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Koizumi-Type Interpolation Theorem and Extrapolation Estimates

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Dedicated to Professor Sumiyuki Koizumi on his seventy-fifth birthday

Abstract. In the present paper, we shall show a Koizumi-type interpolation theorem in order to show the relation between several Yano-type extrapolation estimates.

1. Introduction and Result

Let (Ω, μ) be a σ-finite measure space. We shall consider operators which transform each μ-measurable simple functions to some μ-measurable function. If such operator T satisfies $|T(f+g)| \leq C(|Tf| + |Tg|)$ and $|T(\alpha f)| = |\alpha||Tf|$ μ-*a.e.* for some fixed number $C \geq 1$, any measurable simple functions f, g and scalar α, we call T quasi-linear. If T satisfies $|T(f+g)| \leq |Tf| + |Tg|$ μ-*a.e.* , we call T sub-additive.

In [5], S. Koizumi showed the following interpolation theorem:

FACT A. *Suppose* $1 < p_1 < \infty$. *Let* T *be quasi-linear operator and satisfy*

(W1-p_1) $\qquad T$ *is weak-type* $(1, 1)$ *and strong type* (p_1, p_1) ,

then

$$(K_1) \qquad T : L^1 + L^q(\Omega) \longrightarrow L^1 \log L + L^q(\Omega) \quad \textit{bounded}, \quad \forall q \in (1, p_1] .$$

Orlicz classes appeared in (K_1), the sum of two spaces, are useful to consider extrapolation estimates over infinite measure space. In [8], the author showed

FACT B. *Let* $1 < p_1 < \infty$ $\alpha \geq 0$. *If* T *is a sub-additive operator and satisfies*

$$(Y_\alpha) \qquad \|Tf\|_{L^p(\Omega)} \leq A(p-1)^{-\alpha} \|f\|_{L^p(\Omega)} \quad \textit{for any } f \in L^p(\Omega, \mu),\ 1 < \forall p \leq p_1$$

then

$$(K_\alpha) \qquad T : L^1 + L^q(\Omega) \longrightarrow L^1(\log L)^\alpha + L^q(\Omega) \quad \textit{bounded}, \quad \forall q \in (1, p_1]$$

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which is an extension of Yano's classical extrapolation theorem ([9]). On the other hand, the Marcinkiewicz interpolation theorem asserts (W1-p_1) implies (Y_α) for $\alpha = 1$. So, Fact B is a partial extension of Fact A.

Also the following extrapolation estimate is known ([2][4]):

FACT C. *If we assume the condition (Y_α) for some $\alpha > 0$, then T satisfies*

$$(C_\alpha) \qquad \sup_{r>0} \frac{\int_\Omega (|Tf| - \frac{1}{r})_+ d\mu}{(1+\log^+ r)^\alpha} \le C \int_\Omega |f|(1+\log^+|f|)^\alpha d\mu .$$

for any $f \in L^1(\log L)^\alpha(\Omega)$. Here, $\log^+ r = \max(\log r, 0)$ *and* $(y)_+ = \max(y, 0)$, *for any* $r > 0$ *and* $y \in \mathbb{R}$.

In the present paper, in order to know the relation between two extrapolation estimates (K_α) and (C_α), we shall show the following: In §2, a Koizumi type interpolation theorem

$$\begin{cases} (C_\alpha) \\ L^q \text{ boundedness} \end{cases} \Longrightarrow (K_\alpha) \quad \text{for each } q > 1$$

is given. In §3, we give an extrapolation estimate

$$(K_\alpha) \Longrightarrow (C_\alpha) .$$

Moreover, we shall remark the relation between our interpolation theorem in §2 and original Koizumi's in §4.

2. Koizumi type interpolation theorem

In this section, we shall show the following Koizumi type interpolation theorem.

THEOREM 2.1. *Let (Ω, μ) be a σ-finite measure space. Suppose $1 < q < \infty$ and $\alpha > 0$. Suppose T be a sub-additive operator on $L^1(\log L)^\alpha + L^q(\Omega, \mu)$, there exists some $A > 0$ such that*

$$(2.1) \qquad \left[\int_\Omega |Tf(x)|^q d\mu(x)\right]^{1/q} \le A\left[\int_\Omega |f(x)|^q d\mu(x)\right]^{1/q} \quad \textit{for any} \quad f \in L^q(\Omega, \mu)$$

and there exists some $B > 0$ such that

$$(2.2) \qquad \sup_{r>0} \frac{\int_\Omega (|Tf(x)| - \frac{1}{r})_+ d\mu(x)}{(1+\log^+ r)^\alpha} \le B \int_\Omega |f(x)|(1+\log^+|f(x)|)^\alpha d\mu(x)$$

for any $f \in L^1(\log L)^\alpha$. Then, we conclude that there exists some $C > 0$ such that

$$\begin{aligned}(2.3)\qquad &\int_{|Tf|\le 1}|Tf(x)|^q d\mu(x)+\int_{|Tf|>1}|Tf(x)|d\mu(x)\\ &\le C\left[\int_{|f|\le 1}|f(x)|^q d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^\alpha d\mu(x)\right]\end{aligned}$$

PROOF. We shall prove this with the method of [5] and [7]. First, we divide each function f as

$$g(x)=\begin{cases} f(x)\,, & \text{if } |f(x)|\le 1\\ 0\,, & \text{elsewhere}\end{cases}\qquad \text{and put}\quad h(x)=f(x)-g(x)\,.$$

Now, we immediately get

$$(2.4)\qquad \int_{|Tg|\le 1}|Tg(x)|^q d\mu(x)\le A\int_\Omega |g(x)|^q d\mu(x)=A\int_{|f|\le 1}|f(x)|^q d\mu(x)\,,$$

$$\begin{aligned}(2.5)\qquad \int_{|Tg|>1}|Tg(x)|^1 d\mu(x)&\le \int_{|Tg|>1}|Tg(x)|^q d\mu(x)\\ &\le A\int_\Omega |g(x)|^q d\mu(x)=A\int_{|f|\le 1}|f(x)|^q d\mu(x)\end{aligned}$$

by (2.1).

Put $r=1$ on (2.2) and we have

$$\begin{aligned}(2.6)\qquad \int_{|Th|>1}|Th(x)|d\mu(x)&\le B\int_\Omega |h(x)|(1+\log^+|h(x)|)^\alpha d\mu(x)\\ &=B\int_{|f|>1}|f(x)|(1+\log|f(x)|)^\alpha d\mu(x)\end{aligned}$$

Finally, from (2.2), for $r=2^{n+2}$ $(n=0,1,2\dots)$, we have

$$\int_{|Th|>2^{-n-2}}(|Th(x)|-2^{-n-2})d\mu(x)\le B'(1+n)^\alpha\int_\Omega |h(x)|(1+\log^+|h(x)|)^\alpha d\mu(x)\,.$$

When $|Th(x)|\ge 2^{-n-1}$, $|Th(x)|\le 2(|Th(x)|-2^{-n-2})$. Hence, for each n,

$$\begin{aligned}&\int_{2^{-n-1}<|Th|\le 2^{-n}}|Th(x)|^q d\mu(x)\\ &\le \int_{2^{-n-1}<|Th|\le 2^{-n}}|Th(x)|2^{-n(q-1)}d\mu(x)\\ &\le 2^{-n(q-1)}\int_{2^{-n-1}<|Th|\le 2^{-n}}2(|Th(x)|-2^{-n-2})d\mu(x)\end{aligned}$$

$$\leq 2^{-n(q-1)+1}\int_{|Th|>2^{-n-2}}(|Th(x)|-2^{-n-2})d\mu(x)$$
$$\leq B'(1+n)^{\alpha}2^{-n(q-1)+1}\int_{\Omega}|h(x)|(1+\log^{+}|h(x)|)^{\alpha}d\mu(x)\,.$$

Summing up with respect to n, we have

$$\begin{aligned}\int_{|Th|\leq 1}|Th(x)|^{q}d\mu(x) &\leq C'\int_{\Omega}|h(x)|(1+\log^{+}|h(x)|)^{\alpha}d\mu(x)\\ &= C'\int_{|f|>1}|f(x)|(1+\log^{+}|f(x)|)^{\alpha}d\mu(x)\,.\end{aligned} \tag{2.7}$$

Combining (2.4), (2.5), (2.6) and (2.7), we conclude (2.3) as same as Koizumi's work [5] with the lemma below.

LEMMA 2.2 (S. Koizumi [5]). *Assume* $A \leq B + C$ *and* $A, B, C > 0$ *and* $1 < q < \infty$.
(i) *When* $0 \leq A \leq 1$,

$$A \leq \begin{cases} B + C, & \textit{if } 0 \leq C \leq 1 \\ B + C^{\frac{1}{q}} & \textit{if } C > 1\,. \end{cases} \tag{2.8}$$

(ii) *When* $A > 1$,

$$A \leq \begin{cases} 2^{q}(B^{q} + C^{q}), & \textit{if } 0 \leq C \leq 1 \\ 2^{q}(B^{q} + C) & \textit{if } C > 1\,. \end{cases} \tag{2.9}$$

3. The relation between the extrapolation estimates

In this section, we shall give, so to say, an extrapolation estimate.

PROPOSITION 3.1. *Let* $\alpha \geq 0$. *If the measurable functions* f *and* g *satisfy*

$$\begin{aligned}&\int_{|g|\leq 1}|g(x)|^{q}d\mu(x)+\int_{|g|>1}|g(x)|d\mu(x)\\ &\leq\frac{C}{(q-1)^{\alpha}}\left[\int_{|f|\leq 1}|f(x)|^{q}d\mu(x)+\int_{|f|>1}|f(x)|(1+\log|f(x)|)^{\alpha}d\mu(x)\right]\end{aligned} \tag{3.1}$$

for any $1 < q < p_1$, *then they satisfy*

$$\sup_{r>0}\frac{\int_{\Omega}(|g(x)|-\frac{1}{r})_{+}d\mu(x)}{(1+\log^{+}r)^{\alpha}} \leq C'\int_{\Omega}|f(x)|(1+\log^{+}|f(x)|)^{\alpha}d\mu(x)\,. \tag{3.2}$$

PROOF. It is easy to show that the right hand side of (3.2) dominates the right hand side of (3.1). Then, we have

$$\sup_{0<r<1} \frac{\int_\Omega (|g(x)| - \frac{1}{r})_+ d\mu(x)}{(1+\log^+ r)^\alpha} \le \int_{|g|>1} |g(x)| d\mu(x)$$
$$\le C'' \int_\Omega |f(x)|(1+\log^+ |f(x)|)^\alpha d\mu(x) .$$

Let $r \ge 1$. From (3.1), for $q = 1 + \dfrac{p_1 - 1}{\log r}$,

$$\int_\Omega |f(x)|(1+\log^+ |f(x)|)^\alpha d\mu(x) \ge (q-1)^\alpha \int_{|g|\le 1} |g(x)|^q d\mu(x)$$
$$\ge \left(\frac{p_1 - 1}{\log r}\right)^\alpha \int_{e^{-\log r}<|g|\le 1} |g(x)| \cdot |g(x)|^{\frac{p_1-1}{n}} d\mu(x)$$
$$\ge (p_1-1)^\alpha \frac{\int_{\frac{1}{r}<|g|\le 1} |g(x)| \cdot |g(x)|^{\frac{p_1-1}{\log r}} d\mu(x)}{(\log r)^\alpha} d\mu(x) \approx \frac{\int_{\frac{1}{r}<|g|\le 1} |g(x)|}{(\log r)^\alpha} .$$

Now we conclude

$$\frac{\int_\Omega (|g(x)| - \frac{1}{r})_+ d\mu(x)}{(1+\log^+ r)^\alpha} \le C \int_\Omega |f(x)|(1+\log^+ |f(x)|)^\alpha d\mu(x)$$

for any $r > 0$.

4. Relation between two interpolation theorems

Here, we shall consider the relation between two interpolation theorems, Fact A and Theorem 2.1. M. Carro proved the following relation([3]).

PROPOSITION 4.1. *Let (Ω, μ) be a measure space. The space of mesurable functions on (Ω, μ)*

$$B_1 = \left\{ f : \|f\|_{B_1} = \inf\left(\lambda > 0 : \frac{\int_{\frac{1}{r}}^\infty \mu(|f| > \lambda y) dy}{(1+\log^+ r)} \le 1\right) < \infty \right\}$$

coincides to the space

$$M(t(1+\log^+ t)^{-1}) = \left\{ f : \|f\|_{M(t(1+\log^+ t)^{-1})} = \sup_{t>0} \left(\frac{t f^{**}(t)}{1 + t\log^+ t}\right) < \infty \right\}$$

and their norms are equivalent. Here,

$$f^*(t) = \inf\{\lambda > 0 : \mu(\{x \in \Omega : |f(x)| > \lambda\})\}$$

and $f^{**}(t) = \dfrac{1}{t} \int_0^t f^*(s) ds$.

From the definitions, it is easy to show

$$M(t(1+\log^+ t)^{-1}) = L_{(1,\infty;0,-1)} \supset L_{1,\infty}$$

(also see [6]). Therefore, for sub-linear operator (quasi-linear for $C = 1$ in §1), the boundedness of weak type (1,1) implies (C_α) for $\alpha = 1$ and we may conclude that our Theorem 2.1 is a partial extension of Fact A.

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