} \tag{2.2}$$

LEMMA 2.1. $\phi = \frac{\pi}{2}$.

PROOF. Since the integration of (2.2) gives

$$a = \alpha(x)\rho^{-\frac{1}{2}} e^{\frac{\cos\phi}{2}\int\frac{\rho_x}{\rho}dy} = \beta(y)\rho^{-\frac{1}{2}} e^{\frac{\cos\phi}{2}\int\frac{\rho_y}{\rho}dx}\,, \tag{2.3}$$

where α and β are positive functions of x and y only respectively, we have

$$\left(\frac{\rho_x}{\rho}\right)_x \cos\phi = \left(\frac{\rho_y}{\rho}\right)_y \cos\phi\,, \tag{2.4}$$

which is equivalent to

$$\{(\rho_{xx} - \rho_{yy})\rho - (\rho_x^2 - \rho_y^2)\}\cos\phi = 0\,. \tag{2.5}$$

Combining (2.1), (2.3) and (2.4), we have

$$\left\{\left(\frac{\rho_x}{\rho}\right)_x - \frac{\alpha^2}{\rho} e^{\cos\phi\int\frac{\rho_x}{\rho}dy}\right\}\cos\phi = 0\,, \tag{2.6}$$

$$\left\{\left(\frac{\rho_y}{\rho}\right)_y - \frac{\beta^2}{\rho} e^{\cos\phi\int\frac{\rho_y}{\rho}dx}\right\}\cos\phi = 0\,. \tag{2.7}$$

From (2.6) we have

$$(\rho\rho_{xx} - \rho_x^2 - \alpha^2\rho e^{\cos\phi\int\frac{\rho_x}{\rho}dy})\cos\phi = 0\,. \tag{2.8}$$

Since $\rho_{xy} = 0$, differentiating (2.8) twice by y, we have

$$\{\rho_{xx}\rho_{yy} - (\rho\rho_{yy} + \rho_x\rho_y\cos\phi + \rho_x^2\cos^2\phi)a^2\}\cos\phi = 0\,. \tag{2.9}$$

Similarly from (2.7) we have

$$\{\rho_{xx}\rho_{yy} - (\rho\rho_{xx} + \rho_x\rho_y\cos\phi + \rho_y^2\cos^2\phi)a^2\}\cos\phi = 0\,. \tag{2.10}$$

Combining (2.5), (2.9) and (2.10), we have

$$(\rho_x^2 - \rho_y^2)\cos\phi = 0\,. \tag{2.11}$$

Now we assume that $\phi \neq \frac{\pi}{2}$. From (2.11) we have

$$\rho = C(x \pm y)\,,$$

where $C \in \mathbf{R} \setminus \{0\}$. Then from (2.1) we have

$$a^2 = -\frac{1}{(x \pm y)^2}\,,$$

which is a contradiction.

By Example 1.2 we may assume that $a = \rho^{-\frac{1}{2}}$.

LEMMA 2.2. *ρ is a function of x or y only.*

PROOF. From (2.1) we have

$$(\rho_{xx} + \rho_{yy})\rho - (\rho_x^2 + \rho_y^2) - 2\rho = 0\,. \tag{2.12}$$

Differentiating (2.12) by x and y, we have

$$\rho_x \rho_{yyy} + \rho_y \rho_{xxx} = 0\,. \tag{2.13}$$

Now we assume that $\rho = f(x) + g(y)$, where f and g are functions of x and y only respectively such that $f', g' \neq 0$. From (2.13) we have

$$\begin{cases} (f')^2 = C_1 f^2 + C_2 f + C_3\,, \\ (g')^2 = -C_1 g^2 + C_4 g + C_5\,, \end{cases} \tag{2.14}$$

where $C_1, \cdots, C_5 \in \mathbf{R}$. Combining (2.12) and (2.14), we have

$$\left(-\frac{C_2}{2} + \frac{C_4}{2} - 2\right) f + \left(\frac{C_2}{2} - \frac{C_4}{2} - 2\right) g - (C_3 + C_5) = 0\,,$$

which is a contradiction.

PROOF OF THEOREM 1.1. For simplicity we consider only the case that ρ is a function of x only. Then we can solve (2.12) explicitly:

$$\rho = \frac{1}{C_1^2} \cosh^2(C_1 x + C_2)\,,$$

where $C_1 \in \mathbf{R} \setminus \{0\}$, $C_2 \in \mathbf{R}$. Now we have obtained the fundamental forms concretely:

$$\mathrm{I} = \frac{1}{C_1^2} \cosh^2(C_1 x + C_2)(dx^2 + dy^2)\,, \quad \mathrm{II} = 2dxdy\,.$$

Hence the surface is a piece of a right helicoid. Indeed we have

$$F(x, y) = \frac{1}{C_1^2} T \begin{pmatrix} \sinh(C_1 x + C_2) \sin C_1 y \\ \sinh(C_1 x + C_2) \cos C_1 y \\ y \end{pmatrix} + p\,,$$

where $T \in \mathrm{SO}(3)$, $p \in \mathbf{R}^3$.

By Example 1.2 we have the following:

COROLLARY 2.1. *A minimal Bianchi surface is a piece of a right helicoid.*

References

[1] L. BIANCHI, Sopra alcune nuove classi di superficie e di sistemi tripli ortogonali, Ann. Mat. (2) **18** (1890), 301–358.

[2] L. BIANCHI, *Lezioni di geometria differenziale*, Pisa (1909).

[3] A. I. BOBENKO, Surfaces in terms of 2 by 2 matrices. Old and new integrable cases. *Harmonic maps and integrable systems*, Aspects Math., E23, Vieweg (1994), 83–127.

[4] A. BOBENKO, U. EITNER and A. KITAEV, Surfaces with harmonic inverse mean curvature and Painlevé equations, Geom. Dedicata **68** (1997), 187–227.

[5] W. CHEN and H. LI, Bonnet surfaces and isothermic surfaces, Results Math. **31** (1997), 40–52.

[6] A. G. COLARES and K. KENMOTSU, Isometric deformation of surfaces in $\mathbf{R}^3$ preserving the mean curvature function, Pacific J. Math. **136** (1989), 71–80.

[7] A. S. FOKAS and I. M. GELFAND, Surfaces on Lie groups, on Lie algebras, and their integrability, With an appendix by Juan Carlos Alvarez Paiva, Comm. Math. Phys. **177** (1996), 203–220.

[8] A. FUJIOKA, Surfaces with harmonic inverse mean curvature in space forms, Proc. Amer. Math. Soc. **127** (1999), 3021–3025.

[9] A. FUJIOKA and J. INOGUCHI, Bonnet surfaces with constant curvature, Results Math. **33** (1998), 288–293.

[10] A. FUJIOKA and J. INOGUCHI, On some generalisations of constant mean curvature surfaces, Towards 100 years after Sophus Lie (Kazan, 1998), Lobachevskii J. Math. **3** (1999), 73–95 (electronic).

[11] A. FUJIOKA and J. Inoguchi, Spacelike surfaces with harmonic inverse mean curvature, J. Math. Sci. Univ. Tokyo **7** (2000), 657–698.

[12] D. A. KOROTKIN and V. A. Reznik, Bianchi surfaces in $\mathbf{R}^3$ and deformations of hyperelliptic curves, Math. Notes **52** (1992), 930–937.

[13] D. LEVI and A. SYM, Integrable systems describing surfaces of nonconstant curvature, Phys. Lett. A **149** (1990), 381–387.

[14] I. M. ROUSSOS, Principal curvature preserving isometries of surfaces in ordinary space, Bol. Soc. Brasil. Mat. **18** (1987), 95–105.

[15] W. K. SCHIEF, An infinite hierarchy of symmetries associated with hyperbolic surfaces, Nonlinearity **8** (1995), 1–9.

[16] H. TAKEUCHI, Isometric deformation of surfaces in the hyperbolic 3-manifold preserving the mean curvature, Tokyo J. Math. **18** (1995), 247–258.

Present Address:
GRADUATE SCHOOL OF ECONOMICS, HITOTSUBASHI UNIVERSITY,
NAKA, KUNITACHI, TOKYO, 186–8601 JAPAN.
e-mail: fujioka@math.hit-u.ac.jp