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A Note on Local Reduction Numbers and a^*-Invariants of Graded Rings

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1. Introduction

Let $(A, \mathfrak{m})$ be a Noetherian local ring of dimension d with an infinite residue field, and let I be an ideal of A. Denote by $r(I)$ the *reduction number* of I, by $\ell(I)$ the *analytic spread* of I. An interesting question is the relationship between the Cohen-Macaulay (CM) property of the Rees algebra $R(I) := \bigoplus_{n\geq 0} I^n$ and the associated graded ring $G(I) := \bigoplus_{n\geq 0}(I^n/I^{n+1})$. In the case that A is a CM ring, one approach to this problem was taken first by Goto-Shimoda when I is $\mathfrak{m}$-primary [5] in 1979. The theorem of Goto and Shimoda states:

THEOREM 1.1 [5]. *Let $(A, \mathfrak{m})$ be a CM ring of dimension d with infinite residue field. Let I be an $\mathfrak{m}$-primary ideal. Then $R(I)$ is CM iff $G(I)$ is CM and $r(I) \leq d-1$.*

Next, other authors extended this theorem to ideals having small analytic deviation, see, e.g., [2, 4, 6]. But the most general result was obtained by Johnston-Katz in [9, Theorem 2.3], independently, Aberbach-Huneke-Trung in [1, Theorem 5.1]. Moreover, [1] also gave a similar characterization for the Gorenstein property of $R(I)$ [1, Theorem 5.8]. The method used in [1] is the study of the relationship between the so-called *local reduction numbers* of an ideal I and the a-invariant of $G(I)$ in the case that $G(I)$ is CM.

Set $L_i(I) := \{I \subseteq \mathfrak{p} \in \operatorname{Spec} A \mid \ell(I_\mathfrak{p}) = \operatorname{ht}(\mathfrak{p}) \leq i\};\ i \leq \ell(I)$. The number

$$r_i(I) = \begin{cases} i - \operatorname{ht}(I) & \text{if } i < \operatorname{ht}(I) \\ \max\{r(I_\mathfrak{p}) - \operatorname{ht}(\mathfrak{p}) \mid \mathfrak{p} \in L_i(I)\} + i & \text{if } \operatorname{ht}(I) \leq i \leq \ell(I) \end{cases}$$

is called the *$i-th$ local reduction number* of I (see [1]). These invariants have been shown to play an important role in studying the CM and Gorenstein property of Rees algebras, see, e.g., [1] and [15]. Note that Aberbach-Huneke-Trung's method in [1] yielded important information concerning the local reduction numbers of an ideal I and a-invariant of $G(I)$ in the case that $G(I)$ is CM. The aims of [1] were achieved by the following result.

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THEOREM 1.2 [1, Theorem 4.4]. *Assume that $G(I)$ is a CM ring. Set $\ell = \ell(I)$ and let J be a minimal reduction of I. Then*

(i) $\max\{r_{\ell-1}(I) + 1, r_J(I)\} = r_\ell(I)$.

(ii) $\max\{r_{\ell-1}(I) + 1, a(G(I)) + \ell\} = r_\ell(I)$.

Let $S := \bigoplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over a Noetherian local ring S_0 (i.e., $S = S_0[S_1]$). We denote by S^+ the ideal generated by all homogeneous elements of positive degree of S. Set

$$a_0(S) := \inf\{a \in Z \mid [H^0_{S^+}(S)]_n = 0 \ \text{ for all } \ n > a\}\,.$$

The aim of this note is to show that one can extend Theorem 4.4 and Theorem 5.8 in [1] to rings satisfying the Serre's condition (S_ℓ) (ℓ the analytic spread of S^+). Although most of the arguments in the proofs of [1, Theorem 4.4] can be applied to these rings, there are some critical places which have to be dealt with. For example, the notion of local reduction numbers in [1] and [15] is not suitable in the case of a graded ring S having grade $S^+ < \text{ht}\,(S^+)$ (see the proofs of [1, Theorem 4.4] and [15, Theorem 3.3]). To overcome this difficulty, the notion of a local reduction number is defined as follows.

The number

$$r_i(I) = \begin{cases} \max\{-\text{ht}(I), a_0(G)\} + i & \text{if } \ i < \text{ht}(I) \\ \max\{a_0(G), r(I_\mathfrak{p}) - \text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in L_i(I)\} + i & \text{if } \ \text{ht}(I) \leq i \leq \ell(I) \end{cases}$$

is called the $i-th$ *local reduction number* of I. For any $\mathfrak{p} \in$ Spec S_0 we denote by $S_\mathfrak{p}$ the localization of S at the multiplicatively closed set $S_0\backslash\mathfrak{p}$. Set $\text{ht}(S^+) = h$,

$$L_i(S) := \{\mathfrak{p} \in \ \text{Spec}\ S_0 \mid \ell(S^+_\mathfrak{p}) = \text{ht}(\mathfrak{p}) + h \leq i\};\ \text{for}\ i \leq \ell(S^+).$$

The number

$$r_i(S) = \begin{cases} \max\{-h, a_0(S)\} + i & \text{if } \ i < h \\ \max\{a_0(S), r(S^+_\mathfrak{p}) - \text{ht}(\mathfrak{p}) - h \mid \mathfrak{p} \in L_i(S)\} + i & \text{if } \ h \leq i \leq \ell(S^+) \end{cases}$$

is called the $i-th$ *local reduction number* of S.

Recall that if $G(I)$ is CM, then $a_0(G(I)) = -\infty$. So this new definition reduces to the old definition [1] in the case that the graded ring is CM. Set

$$a^*(S) := \inf\{a \in Z \mid [H^i_M(S)]_n = 0 \quad \text{for all} \quad n > a \quad \text{and} \quad i \leq d\}$$

where $d = \dim S$ and M is the maximal graded ideal of S.

By studying the relationship between the local reduction numbers and the a^*-invariant of a graded ring, we get interesting results. Now, we summarize some important results of this paper. The first main result of this paper is the following theorem.

THEOREM 2.7. *Let J be an arbitrary minimal reduction of S^+ generated by forms of degree* 1 *and $\ell(S^+) = \ell$. Suppose that S satisfies the Serre's condition (S_ℓ). Then*

(i) $\max\{r_{\ell-1}(S)+1, r_J(S^+)\} = r_\ell(S)$.
(ii) $\max\{r_{\ell-1}(S)+1, a^*(S)+\ell\} = r_\ell(S)$.

This result has some interesting consequences. First, we obtain Theorem 3.2 which is a generalization of Theorem 4.4 [1]. Especially, if I is equimultiple (i.e., ht $(I) = \ell(I)$), then we have the following result.

THEOREM 3.3. *Let I be an equimultiple ideal of A with $h = \mathrm{ht}(I) > 0$ and let J be a minimal reduction of I. Suppose that $G(I)$ satisfies the Serre's condition (S_h) and A is equidimensional and catenary. Then $r_J(I) = r_h(I) = a^*(G(I)) + h$.*

By using the above results to study the Gorenstein property of $R(I)$, we get the following second main result which is a generalization of [1, Theorem 5.8].

THEOREM 4.1. *Let I be an ideal of A. Suppose that* grade $I \geq 2$, *A satisfies the Serre's condition $(S_{\ell(I)})$. Then $R(I)$ is Gorenstein if and only if the following conditions are satisfied*:
(i) *$G(I)$ is Gorenstein.*
(ii) $r(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) - 2$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\mathrm{ht}(\mathfrak{p}/I) = 0$.
(iii) $r(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) - 2$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\ell(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) \leq \ell(I)$.
In this case, A is a Gorenstein ring.

In addition assume that I is equimultiple, then as an immediate consequence of Theorem 4.1 we have the following theorem.

THEOREM 4.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with* $\dim A = d \geq 2$, *I an equimultiple ideal of A with* grade $I \geq 2$. *Assume that A satisfies the Serre's condition $(S_{\mathrm{ht}(I)})$. Then $R(I)$ is a Gorenstein ring if and only if $G(I)$ is Gorenstein and* $r(I) = \mathrm{ht}(I) - 2$. *In this case, A is Gorenstein.*

Finally, we close the paper with a different proof of the following theorem.

THEOREM 4.3 [15, Theorem 4.1]. *Let I be an ideal of A with* $\mathrm{ht}(I) > 0$. *Suppose that* $\dim A = d$. *Then $R(I)$ is CM if and only if the following conditions are satisfied*:
(i) $[H^i_M(G(I))]_n = 0$ *for all* $n \neq -1, i = 0, \ldots, d-1$.
(ii) $r(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) - 1$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\ell(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$.

We emphasize that Johnston-Katz's method in [9] yielded important information concerning the so-called *Castelnuovo regularity of a graded ring S with respect to an ideal of S*, see [9, Proposition 2.1]. Using Proposition 2.1 [9], we can give a single proof of Theorem 4.3.

2. Local reduction numbers of a graded algebra

In this section, we define the notion of local reduction numbers and give the proof of Theorem 2.7.

Let I be an ideal of A. An ideal J is called a *reduction* of I if $J \subseteq I$ and $I^{n+1} = JI^n$ for some non-negative integer n. A reduction J is called a *minimal reduction* if it does not properly contain any other reduction of I. These notions were introduced by Northcott and Rees [10]. They proved that every minimal reduction of I is minimally generated by $\ell = \ell(I) := \dim \bigoplus_{n\geq 0}(I^n/\mathfrak{m}I^n)$ elements, and $\ell(I)$ is called the analytic spread of I. It is well-known that $\mathrm{ht}(I) \leq \ell(I) \leq \dim A$ and the difference $\mathrm{ad}(I) := \ell(I) - \mathrm{ht}(I)$ is called the *analytic deviation* of I. In the case $\mathrm{ad}(I) = 0$, the ideal I is called *equimultiple.* Let J be a reduction of I. The least integer n such that $I^{n+1} = JI^n$ is called the *reduction number* of I *with respect to* J and we denote it by $r_J(I)$. The *reduction number* of I is defined by $r(I) := \min\{r_J(I) \mid J$ is a minimal reduction of $I\}$.

Throughout this paper let $S := \bigoplus_{n\geq 0} S_n$ be a finitely generated standard graded algebra over a Noetherian local ring S_0 having an infinite residue field. Recall that an graded ideal J *generated by* 1*-forms* of S is a *reduction* of S^+ if $J_n = S_n$ for some integer n. The least integer n such that $J_{n+1} = S_{n+1}$ is called the reduction number of S^+ with respect to J and we denote it by $r_J(S^+)$. Set

$$a_i(S) := \inf\{a \in Z \mid [H^i_{S^+}(S)]_n = 0 \quad \text{for all } \ n > a\}\,,$$

$$a^i(S) := \inf\{a \in Z \mid [H^i_M(S)]_n = 0 \quad \text{for all } \ n > a\}\,,$$

$$a^*(S) := \max\{a^i(S); \quad \text{for all } \ i \leq \dim S\}$$

where $d = \dim S$ and M is the maximal graded ideal of S.

Note that $a^d(S) = a(S)$, which is called the *a-invariant* of S [3] and if S is CM then $a^*(S) = a(S)$.

DEFINITION. Set $L_i(I) := \{I \subseteq \mathfrak{p} \in \mathrm{Spec}\, A \mid \ell(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) \leq i\}$; $i \leq \ell(I)$, $\mathrm{ht}\,(I) = h$. The number

$$r_i(I) = \begin{cases} \max\{-h, a_0(G(I))\} + i & \text{if } \ i < h \\ \max\{a_0(G(I)), r(I_\mathfrak{p}) - \mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in L_i(I)\} + i & \text{if } \ h \leq i \leq \ell(I) \end{cases}$$

is called the *$i-th$ local reduction number* of I.

For any $\mathfrak{p} \in \mathrm{Spec}\, S_0$ we denote by $S_\mathfrak{p}$ the localization of S at the multiplicatively closed set $S_0\backslash\mathfrak{p}$. Set $\mathrm{ht}(S^+) = h$,

$$L_i(S) = \{\mathfrak{p} \in \ \mathrm{Spec}\, S_0 \mid \ell(S^+_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p}) + h \leq i\};\ \text{for}\ i \leq \ell(S^+).$$

The number

$$r_i(S) = \begin{cases} \max\{-h, a_0(S)\} + i & \text{if } \ i < h \\ \max\{a_0(S), r(S^+_\mathfrak{p}) - \mathrm{ht}(\mathfrak{p}) - h \mid \mathfrak{p} \in L_i(S)\} + i & \text{if } \ h \leq i \leq \ell(S^+) \end{cases}$$

is called the *$i-th$ local reduction number* of S.

A sequence $x_1, \ldots, x_r$ of homogeneous elements of S is called $[t_1, \ldots, t_r]$*-regular* if $[(x_1, \ldots, x_{i-1}) : x_i]_n = (x_1, \ldots, x_{i-1})_n$ for all $n \geq t_i, i = 1, \ldots, r$ [1], [11]. If all

$t_1, \dots, t_r$ are finite then $x_1, \dots, x_r$ is called a *filter-regular* sequence [11]. By [11, Lemma 3.1], every minimal reduction of S^+ can be minimally generated by a filter-regular sequence of homogeneous forms of degree 1.

Let J be a minimal reduction of S^+. We denote by $S(J)$ the least number t such that there exists a homogeneous minimal generating set $x_1, \dots, x_\ell$ of J which is $[t+1, \dots, t+\ell]$-regular on S [1].

In [14, Theorem 2.2] Trung gave the following characterization of $a^*(S)$.

LEMMA 2.1 [14, Theorem 2.2]. *Let J be an arbitrary reduction of S^+ generated by forms of degree* 1. *Then* $a^*(S) = \max\{S(J), r_J(S^+) - \ell(S^+)\}$.

This lemma is a bridge between the a^*-invariant and the local reduction numbers, which is an important key to the proof of Theorem 2.7.

LEMMA 2.2 [15, Proposition 2.1 (iv)]. *Let J be a minimal reduction of S^+. If Y is a minimal reduction of S^+ such that $r(S^+) = r_Y(S^+)$ then*

$$r_J(S^+) \leq \max\{S(J) + \ell(S^+), S(Y) + \ell(S^+)\}.$$

Although the notion of local reduction numbers is generalized, it is an easy check that the following result holds.

REMARK 2.3.

(i) $r_{i+1}(S) = r_i(S) + 1$ for all $i < h - 1$,

$$r_{i+1}(S) = \max\{r_i(S) + 1, r(S^+_{\mathfrak{p}}) \mid \mathfrak{p} \in L_{i+1}(S) \backslash L_i(S)\}, h \leq i \leq \ell(S^+) - 1,$$

$$r_h(S) = \max\{a_0(S) + h, r(S^+_{\mathfrak{p}}) \mid \mathrm{ht}(\mathfrak{p}) = 0\}$$
$$\geq \max\{0, a_0(S) + h\} = r_{h-1}(S) + 1.$$

(ii) $r_i(S_{\mathfrak{p}}) \leq r_i(S), i \leq \ell(S^+)$.

Let us consider the following conditions.

$(C_i) : [(x_1, \dots, x_i) : x_{i+1}]_n = (x_1, \dots, x_i)_n$ for all $n \geq r_i(S) + 1, 0 \leq i < \ell$.

$(C_\ell) : r_J(S^+) \leq r_\ell(S)$.

Using the same arguments as in the proofs of Lemma 3.4 and Lemma 3.5 in [1], we get the following lemmas.

LEMMA 2.4. *Let J be a minimal reduction of S^+ generated by forms of degree* 1 *and $\ell(S^+) = \ell$. Suppose that S satisfies the Serre's condition (S_ℓ) and $J = (x_1, \dots, x_\ell)$. Fix i such that $0 \leq i < \ell$. Assume that the sequence $x_1, \dots, x_\ell$ satisfies (C_j) for all $0 \leq j < i$. Let $\mathfrak{P} = \mathfrak{p} + S^+$ for $\mathfrak{p} \in \mathrm{Spec}\, S_0$ with $\mathrm{ht}(\mathfrak{P}) > i$. Then*

$$[H^k_{\mathfrak{P}_{\mathfrak{p}}}(S_{\mathfrak{p}}/(x_1, \dots, x_i)_{\mathfrak{p}})]_n = 0$$

for all $n \geq r_{i-1}(S) + 2$ and all $k < \min\{\mathrm{ht}\, \mathfrak{P}, \ell\} - i$.

LEMMA 2.5. *Let S, J, ℓ be as in Lemma 2.4. Let U_i be the intersection of the primary components of the ideal $(x_1, \dots, x_i)$ such that their associated prime ideals contain S^+ and have the height at most i,*

$$V_i = \bigcup_{n\geq 0}[(x_1, \dots, x_i) : (S^+)^n] .$$

Assume that (C_j) holds for all $0 \leq j < i < \ell$. Then $[U_i \cap V_i]_n = (x_1, \dots, x_i)_n$ for all $n \geq r_{i-1} + 2$.

Note that when S satisfies the Serre's condition (S_ℓ), the proofs of Lemma 2.4 and Lemma 2.5 only use the increasing property of a sequence of local reduction numbers as in [1] and [15]. But this property is also true for our new local reduction numbers by Remark 2.3. So using the same arguments as in [1] and [15] we get the above lemmas.

Satisfactory tools for the proof of Theorem 2.7 are Lemma 2.1 and the following proposition.

PROPOSITION 2.6. *Let J be a minimal reduction of S^+ generated by forms of degree 1 and $\ell(S^+) = \ell$. Suppose that S satisfies the Serre's condition (S_ℓ). Then*

(i) *For any filter-regular sequence $x_1, \dots, x_\ell$ of S which generates J, $x_1, \dots, x_\ell$ is $[r_0(S) + 1, \dots, r_{\ell-1}(S) + 1]$ -regular.*

(ii) $r_J(S^+) \leq r_\ell(S)$.

By using the above lemmas and the notion of our local reduction numbers and by arguing as in the proof of [1, Theorem 3.2], we see that the key to the proof of this proposition is the starting point of the following inductive argument.

PROOF. Set $J_i = (x_1, \dots, x_i)$, $\operatorname{ht}(S^+) = h$. Note that

$$r_0(S) = \begin{cases} \max\{-h, a_0(S)\} \text{ if } 0 < h \\ \max\{a_0(S), r(S^+_{\mathfrak{p}}) - \operatorname{ht}(\mathfrak{p}) - h \mid \mathfrak{p} \in L_0(S)\} \quad \text{if } h \leq 0 \leq \ell(S^+) . \end{cases}$$

The proof is by induction on $d = \dim S$. If $d = 0$ then $\ell = h = 0$, so we only need to check the condition (C_0). In this case, we have $J = 0$. Consequently

$$r_0(S) = \max\{a_0(S), r(S^+_{\mathfrak{p}}) \mid \mathfrak{p} \in L_0(S)\} = a_0(S) = r(S^+) = r_J(S^+) .$$

Hence (C_0) holds. Thus, the result is true for $d = 0$.

Assume now that $d > 0$. Suppose that the result has been proved for $\dim S < d$. We need to show that the result is true for d.

We first will prove by induction on i that (C_i) holds whenever $i < \ell = \ell(S^+)$.

We need to check the condition (C_0). We shall see that the conclusion follows from the following cases.

Note that if $\operatorname{ht}(S^+) = 0$ then $r_0(S) = \max\{a_0(S), r(S^+_{\mathfrak{p}}) \mid \mathfrak{p} \in L_0(S)\}$ and if $h = \operatorname{ht}(S^+) > 0$ then $r_0(S) = \max\{-h, a_0(S)\}$.

Suppose that $h = \operatorname{ht}(S^+) = 0$.

If $L_0(S) = \emptyset$. Then $r_0(S) = a_0(S)$. Thus, (C_0) is true.

If $L_0(S) \neq \emptyset$. By $\mathfrak{p} \in L_0(S)$, $\ell(S^+_{\mathfrak{p}}) = 0$, $S^+_{\mathfrak{p}}$ is nilpotent. From this it follows that $r(S^+_{\mathfrak{p}}) = a_0(S_{\mathfrak{p}}) \leq a_0(S)$. Hence $r_0(S) = a_0(S)$, so (C_0) is also true.

Now suppose that $h = \operatorname{ht}(S^+) > 0$.

If $H^0_{S^+}(S) = 0$. Then $r_0(S) = -h$ and $0 : x_1 = 0$. Hence (C_0) is true.

If $H^0_{S^+}(S) \neq 0$. Then $a_0(S) \geq 0$ and $r_0(S) = a_0(S)$. So (C_0) is true.

Thus, the condition (C_0) has been proved.

Assume that $i \geq 1$ and assume by induction that (C_j) holds for all $0 \leq j < i$. As the next step, we claim that (C_i) is true. By Lemma 2.5 we know that

$$(x_1, x_2, \ldots, x_i)_n = [U_i \cap V_i]_n \quad \text{for all } n \geq r_i(S) + 1$$

since $r_i(S) + 1 \geq r_{i-1}(S) + 2$. Let $\mathfrak{P} \in \operatorname{Ass}(S/U_i)$ and $\mathfrak{p} = \mathfrak{P} \cap S_0$. Since $\operatorname{ht}(\mathfrak{P}) \leq i < \ell$, it follows that $\dim S_{\mathfrak{p}} \leq i < \ell$. Then by [1, Corollary 2.2 and Lemma 2.4], $(J_i)_{\mathfrak{p}}$ is a reduction of $S^+_{\mathfrak{p}}$. Since $\dim S_{\mathfrak{p}} \leq i < \ell \leq \dim S = d$, applying the inductive hypothesis to $S_{\mathfrak{p}}$ we have

$$r_{(J_i)_{\mathfrak{p}}}(S^+_{\mathfrak{p}}) \leq r_k(S_{\mathfrak{p}}) \leq r_k(S) \leq r_i(S)$$

for $k = \min\{i, \ell(S^+_{\mathfrak{p}})\}$. Thus,

$$[(J_i)_{\mathfrak{p}}]_n = [S_{\mathfrak{p}}]_n \text{ for all } n \geq r_i(S) + 1 .$$

By [1, Lemma 3.3] we get $[U_i]_n = [S_i]_n$ for all $n \geq r_i(S) + 1$.

Next, the same arguments as in the proof of [1, Theorem 3.2] show (C_i) if $i < \ell$. Thus, the induction on i is complete. So we have proved that (C_i) is true for all $i < \ell$. This implies that $S(J) \leq r_{\ell-1}(S) - \ell + 1$. Then by Lemma 2.2, we get $r_J(S^+) \leq r_{\ell-1}(S) + 1$. By this fact and $r_{\ell-1}(S) + 1 \leq r_\ell(S)$, it follows that $r_J(S^+) \leq r_\ell(S)$. Hence (C_ℓ) holds. So the induction on d is complete. Proposition 2.6 has been proved.

THEOREM 2.7. *Let J be an arbitrary minimal reduction of S^+ generated by forms of degree* 1 *and $\ell(S^+) = \ell$. Suppose that S satisfies the Serre's condition (S_ℓ). Then*

(i) $\max\{r_{\ell-1}(S) + 1, r_J(S^+)\} = r_\ell(S)$.

(ii) $\max\{r_{\ell-1}(S) + 1, a^*(S) + \ell\} = r_\ell(S)$.

PROOF. By Remark 2.3 (i), we get

$$\begin{aligned} r_\ell(S) &= \max\{r_{\ell-1}(S) + 1, r(S^+_{\mathfrak{p}}) \mid \mathfrak{p} \in L_\ell(S) \backslash L_{\ell-1}(S)\} \\ &\leq \max\{r_{\ell-1}(S) + 1, r_J(S^+)\} . \end{aligned}$$

By Proposition 2.6, we have $r_J(S^+) \leq r_\ell(S)$. Using the above facts and $r_{\ell-1}(S) + 1 \leq r_\ell(S)$ we can deduce that

$$r_\ell(S) = \max\{r_{\ell-1}(S) + 1, r_J(S^+)\} .$$

Hence (i) of Theorem 2.7 holds. By Lemma 2.1, we get

$$a^*(S) + \ell = \max\{S(J) + \ell, r_J(S^+)\} .$$

Therefore

$$\max\{r_{\ell-1}(S)+1, a^*(S)+\ell\} = \max\{S(J)+\ell, r_{\ell-1}(S)+1, r_J(S^+)\}\,.$$

By Proposition 2.6 we have $S(J)+\ell \le r_{\ell-1}(S)+1$. Using this inequality we get

$$\max\{S(J)+\ell, r_{\ell-1}(S)+1, r_J(S^+)\} = \max\{r_{\ell-1}(S)+1, r_J(S^+)\}\,.$$

Combining the above facts we obtain

$$\max\{r_{\ell-1}(S)+1, a^*(S)+\ell\} = \max\{r_{\ell-1}(S)+1, r_J(S^+)\}\,.$$

This equality together with the equality (i) proves that

$$\max\{r_{\ell-1}(S)+1, a^*(S)+\ell\} = \max\{r_{\ell-1}(S)+1, r_J(S^+)\} = r_\ell(S)\,.$$

Thus, (ii) of Theorem 2.7 holds. The proof of Theorem 2.7 is now completed.

3. Local reduction numbers of ideals

In this section we give some interesting applications of Theorem 2.7.

We will be concerned with a Noetherian local ring $(A, \mathfrak{m}, k)$ of $\dim A = d > 0$ having an infinite residue field k. Set $G = G(I)$ and $G^+ = \bigoplus_{n\ge 1}(I^n/I^{n+1})$. For any prime ideal $\mathfrak{p}$ of A we denote by $G_\mathfrak{p}$ the localization of G at the multiplicative closed set $A\backslash\mathfrak{p}$. Set $R = R(I)$. It can be verified that if $\mathfrak{P} = \mathfrak{p} + R^+(I)$ then $\dim G_\mathfrak{P} = \dim G_\mathfrak{p}$ and $G(I_\mathfrak{p}) = G(I)_\mathfrak{p}$. Set $h = \mathrm{ht}\,(I)$, $\ell = \ell(I)$.

REMARK 3.1. If A is equidimensional and catenary and $\mathrm{ht}(I) = h$, then

$$r_i(G) = \begin{cases} \max\{-h, a_0(G)\}+i & \text{if } i < h \\ \max\{a_0(G), r(G^+_\mathfrak{p}) - \mathrm{ht}(\mathfrak{p}) \mid \mathfrak{p} \in L_i(I)\}+i & \text{if } h \le i \le \ell \end{cases}$$

and since $r(G^+_\mathfrak{p}) = r(I_\mathfrak{p})$ for every $\mathfrak{p} \in$ Spec A, we have $r_i(G) = r_i(I)$ for all $i \le \ell$.

Then as immediate consequences of Theorem 2.7 we have the following results.

THEOREM 3.2. *Let I be an ideal of A with* $\mathrm{ht}(I) > 0$. *Set $\ell = \ell(I)$. Let J be a minimal reduction of I. Suppose that $G(I)$ satisfies the Serre's condition (S_ℓ) and A is equidimensional and catenary. Then*

(i) $\max\{r_{\ell-1}(I)+1, r_J(I)\} = r_\ell(I)$.
(ii) $\max\{r_{\ell-1}(I)+1, a^*(G)+\ell\} = r_\ell(I)$.

In the case that A and $G(I)$ are CM, we get Theorem 4.4 in [1]. And in the case that I is equimultiple, we get the following theorem.

THEOREM 3.3. *Let I be an equimultiple ideal of A with $h = \mathrm{ht}(I) > 0$ and let J be a minimal reduction of I. Suppose that $G(I)$ satisfies the Serre's condition (S_h) and A is equidimensional and catenary. Then $r_J(I) = r_h(I) = a^*(G)+h$.*

PROOF. We see that if I is an equimultiple ideal of A with $\mathrm{ht}(I) = h > 0$, then $L_i(I) = \emptyset$ for all $i < h$. From this it follows that

$$\max\{r_{h-1}(I) + 1, r_J(I)\} = \max\{0, a_0(G) + h, r_J(I)\}$$
$$= \max\{a_0(G) + h, r_J(I)\} = r_h(I)$$

and

$$\max\{r_{h-1}(I) + 1, a^*(G) + h\} = \max\{0, a_0(G) + h, a^*(G) + h\} = r_h(I)$$

By [14, Corollary 2.8] we have $a^*(G) \geq a_0(G)$. Hence

$$r_J(I) = \max\{0, r_J(I)\} = r_h(I) = \max\{0, a^*(G) + h\}\,.$$

By Lemma 2.1, $a^*(G) + h \geq r_J(I)$. So $r_h(I) = r_J(I) = a^*(G) + h$.

Note that

$$r_h(I) = \max\{a_0(G) + h, r(I_{\mathfrak{p}}) \mid I \subseteq \mathfrak{p} \in \mathrm{Spec}\, A \text{ with } \mathrm{ht}\,(\mathfrak{p}) = h\}\,.$$

As a consequence of Theorem 3.3 we have the following proposition.

PROPOSITION 3.4. *Let I be an equimultiple ideal of of A with $h = \mathrm{ht}(I) > 0$ and let J be a minimal reduction of I. Suppose that grade $G(I)^+ \geq h$ and A is equidimensional and catenary. Then*

$$r_J(I) = a^*(G(I)) + h = \max\{r(I_{\mathfrak{p}}) \mid I \subseteq \mathfrak{p} \in \mathrm{Spec}\, A \textit{ with ht } (\mathfrak{p}) = h\}\,.$$

PROOF. Since grade $G(I)^+ \geq h$, $G(I)$ satisfies the Serre's condition (S_h). By Theorem 3.3, $r_J(I) = a^*(G) + h$. Since grade $G^+ > 0$,

$$r_h(I) = \max\{r(I_{\mathfrak{p}}) \mid I \subseteq \mathfrak{p} \in \mathrm{Spec}\, A \text{ with } \mathrm{ht}\,(\mathfrak{p}) = h\}\,.$$

In the case that $G(I)$ is CM, from Proposition 3.4 we immediately get the following corollary.

COROLLARY 3.5. *Let I be an equimultiple ideal of A and let J be a minimal reduction of I. Suppose that $G = G(I)$ is CM and A is equidimensional and catenary. Then*

$$r_J(I) = a(G) + h = \max\{r(I_{\mathfrak{p}}) \mid I \subseteq \mathfrak{p} \in \mathrm{Spec}\, A \textit{ with } \mathrm{ht}\,(\mathfrak{p}) = h\}\,.$$

And I is an ideal of the principal class if and only if $a(G) = -h$.

4. On the Cohen-Macaulayness and Gorensteinness of Rees algebras

In this section, we first will use the results of Section 3 to study the Gorenstein property of the Rees algebra $R(I)$. Next, we give a different proof of Theorem 4.1 [15].

THEOREM 4.1. *Suppose that I is an ideal with* grade $I \geq 2$, *A satisfies the Serre's condition $(S_{\ell(I)})$. Then $R(I)$ is Gorenstein if and only if the following conditions are satisfied:*

(i) *$G(I)$ is Gorenstein.*
(ii) $r(I_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) - 2$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\mathrm{ht}(\mathfrak{p}/I) = 0$.
(iii) $r(I_{\mathfrak{p}}) \leq \mathrm{ht}(\mathfrak{p}) - 2$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\ell(I_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) \leq \ell(I)$.
In this case, A is a Gorenstein ring.

PROOF. ($\Rightarrow$) Set $R = R(I)$ and $G = G(I)$. Let $\mathfrak{P}$ be a prime ideal of R with $\mathfrak{P} \cap A = \mathfrak{p} \in$ Spec A. If $\mathfrak{P} \not\supseteq R^+$ then $G_{\mathfrak{P}}$ is CM by [12, Proposition 3.3]. If $\mathfrak{P} \supseteq R^+$ then from R is CM, by [12, Theorem 2.1] we get $[H^i_{\mathfrak{P}_{\mathfrak{p}}}(G_{\mathfrak{p}})] \simeq H^i_{\mathfrak{p}A_{\mathfrak{p}}}(A_{\mathfrak{p}})$ for all $i < \dim A_{\mathfrak{p}}$. Combining the above facts with the property of A, it follows that G satisfies the Serre's condition (S_ℓ). Let $\mathfrak{p} \sqsupseteq I$ be a minimal prime ideal of I. After localizing we get that $I_{\mathfrak{p}}$ is $\mathfrak{p}A_{\mathfrak{p}}$-primary. $R_{\mathfrak{p}}$ Gorenstein, by [12, Theorem 2.1] we have $a^i(G_{\mathfrak{p}}) < 0$ for all $i < \mathrm{ht}\,(\mathfrak{p}) = \mathrm{ht}\,(I) := h$. And by [8], $a^h(G_{\mathfrak{p}}) = a(G_{\mathfrak{p}}) = -2$. By [11, Proposition 3.2] we get $r(I_{\mathfrak{p}}) = a(G_{\mathfrak{p}}) + h = h - 2$. This gives (ii). We now prove (iii) by induction on $\mathrm{ht}(\mathfrak{p})$ with $I \subseteq \mathfrak{p}$ and $\ell(I_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) \leq \ell(I)$. Assume by induction that (iii) holds for all $\mathfrak{q} \sqsupseteq I$ with $\ell(I_{\mathfrak{q}}) = \mathrm{ht}(\mathfrak{q}) < \mathrm{ht}(\mathfrak{p})$. Set $\ell(I_{\mathfrak{p}}) := \ell_{\mathfrak{p}}$. Since A satisfies the Serre's condition (S_ℓ), $A_{\mathfrak{p}}$ is CM. Note that $R_{\mathfrak{p}}$ is Gorenstein. Hence, by [13, Corollary 3.5], $G_{\mathfrak{p}}$ is CM and $a^*(G_{\mathfrak{p}}) = a(G_{\mathfrak{p}}) = -2$. Applying Theorem 3.2 we get

$$r_{\ell_{\mathfrak{p}}}(I_{\mathfrak{p}}) = \max\{r_{\ell_{\mathfrak{p}}-1}(I_{\mathfrak{p}}) + 1, a^*(G_{\mathfrak{p}}) + \ell_{\mathfrak{p}}\}.$$

Since

$$r_{\ell_{\mathfrak{p}}-1}(I_{\mathfrak{p}}) = \begin{cases} \max\{-\mathrm{ht}\,(I_{\mathfrak{p}}), a_0(G_{\mathfrak{p}})\} + \ell_{\mathfrak{p}} - 1 & \text{if } \ell_{\mathfrak{p}} \leq \mathrm{ht}(I_{\mathfrak{p}}) \\ \max\{a_0(G_{\mathfrak{p}}), r(I_{\mathfrak{q}}) - \mathrm{ht}(\mathfrak{q}) \mid \mathfrak{q} \in L_{\ell_{\mathfrak{p}}-1}(I_{\mathfrak{p}})\} + \ell_{\mathfrak{p}} - 1 & \text{if } \ell_{\mathfrak{p}} > \mathrm{ht}(I_{\mathfrak{p}}) \end{cases}$$

and grade $G^+_{\mathfrak{p}} > 0$, by the inductive hypothesis $r(I_{\mathfrak{q}}) \leq \mathrm{ht}(\mathfrak{q}) - 2$ we have

$$r_{\ell_{\mathfrak{p}}-1}(I_{\mathfrak{p}}) \leq \ell_{\mathfrak{p}} - 3\,.$$

Combining this inequality with $a^*(G_{\mathfrak{p}}) = -2$ and Theorem 3.2 (ii) we get

$$r_{\ell_{\mathfrak{p}}}(I_{\mathfrak{p}}) \leq \ell_{\mathfrak{p}} - 2 = \mathrm{ht}(\mathfrak{p}) - 2\,.$$

Hence, by $r(I_{\mathfrak{p}}) \leq r_{\ell_{\mathfrak{p}}}(I_{\mathfrak{p}})$ we get $r(I_{\mathfrak{p}}) \leq \mathrm{ht}(\mathfrak{p}) - 2$. Thus, (iii) holds. By the condition (iii) we know that $r(I_{\mathfrak{p}}) \leq \mathrm{ht}(\mathfrak{p}) - 2$ for every prime ideal $\mathfrak{p} \sqsupseteq I$ with

$$\ell(I_{\mathfrak{p}}) = \mathrm{ht}(\mathfrak{p}) \leq \ell(I)\,.$$

Therefore $r_i(I) \leq i - 2$ for all $i \leq \ell(I)$. Applying Theorem 3.2 we get $a^*(G) \leq -2$. Since $R(I)$ CM, by [12, Theorem 2.1] we have

$$[H^i_M(G)]_n = 0 \quad \text{for all } n \neq -1\,, \quad i = 0, \ldots, d-1$$

where M is the maximal graded ideal of $R(I)$. Hence $H^i_M(G) = 0$ for all $i < d$. Thus, G is CM. Hence A is a CM ring by [12]. $R(I)$ is Gorenstein, by [13, Corollary 3.5] we get that A and G are Gorenstein.

($\Leftarrow$) The proof is immediate from [1, Theorem 5.8].

If I is equimultiple, then from Theorem 4.1 and Corollary 3.5 we immediately get the following result.

THEOREM 4.2. *Let $(A, \mathfrak{m})$ be a Noetherian local ring with* $\dim A = d \geq 2$, *I an equimultiple ideal of A with* grade $I \geq 2$. *Assume that A satisfies the Serre's condition $(S_{\mathrm{ht}(I)})$. Then $R(I)$ is a Gorenstein ring if and only if $G(I)$ is Gorenstein and $r(I) = \mathrm{ht}(I) - 2$. In this case, A is Gorenstein.*

By the study of the relationship between the local reduction numbers of an ideal I and the a^*-invariant of the Rees algebra $R(I)$ satisfying the Serre's condition $(S_{\ell(I)})$, Viet in [15] gave the following theorem.

THEOREM 4.3 [15, Theorem 4.1]. *Let I be an ideal of A with* $\mathrm{ht}(I) > 0$. *Suppose that* $\dim A = d$ *and M is the maximal graded ideal of $R(I)$. Then $R(I)$ is CM if and only if the following conditions are satisfied:*

(i) $[H^i_M(G(I))]_n = 0$ *for all* $n \neq -1, i = 0, \ldots, d-1$.

(ii) $r(I_\mathfrak{p}) \leq \mathrm{ht}(\mathfrak{p}) - 1$ *for every prime ideal* $\mathfrak{p} \sqsupseteq I$ *with* $\ell(I_\mathfrak{p}) = \mathrm{ht}(\mathfrak{p})$.

We would like to point out that by applying [11, Proposition 3.2] and the following lemma [9, Proposition 2.1], we can give a different proof of this theorem.

LEMMA 1 [9, Proposition 2.1]. *Let $S := \bigoplus_{n \geq 0} S_n$ be a finitely generated standard graded algebra over a Noetherian local ring S_0. Set $d(p) = \dim S_p$ and $\mathfrak{p} = p + S^+$ with $p \in$ Spec S_0. Suppose that $\dim S = d$. Then*

(i) *Let r be an integer such that $H^i_\mathfrak{p}(S_p)_j = 0$, for all $j \geq r$, $p \in$ Spec S_0, and all i, $0 \leq i \leq d(p)$. Then for any homogeneous ideal $\mathfrak{J}$ which contains S^+, $H^i_\mathfrak{J}(S)_j = 0$ for all $j \geq r$ and $0 \leq i \leq d$.*

(ii) *Conversely, suppose that s be an integer satisfying $H^i_{S^+}(S)_j = 0$ for all $j \geq s$ and $0 \leq i \leq d$. Then, for all $p \in$ Spec S_0 and $0 \leq i \leq d(p)$, $H^i_\mathfrak{p}(S_p)_j = 0$ for all $j \geq s$.*

(iii) *Let t be an integer such that $H^{d(p)}_{S^+_p}(S_p)_j = 0$, for all $j \geq t$ and $p \in$ Spec S_0. Then for any homogeneous ideal $\mathfrak{J}$ which contains S^+, $H^d_\mathfrak{J}(S)_j = 0$, for all $j \geq t$.*

PROOF OF THEOREM 4.3. ($\Rightarrow$) Set $R = R(I)$ and $G = G(I)$, $d(p) = \dim A_p$,

$$a_i(G) := \inf\{a \in Z \mid [H^i_{G^+}(G)]_n = 0 \quad \text{for all} \quad n > a\},$$

$$a^i(G) := \inf\{a \in Z \mid [H^i_M(G)]_n = 0 \quad \text{for all} \quad n > a\},$$

$$a^*(G) := \max\{a^i(G); \quad \text{for all} \ \ i \leq \dim G\},$$

$$L(I) := \{I \subseteq p \in \text{Spec } S_0 \mid \ell(I_p) = \dim A_p\}.$$

Suppose that R is CM and $p \in L(I)$. By [12, Theorem 2.1] we get (i) and R_p is also CM. By [12, Theorem 2.1], $a^*(G_p) < 0$. For all $q \subseteq p$ and $q \in$ Spec A, set $q^* = qA_p$, we have

$(R_p)_{q^*}$ CM. Hence $a^*((G_p)_{q^*}) < 0$. By Lemma (i), $a_i(G_p) < 0$ for all i. Hence, by [11, Proposition 3.2], $r(I_p) \leq \ell(I_p) - 1 = \mathrm{ht}(p) - 1$. So (ii) holds.

($\Leftarrow$) First we show that $a_d(p)(G_p) < 0$ for all $p \in$ Spec A. If $p \notin L(I)$, then $\ell(I_p) < d(p)$, so $a_d(p)(G_p) = -\infty$. If $p \in L(I)$, by (ii) we have

$$r(I_p) \leq \mathrm{ht}(p) - 1 = d(p) - 1 .$$

Hence, by [11, Proposition 3.2] we get $a_d(p)(G_p) < 0$. So $a_d(p)(G_p) < 0$ for all $p \in$ Spec A. By Lemma(iii), $a(G) < 0$. Thus, R is CM by [12, Theorem 2.1]. The proof is complete.

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