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Barnes' Double Zeta Function, the Dedekind Sum and Ramanujan's Formula

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1. Introduction

Let $\zeta_2(s; w; \omega_1, \omega_2)$ be the Barnes double zeta function [2], [9].

In the present paper, we show that the residue computation of the contour integral representation of $\zeta_2(s; 1; 1, \omega)$ yields

(1) the reciprocity formula of Apostol's generalized Dedekind sum [1] for rational ω, and

(2) Ramanujan's formula for values of Riemann zeta function at positive odd arguments [6], [8] as the limit case of the formula obtained for irrational ω. This shows that, in a sense, the Dedekind sum and Ramanujan's formula live on the same ground provided by the Barnes double zeta function.

As for (1), more generally, we shall derive the reciprocity formula for the Apostol-Rademacher Dedekind sum using more general ζ_2.

In [10], the authors investigated three kinds of Dedekind sums of Apostol and Apostol-Rademacher type, by computing values of Barnes' double zeta functions at non-positive integers and derived their reciprocity laws. Their method is algebraic. Our method is analytic.

In (2) the formula can be viewed as a limit case $x + i\omega \to i\omega$ (ω: irrational). This seems a new view point for Ramanujan's formula.

So, as for limit cases of (1), (2), we may think of a proof of the reciprocity formula of Gaussian sum [12], of Riemann's Fragmente [11] in which Riemann considered the limit cases of formulas in Jacobi [7], and of Dedekind's Erläuterungen [5] to it. This point of "limiting" view will be important for further investigation of the Dedekind sum and Ramanujan's formula.

Our method is very powerful. Barnes' multiple Riemann zeta functions of various types relate with Dedekind sums of various types.

In a subsequent paper, we shall consider Barnes' triple Riemann zeta function. Then, in particular, we can derive the formula to be called "triple term formula" for Apostol's Dedekind sum, which is different from Rademacher's for ordinary Dedekind sum.

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2. Barnes' double zeta funtion

We define the Barnes double zeta function [2] (cf. [9]) by

$$\zeta_2(s; w; \omega_1, \omega_2) = \sum_{m,n=0}^{\infty} (w + m\omega_1 + n\omega_2)^{-s}, \quad \mathrm{Re}\, s > 2,$$

for complex numbers $w \neq 0, \omega_1, \omega_2$ with positive real parts. Here, for a complex number $z \notin (-\infty, 0]$, we put

$$z^s = e^{s \log z}, \quad \log z = \log |z| + i \arg z, \quad -\pi < \arg z < \pi .$$

The function ζ_2 has the contour integral representation

$$\zeta_2(s; w; \omega_1, \omega_2) = \frac{\Gamma(1-s)e^{-s\pi i}}{2\pi i} \int_{I(\lambda,\infty)} \frac{e^{-wt} t^{s-1} dt}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})}, \tag{2.1}$$

through which ζ_2 is continued analytically to the whole complex plane except for simple poles at $s = 1$ and $s = 2$.

Here $I(\lambda, \infty)$ is the contour consisting of the real line from ∞ to λ, the circle $U(\lambda)$ around the origin counterclockwise from λ to λ and the real line from λ to ∞.

We have the expression

$$\frac{e^{-wt} t}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})} = \frac{1}{\omega_1 \omega_2 t} + \left(\frac{\omega_1 + \omega_2}{2\omega_1 \omega_2} - \frac{w}{\omega_1 \omega_2} \right) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} {}_2S'_n(w; \omega_1, \omega_2) t^n}{n!} \tag{2.2}$$

for $|t| < |2\pi/\omega_1|, |2\pi/\omega_2|$. Here ${}_2S'_n(w; \omega_1, \omega_2)$ is the derivative of the polynomial ${}_2S_n(w; \omega_1, \omega_2)$, with respect to w, which is called the Barnes double n-th Bernoulli polynomial [2] (cf. [9]).

The connection of ${}_2S'_n(w; \omega_1, \omega_2)$ with ordinary Bernoulli polynomial $B_r(w)$ is given by

$${}_2S'_n(w; \omega_1, \omega_2) = \frac{({}^1B\omega_1 + {}^2B\omega_2 + w)^{n+1}}{(n+1)\omega_1 \omega_2}, \tag{2.3}$$

where ${}^1B = {}^2B$ is the ordinary Bernoulli number and in the multinomial expansion of the numerator,

$$({}^iB)^j = (\text{the } j\text{-th power of } {}^iB) = B_j$$

but

$$({}^{i}B)^{j} \cdot ({}^{i'}B)^{k} \neq B_{j+k} \quad \text{for } i \neq i' .$$

Then we have

$$\zeta_2(1-p; w; \omega_1, \omega_2) = \frac{{}_2S'_p(w; \omega_1, \omega_2)}{p} \tag{2.4}$$

for a positive integer p, by the residue calculus around the origin.

3. Generalized Dedekind sum in the sense of Apostol-Rademacher

In [1], Apostol defined and investigated the generalized Dedekind sum

$$s_p(h,k) = \sum_{\mu=1}^{k-1} \frac{\mu}{k} \bar{B}_p\left(\frac{h\mu}{k}\right), \quad h\,,\ \ k \in \mathbf{Z}^+\,, \quad (h,k) = 1\,,$$

where

$$\bar{B}_p(x) = B_p(x - [x]) \quad \text{for } p > 1 \quad \text{and} \quad p = 1\,,\ x \notin \mathbf{Z}\,,$$
$$\bar{B}_1(x) = 0 \quad \text{for } x \in \mathbf{Z}\,.$$

We shall quote some of his results:
First we note that for odd $p = 1$,

$$({}^{1}Bh - {}^{2}Bk)^{p+1} = ({}^{1}Bh + {}^{2}Bk)^{p+1}$$

holds, because the left hand side is

$$\sum_{s=0}^{p+1} \binom{p+1}{s} (-1)^s B_s h^s B_{p+1-s} k^{p+1-s} = \sum_{s=0}^{p+1} \binom{p+1}{s} B_s h^s B_{p+1-s} k^{p+1-s}$$

since $B_s = 0$ for odd $s > 1$ and the last formula equals $({}^{1}Bh + {}^{2}Bk)^{p+1}$.
For odd p, Apostol proved the reciprocity law

$$(p+1)\{hk^p s_p(h,k) + kh^p s_p(k,h)\} = ({}^{1}Bh - {}^{2}Bk)^{p+1} + pB_{p+1} \tag{3.1}$$

and gives the representation of the Dedekind sum by Lambert series:
for odd $p \geqq 1$,

$$s_p(h,k) = \frac{p!}{(2\pi i)^p} \sum_{\substack{n=1 \\ n \not\equiv 0 \ (\mathrm{mod}\, k)}}^{\infty} \frac{1}{n^p} \left\{ \frac{e^{2\pi i n h/k}}{1 - e^{2\pi i n h/k}} - \frac{e^{-2\pi i n h/k}}{1 - e^{-2\pi i n h/k}} \right\}, \tag{3.2}$$

and for even $p \geqq 2$,

$$s_p(h,k) = \frac{k-1}{k^p}\cdot\frac{B_p}{2} - \frac{p!}{(2\pi i)^p}\sum_{\substack{n=1\\ n\not\equiv 0\,(\mathrm{mod}\,k)}}^{\infty}\frac{1}{n^p}\left\{\frac{e^{2\pi inh/k}}{1-e^{2\pi inh/k}} + \frac{e^{-2\pi inh/k}}{1-e^{-2\pi inh/k}}\right\} \tag{3.3}$$
$$= \frac{k-k^p}{k^p}\cdot\frac{B_p}{2}\,.$$

In the present paper, more generally, we consider Apostol's Dedekind sum of Rademacher type (=Apostol-Rademacher Dedekind sum):

$$s_p(h,k;x,y) = \sum_{\nu=0}^{k-1}\bar{B}_1\left(\frac{\nu+y}{k}\right)\bar{B}_p\left(\frac{h(\nu+y)}{k}+x\right)$$

where x, y are real numbers and we derive its reciprocity formula, from which (3.1) follows.

In "Erläuterungen" [5], Dedekind already obtained (3.2) for $p = 1$ and showed that the series on the right of (3.2) is convergent for $p = 1$.

The points of deriving (3.2) are

$$\sum_{\mu=1}^{k-1}\mu x^{\mu} = \frac{k}{x-1}\,, \quad \text{for } x \neq 1\,, \quad x^k = 1\,, \tag{3.4}$$

and

$$\bar{B}_p(x) = -p!(2\pi i)^{-p}\sum_{m=-\infty}^{\infty}{}' m^{-p}e^{2\pi imx} \tag{3.5}$$

where $\sum'$ means the sum except for 0.

4. Ramanujan's formula for $\zeta(2\nu+1)$

We put, for $x > 0$, $\nu \in \mathbf{Z}$, $\nu > 0$,

$$R_\nu(x) = \frac{1}{(4\pi x)^\nu}\left\{\frac{1}{2}\zeta(2\nu+1) + \sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(e^{2\pi mx}-1)}\right\} + \frac{B_{2\nu+2}}{(2\nu+2)!}\pi^{\nu+1}x^{\nu+1}$$
$$+\pi^{\nu+1}\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^k\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}x^{\nu+1-2k}$$
$$= \frac{-1}{(4\pi x)^\nu}\left\{\frac{1}{2}\zeta(2\nu+1) - \sum_{m=1}^{\infty}\frac{1}{m^{2\nu+1}(1-e^{-2\pi mx})}\right\} + \frac{B_{2\nu+2}}{(2\nu+2)!}\pi^{\nu+1}x^{\nu+1}$$
$$+\pi^{\nu+1}\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^k\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}x^{\nu+1-2k}$$

where for odd ν, the term corresponding to $k = \frac{1}{2}(\nu + 1)$ is multiplied by $\frac{1}{2}$.

Then Ramanujan's formula asserts that

$$R_\nu(x) = -R_\nu(-1/x)\,. \tag{4.1}$$

This formula has been proved by several authors. A nice story of Ramanujan's formula can be found in Berndt's [3], Chapt. 14.

For Lambert series

$$\sum_{n=1}^{\infty} \frac{a_n z^n}{1 - z^n}\,, \tag{4.2}$$

it is known, in general, that

$$\text{(4.2) is convergent, for any } |z| \neq 1, \text{ if } \sum_{n=1}^{\infty} a_n \text{ is convergent}\,,$$

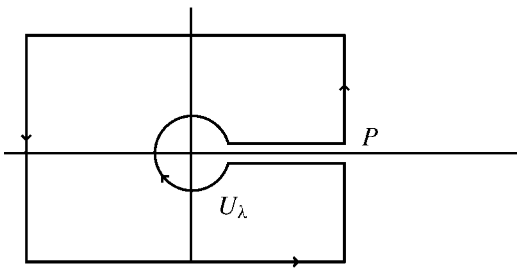
and for any positive δ, $0 < \delta < 1$, (4.2) is uniformly convergent for $|z| \leqq 1 - \delta$ and for $|z| \geqq 1 + \delta$. Hence we do not mention the convergence of Lambert series appearing in the sequel.

5. Deriving Ramanujan's formula

Let ω be a positive irrational number (a complex number $\notin \mathbf{Q}$ with positive real part). We compute

$$\frac{1}{2\pi i} \int_{I(\lambda,\infty)} f(t) t^{s-1} dt\,, \quad \text{with } f(t) = \frac{e^{-t}}{(1 - e^{-t})(1 - e^{-\omega t})}$$

by integrating on the path descibed below and letting P tend to ∞. Here poles



of the integrand are not on the path and U_λ does not contain any pole of the integrand except for 0.

Then for $P \to \infty$, the integral on the side of the square goes to 0. Its proof is the same as in Siegel [12] using the larger one of $|1 - e^{-t}|^{-1}$ and $|1 - e^{-\omega t|^{-1}}$, so we do not reproduce it here.

Thus we have

$$-\zeta_2(s;1;1,\omega) = \Gamma(1-s)e^{-s\pi i}\sum_{\text{all poles}} \text{Residue of } f(t)\,, \tag{5.1}$$

provided the right hand side converges.

Now

$$2\pi i n\,,\quad n=\pm1,\pm2,\pm3,\cdots$$

and

$$2\pi i n\frac{1}{\omega}\,,\quad n=\pm1,\pm2,\pm3,\cdots$$

are poles of the first order of the integrand.

We have

$$t^{s-1}=\begin{cases}(2\pi n)^{s-1}e^{\pi i(s-1)/2} & t=2\pi i n\,,\quad n>0\,,\\ (2\pi n)^{s-1}e^{3\pi i(s-1)/2} & t=-2\pi i n\,,\quad n>0\,.\end{cases}$$

For ω with $\operatorname{Im}\omega>0$ (<0), we have $\operatorname{Im}\left(\frac{1}{\omega}\right)<0$ (>0).

Then for $n>0$, $\operatorname{Im}\omega>0$,

$$\arg(in/\omega)=\arg(n/\omega)-\frac{3}{2}\pi\,,\quad \arg(-in/\omega)=\arg(n/\omega)-\frac{1}{2}\pi$$

and for $n>0$, $\operatorname{Im}\omega<0$,

$$\arg(in/\omega)=\arg(n/\omega)+\frac{1}{2}\pi\,,\quad \arg(-in/\omega)=\arg(n/\omega)+\frac{3}{2}\pi\,.$$

Hence we have, for $\operatorname{Im}\omega>0$,

$$t^{s-1}=\begin{cases}(2\pi n/\omega)^{s-1}e^{-3\pi i(s-1)/2} & t=2\pi i n/\omega\,,\quad n>0\,,\\ (2\pi n/\omega)^{s-1}e^{-\pi i(s-1)/2} & t=-2\pi i n/\omega\,,\quad n>0\,,\end{cases}$$

and for $\operatorname{Im}\omega<0$,

$$t^{s-1}=\begin{cases}(2\pi n/\omega)^{s-1}e^{\pi i(s-1)/2} & t=2\pi i n/\omega\,,\quad n>0\,,\\ (2\pi n/\omega)^{s-1}e^{3\pi i(s-1)/2} & t=-2\pi i n/\omega\,,\quad n>0\,.\end{cases}$$

Then

$$\text{(i)}\qquad \text{Residue at } 2\pi i n=\frac{(2\pi)^{s-1}e^{\pi i(s-1)/2}}{n^{1-s}(1-e^{-2\pi i n\omega})}\,,\quad n>0$$

$$\text{(ii)}\qquad \text{Residue at } -2\pi i n=\frac{(2\pi)^{s-1}e^{3\pi i(s-1)/2}}{n^{1-s}(1-e^{2\pi i n\omega})}\,,\quad n>0$$

$$= (2\pi)^{s-1}\frac{e^{3\pi i(s-1)/2}}{n^{1-s}} - \frac{(2\pi)^{s-1}e^{3\pi i(s-1)/2}}{n^{1-s}(1-e^{-2\pi in\omega})}\,,$$

(iii) for $\operatorname{Im}\omega > 0$,

$$\begin{aligned}\text{Residue at } 2\pi in\frac{1}{\omega} &= \frac{(2\pi)^{s-1}e^{-3\pi i(s-1)/2}e^{-2\pi in/\omega}}{n^{1-s}\omega^s(1-e^{-2\pi in/\omega})}\,,\quad n>0\\ &= -\frac{(2\pi)^{s-1}}{\omega^s}\frac{e^{-3\pi i(s-1)/2}}{n^{1-s}} + \frac{(2\pi)^{s-1}e^{-3\pi i(s-1)/2}}{\omega^s n^{1-s}(1-e^{-2\pi in/\omega})}\,,\end{aligned}$$

(iv) for $\operatorname{Im}\omega < 0$,

$$\begin{aligned}\text{Residue at } 2\pi in\frac{1}{\omega} &= \frac{(2\pi)^{s-1}e^{\pi i(s-1)/2}e^{-2\pi in/\omega}}{n^{1-s}\omega^s(1-e^{-2\pi in/\omega})}\,,\quad n>0\\ &= -\frac{(2\pi)^{s-1}}{\omega^s}\frac{e^{\pi i(s-1)/2}}{n^{1-s}} + \frac{(2\pi)^{s-1}e^{\pi i(s-1)/2}}{\omega^s n^{1-s}(1-e^{-2\pi in/\omega})}\,,\end{aligned}$$

(v) for $\operatorname{Im}\omega > 0$,

$$\begin{aligned}\text{Residue at } -2\pi in\frac{1}{\omega} &= \frac{(2\pi)^{s-1}e^{-\pi i(s-1)/2}e^{2\pi in/\omega}}{n^{1-s}\omega^s(1-e^{2\pi in/\omega})}\,,\quad n>0\\ &= \frac{-(2\pi)^{s-1}e^{-\pi i(s-1)/2}}{n^{1-s}\omega^s(1-e^{-2\pi in/\omega})}\,,\end{aligned}$$

(vi) for $\operatorname{Im}\omega < 0$,

$$\begin{aligned}\text{Residue at } -2\pi in\frac{1}{\omega} &= \frac{(2\pi)^{s-1}e^{3\pi i(s-1)/2}e^{2\pi in/\omega}}{n^{1-s}\omega^s(1-e^{2\pi in/\omega})}\quad (n>0)\\ &= \frac{-(2\pi)^{s-1}e^{3\pi i(s-1)/2}}{n^{1-s}\omega^s(1-e^{-2\pi in/\omega})}\,.\end{aligned}$$

Assume that $\operatorname{Re} s < 0$. Then the residue computation of the right hand side of (5.1) gives

$$(2\pi)^{s-1}\sum_{n=1}^{\infty}\frac{(e^{\pi i(s-1)/2}-e^{3\pi i(s-1)/2})}{n^{1-s}(1-e^{-2\pi in\omega})} + (2\pi)^{s-1}\zeta(1-s)e^{3\pi i(s-1)/2}$$
$$\begin{cases} +\dfrac{(2\pi)^{s-1}}{\omega^s}\displaystyle\sum_{n=1}^{\infty}\frac{(e^{-3\pi i(s-1)/2}-e^{-\pi i(s-1)/2})}{n^{1-s}(1-e^{-2\pi in/\omega})} - \frac{(2\pi)^{s-1}}{\omega^s}\zeta(1-s)e^{\pi i(s-1)/2} & \operatorname{Im}\omega>0\,,\\ +\dfrac{(2\pi)^{s-1}}{\omega^s}\displaystyle\sum_{n=1}^{\infty}\frac{(e^{\pi i(s-1)/2}-e^{3\pi i(s-1)/2})}{n^{1-s}(1-e^{-2\pi in/\omega})} - \frac{(2\pi)^{s-1}}{\omega^s}\zeta(1-s)e^{\pi i(s-1)/2} & \operatorname{Im}\omega<0\,.\end{cases}$$

Thus we have

THEOREM 1. *For* Re $s<0$ *and irrational* ω (*with positive real part*),

$$
\begin{aligned}
-\zeta(s;1;1,\omega) &= \Gamma(1-s)e^{-\pi is}(2\pi)^{s-1}\\
&\cdot\Bigg\{\zeta(1-s)e^{3\pi i(s-1)/2}+\sum_{n=1}^{\infty}\frac{(e^{\pi i(s-1)/2}-e^{3\pi i(s-1)/2})}{n^{1-s}(1-e^{-2\pi in\omega})}\\
&+\begin{cases}-\zeta(1-s)\dfrac{e^{\pi i(s-1)/2}}{\omega^s}+\displaystyle\sum_{n=1}^{\infty}\frac{(e^{-3\pi i(s-1)/2}-e^{-\pi i(s-1)/2})}{\omega^s n^{1-s}(1-e^{-2\pi in/\omega})}\Bigg\} & \text{for } \operatorname{Im}\omega>0,\\[2ex] -\zeta(1-s)\dfrac{e^{\pi i(s-1)/2}}{\omega^s}+\displaystyle\sum_{n=1}^{\infty}\frac{(e^{\pi i(s-1)/2}-e^{3\pi i(s-1)/2})}{\omega^s n^{1-s}(1-e^{-2\pi in/\omega})}\Bigg\} & \text{for } \operatorname{Im}\omega<0.\end{cases}
\end{aligned}
\tag{5.2}
$$

Now for $s=-2\nu$, $\nu\geqq 1$, $\nu\in\mathbf{Z}$, the left hand side of (5.2) is

$$
\begin{aligned}
-\zeta(-2\nu;1,1;\omega) &= \frac{{}_2S'_{2\nu+1}(1;1,\omega)}{2\nu+1}=\frac{({}^1B+{}^2B\omega+1)^{2\nu+2}}{(2\nu+1)(2\nu+2)\omega}\\
&=\frac{1}{(2\nu+1)(2\nu+2)\omega}\sum_{k=0}^{2\nu+2}\frac{(B+1)^k}{k!}\cdot\frac{B_{2\nu+2-k}\omega^{2\nu+2-k}}{(2\nu+2-k)!}(2\nu+2)!\\
&=\frac{(2\nu)!}{\omega}\sum_{k=0}^{\nu+1}\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}\omega^{2\nu+2-2k}}{(2\nu+2-2k)!}
\end{aligned}
\tag{5.3}
$$

since

$$B_k=0 \quad \text{for odd } k>1 \quad \text{and} \quad (B+1)^k=B_k \ \text{ for } k\geqq 2.$$

The right hand side of (5.2) is

$$
\begin{aligned}
&\frac{2(-1)^\nu(2\nu)!i}{(2\pi)^{2\nu+1}}\Bigg\{\frac{1}{2}\zeta(2\nu+1)-\sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}(1-e^{-2\pi in\omega})}\\
&\qquad+\omega^{2\nu}\frac{1}{2}\zeta(2\nu+1)-\omega^{2\nu}\sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}(1-e^{-2\pi in/\omega})}\Bigg\}.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{(4\pi z)^\nu}\Bigg\{\frac{1}{2}\zeta(2\nu+1)-\sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}(1-e^{-2\pi nz})}\Bigg\}\\
&\qquad+\left(\frac{-z}{4\pi}\right)^\nu\Bigg\{\frac{1}{2}\zeta(2\nu+1)-\sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}(1-e^{2\pi n/z})}\Bigg\}\\
&=\pi^{\nu+1}\sum_{k=0}^{\nu+1}(-1)^k\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}z^{\nu+1-2k},
\end{aligned}
\tag{5.4}
$$

and the last is

$$= \frac{B_{2\nu+2}}{(2\nu+2)!}\left\{z^{\nu+1}\pi^{\nu+1} + \frac{(-\pi)^{\nu+1}}{z^{\nu+1}}\right\}$$
$$+ \pi^{\nu+1}\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^k \frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}\left(z^{\nu+1-2k} + \left(\frac{-1}{z}\right)^{\nu+1-2k}\right),$$

where for odd ν, the term corresponding to $k = \frac{1}{2}(\nu+1)$ is multiplied by $\frac{1}{2}$.

For z, with $\operatorname{Im} z > 0$, we put

$$F_\nu(z) = \frac{-1}{(4\pi z)^\nu}\left\{\frac{1}{2}\zeta(2\nu+1) - \sum_{n=1}^{\infty}\frac{1}{n^{2\nu+1}(1-e^{-2\pi n z})}\right\}$$
$$+ \frac{B_{2\nu+2}}{(2\nu+2)!}z^{\nu+1}\pi^{\nu+1} + \pi^{\nu+1}\sum_{k=1}^{[\frac{1}{2}(\nu+1)]}(-1)^k\frac{B_{2k}}{(2k)!}\cdot\frac{B_{2\nu+2-2k}}{(2\nu+2-2k)!}z^{\nu+1-2k},$$

where for odd ν, the term corresponding to $k = \frac{1}{2}(\nu+1)$ is multiplied by $\frac{1}{2}$.

Then we have

THEOREM 2.

$$F_\nu(z) = -F_\nu\left(-\frac{1}{z}\right).$$

COROLLARY (Ramanujan's formula).

$$R_\nu(x) = -R_\nu\left(-\frac{1}{x}\right).$$

PROOF. Take the limit $y \to 0$, $z = x + iy$, in Theorem 2.

This seems to be a new proof of Ramanujan's formula.

6. Deriving the reciprocity formula for Apostol Dedekind sum

Let h, k be positive integers with $(h,k) = 1$. Let x, y be non-negative real numbers with $(x,y) \neq (0,0)$. For simplicity, we assume $1 - s = p > 0$, $p \in \mathbf{Z}$. Put

$$f(t) = \frac{e^{-(x/h+y/k)t}}{(1-e^{-t/h})(1-e^{-t/k})}.$$

Here we consider

$$-\zeta_2(1-p; x/h+y/k; 1/h, 1/k)\frac{(-1)^{p-1}}{\Gamma(p)} = \sum_{\text{all poles}} \text{Residue of } f(t)t^{-p}. \tag{6.1}$$

This is obtained in the same way to get (5.1).

Now poles of $f(t)$ are given by

$$\text{(Type I)} \qquad t = -2\pi inh\,, \quad n \not\equiv 0 \ (\mathrm{mod}\, k)$$

$$\text{(Type II)} \qquad t = -2\pi ink\,, \quad n \not\equiv 0 \ (\mathrm{mod}\, h)$$

and

$$\text{(Type III)} \qquad t = -2\pi inhk\,, \quad n = 0, \pm 1, \pm 2, \cdots .$$

Poles of (Type I) and (Type II) are of the first order and poles of (Type III) are of the second order.

Then for residues of $f(t)t^{-p}$, we have

$$\text{(Type I)} \qquad \text{Res at } t = -2\pi inh \text{ is } \frac{(-2\pi i)^{-p} e^{2\pi inh(x/h+y/k)}}{h^{p-1} n^p (1 - e^{2\pi inh/k})}\,,$$

$$\text{(Type II)} \qquad \text{Res at } t = -2\pi ink \text{ is } \frac{(-2\pi i)^{-p} e^{2\pi ink(x/h+y/k)}}{k^{p-1} n^p (1 - e^{2\pi ink/h})}\,.$$

At $t = -2\pi inhk$, which is a pole of type (III), we have the expansions

$$\frac{e^{-(x/h+y/k)t}}{(1-e^{-t/h})(1-e^{-t/k})} = \frac{hke^{2\pi in(kx+hy)}}{(t+2\pi inhk)^2} + \frac{\left(kx+hy-\frac{1}{2}(h+k)\right)e^{2\pi in(kx+hy)}}{t+2\pi inhk} + \cdots$$

and

$$t^{-p} = (-2\pi inhk)^{-p} - p(-2\pi inhk)^{-p-1}(t+2\pi inhk) + \cdots .$$

Hence

$$\text{Res at } t = -2\pi inhk \text{ is } \frac{\left(kx+hy-\frac{1}{2}(h+k)\right)(-2\pi i)^{-p} e^{2\pi in(kx+hy)}}{(hk)^p n^p} - p\frac{(-2\pi i)^{-p-1} e^{2\pi in(kx+hy)}}{(hk)^p n^{p+1}}\,.$$

Therefore, the right hand side of (6.1) becomes

$$
\begin{aligned}
&\frac{(-1)^p}{(2\pi i)^p h^{p-1}} \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{e^{2\pi i n h(x/h+y/k)}}{n^p(1-e^{2\pi i n h/k})} \\
&\quad + \frac{(-1)^p}{(2\pi i)^p k^{p-1}} \sum_{\substack{n=-\infty \\ (h)}}^{\infty}{}' \frac{e^{2\pi i n k(x/h+y/k)}}{n^p(1-e^{2\pi i n k/h})} \\
&\quad + \frac{(-1)^p\left(kx+hy-\frac{1}{2}(h+k)\right)}{(2\pi i)^p (hk)^p} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n(kx+hy)}}{n^p} \\
&\quad - \frac{(-1)^{p+1} p}{(2\pi i)^{p+1}(hk)^p} \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n(kx+hy)}}{n^{p+1}},
\end{aligned}
\tag{6.2}
$$

where (h), (k) under $\sum$ mean "$n\not\equiv 0 \pmod h$" and "$n\not\equiv 0 \pmod k$" respectively.

In (6.2), we have by (3.4) and (3.5),

$$
\begin{aligned}
\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{e^{2\pi i n h(x/h+y/k)}}{n^p(1-e^{2\pi i n h/k})} &= -\sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{e^{2\pi i n(x+(h/k)y)}}{n^p} \sum_{\nu=1}^{k-1} \frac{\nu}{k} e^{2\pi i n h\nu/k} \\
&= -\sum_{\nu=1}^{k-1} \frac{\nu}{k} \sum_{\substack{n=-\infty \\ (k)}}^{\infty}{}' \frac{e^{2\pi i n\{x+(h/k)(y+\nu)\}}}{n^p} \\
&= -\sum_{\nu=1}^{k-1} \frac{\nu}{k} \left\{ \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n\{x+(h/k)(y+\nu)\}}}{n^p} - \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi i n(kx+hy)}}{(kn)^p} \right\} \\
&= \frac{(2\pi i)^p}{p!} \left\{ \sum_{\nu=1}^{k-1} \frac{\nu}{k} \bar{B}_p((h/k)(y+\nu)+x) - \frac{\frac{1}{2}(k-1)}{k^p} \bar{B}_p(kx+hy) \right\}.
\end{aligned}
\tag{6.3}
$$

Here we rewrite Apostol-Rademacher Dedekind sum as follows: For $1 > y \geqq 0$

$$
\begin{aligned}
s_p(h,k;x,y) &= \sum_{\nu=0}^{k-1} \bar{B}_1\left(\frac{y+\nu}{k}\right) \bar{B}_p\left(\frac{h}{k}(y+\nu)+x\right) \\
&= \sum_{\nu=1}^{k-1} \frac{\nu}{k} \bar{B}_p\left(\frac{h}{k}(y+\nu)+x\right) + \left(\frac{y}{k}-\frac{1}{2}\right) \sum_{\nu=0}^{k-1} \bar{B}_p\left(\frac{h}{k}(y+\nu)+x\right).
\end{aligned}
\tag{6.4}
$$

It is known, by Lemma 3.2 (2) of [10], that

$$\bar{B}_p(hv) = h^{p-1}\sum_{\nu=0}^{h-1}\bar{B}_p\left(v+\frac{\mu}{h}\right).$$

Hence

$$\sum_{\nu=0}^{k-1}\bar{B}_p\left(\frac{h}{k}(y+\nu)+x\right) = h^{p-1}\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{x+\mu}{h}+\frac{y+\nu}{k}\right).$$

Put this into (6.4) and put the formula thus obtained into (6.3). Then we have, for the first sum of (6.2),

$$\begin{aligned}(6.5)\qquad &\frac{(-1)^p}{(2\pi i)^p h^{p-1}}\sum_{\substack{n=-\infty\\(k)}}^{\infty}{}' \frac{e^{2\pi inh(x/h+y/k)}}{n^p(1-e^{2\pi inh/k})}\\ &= \frac{(-1)^p}{p!h^{p-1}}\left\{s_p(h,k;x,y)-\frac{\frac{1}{2}(k-1)}{k^p}\bar{B}_p(kx+hy)\right\}\\ &\quad -\frac{(-1)^p}{p!}\left(\frac{y}{k}-\frac{1}{2}\right)\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{x+\mu}{h}+\frac{y+\nu}{k}\right).\end{aligned}$$

In the same way for the second sum of (6.2), we have, for $1 > x \geqq 0$,

$$\begin{aligned}(6.6)\qquad &\frac{(-1)^p}{(2\pi i)^p k^{p-1}}\sum_{\substack{n=-\infty\\(h)}}^{\infty}{}' \frac{e^{2\pi ink(x/h+y/k)}}{n^p(1-e^{2\pi ink/h})}\\ &= \frac{(-1)^p}{p!k^{p-1}}\left\{s_p(k,h;y,x)-\frac{\frac{1}{2}(h-1)}{h^p}\bar{B}_p(kx+hy\right\}\\ &\quad -\frac{(-1)^p}{p!}\left(\frac{x}{h}-\frac{1}{2}\right)\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{x+\mu}{h}+\frac{y+\nu}{k}\right).\end{aligned}$$

Further, in (6.2), we have

$$(6.7)\qquad \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi in(kx+hy)}}{n^p} = -\frac{(2\pi i)^p}{p!}\bar{B}_p(kx+hy)\,,$$

$$(6.8)\qquad \sum_{n=-\infty}^{\infty}{}' \frac{e^{2\pi in(kx+hy)}}{n^{p+1}} = -\frac{(2\pi i)^{p+1}}{(p+1)!}\bar{B}_{p+1}(kx+hy)$$

and the left hand side of (6.1) equals

$$(-1)^p \frac{hk\left({}^1B\frac{1}{h} + {}^2B\frac{1}{k} + \frac{x}{h} + \frac{y}{k}\right)^{p+1}}{(p+1)!} .$$

By the formulas just above, (6.7), (6.8), (6.5) and (6.6), we obtain

THEOREM 3. *For* $1 > x \geqq 0$, $1 > y \geqq 0$, $(x, y) \neq (0, 0)$, *and* $(h, k) = 1$,

$$\begin{aligned}(p+1)\{hk^p s_p(h, k; x, y) + kh^p s_p(k, h; y, x)\} \\ = ({}^1Bh + {}^2Bk + kx + hy)^{p+1} + p\bar{B}_{p+1}(kx + hy) \\ + (p+1)(hk)^p\left(\frac{x}{h} + \frac{y}{k} - 1\right)\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{x+\mu}{h} + \frac{y+\nu}{k}\right) \\ - (p+1)(kx + hy - hk)\bar{B}_p(kx + hy) .\end{aligned}$$

Note that for $p = 1$, the formula in the Theorem becomes: for $0 < x < 1$, $0 < y < 1$, $kx + hy \in \mathbf{Z}$, and $(h, k) = 1$,

$$\begin{aligned}s_1(h, k; x, y) + s_1(k, h; y, x) = B_1(x)B_1(y) \\ + \frac{1}{2}\left\{\frac{h}{k}\bar{B}_2(y) + \frac{1}{hk}\bar{B}_2(kx + hy) + \frac{k}{h}\bar{B}_2(x)\right\} .\end{aligned} \tag{6.9}$$

Here we used a consequence of the proof of the formula (2) of Lemma 3.2. [10],

$$\sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{x+\mu}{h} + \frac{y+\nu}{k}\right) = \sum_{\nu=0}^{k-1}\sum_{\mu=0}^{h-1}\bar{B}_p\left(\frac{k\mu + h\nu + \alpha}{hk}\right) = \frac{B_p}{(hk)^{p-1}} ,$$

with $\alpha = kx + hy \in \mathbf{Z}$.

Under our restriction on x and y, the formula (6.9) coincides with Rademacher's reciprocity formula in Theorem 4.1 [4].

So far, it is assumed that $x \geqq 0$, $y \geqq 0$, $(x, y) \neq (0, 0)$. In the formula of Theorem 3, put $y = 0$. Then both hands are polynomials of x near $0+$. Hence letting x tend to $0+$, we have the following

COROLLARY 1 (Reciprocity formula for Apostol's Dedekind sum). *For odd* $p \geqq 1$ *and* $(h, k) = 1$,

$$(p+1)\{hk^p s_p(h, k) + kh^p s_p(k, h)\} = ({}^1Bh - {}^2Bk)^{p+1} + pB_{p+1} .$$

PROOF OF COROLLARY. For $y = 0$, $x \to 0+$, the formula in the Theorem becomes

$$(6.10)\quad \begin{aligned} &(p+1)\{hk^p s_p(h,k;0,0) + kh^p s_p(k,h;0,0)\} \\ &\quad = ({}^1Bh + {}^2Bk)^{p+1} + pB_{p+1} \\ &\qquad - (p+1)(hk)^p \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{h-1} \bar{B}_p\left(\frac{\mu}{h} + \frac{\nu}{k}\right) - (p+1)hkB_p\,. \end{aligned}$$

By definition,

$$s_p(h,k;0,0) = s_p(h,k) + \frac{1}{2} \sum_{\nu=0}^{k-1} \bar{B}_p(h\nu/k)$$

and by Lemma 3.2. (2) of [10],

$$\bar{B}_p(h\nu/k) = h^{p-1} \sum_{\mu=0}^{h-1} \bar{B}_p\left(\frac{\mu}{h} + \frac{\nu}{k}\right).$$

Hence

$$(6.11)\quad hk^p s_p(h,k;0,0) = hk^p s_p(h,k) - \frac{1}{2}(hk)^p \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{h-1} \bar{B}_p\left(\frac{\mu}{h} + \frac{\nu}{k}\right).$$

Similarly,

$$(6.12)\quad kh^p s_p(k,h;0,0) = kh^p s_p(k,h) - \frac{1}{2}(hk)^p \sum_{\nu=0}^{k-1} \sum_{\mu=0}^{h-1} \bar{B}_p\left(\frac{\mu}{h} + \frac{\nu}{k}\right).$$

Put (6.11) and (6.12) into (6.10). Then the double sum

$$\sum_{\nu=0}^{k-1} \sum_{\mu=0}^{h-1} \bar{B}_p\left(\frac{\mu}{h} + \frac{\nu}{k}\right)$$

disappears and (6.10) becomes

$$(6.13)\quad \begin{aligned} &(p+1)\{hk^p s_p(h,k) + kh^p s_p(k,h)\} \\ &\quad = ({}^1Bh + {}^2Bk)^{p+1} + pB_{p+1} - (p+1)hkB_p\,. \end{aligned}$$

Then for odd $p > 1$, we have the Corollary, since $B_p = 0$ and

$$({}^1Bh + {}^2Bk)^{p+1} = ({}^1Bh - {}^2Bk)^{p+1}\,.$$

Now for $p = 1$,

$$2\{s_p(h,k)+s_p(k,h)\} = ({}^1Bh +{}^2 Bk)^2 + pB_{p+1} + 2hkB_1$$
$$= ({}^1Bh -{}^2 Bk)^2 + pB_{p+1} .$$

Hence our formula holds for $p = 1$.

Put

$$s_p(\alpha; h, k) = \sum_{\nu=1}^{k-1} \frac{\nu}{k} \bar{B}_p\left(\frac{\alpha + h\nu}{k}\right).$$

This is called the shifted Dedekind sum in [10].

COROLLARY 2. *Let α be a real number such that $0 \leqq \alpha < h+k$. Then for positive integer $p \geqq 1$ and $(h,k)=1$,*

$$\frac{1}{p}\{k^{p-1}s_p(\alpha;h,k)+h^{p-1}s_p(\alpha;k,h)\}$$
$$= \frac{({}^1Bh +{}^2 Bk + \alpha)^{p+1}}{p(p+1)hk} + \frac{\bar{B}_{p+1}(\alpha)}{(p+1)hk} - \frac{1}{p}\left(1 - \frac{\alpha}{hk}\right)\bar{B}_p(\alpha) .$$

This is nothing but Theorem 3.4 in [10].

PROOF. The case $\alpha = 0$ is stated in Corollary 1.

We take $\alpha = kx + hy$ in (6.2). Hence $0 \leqq \alpha < h + k$.

Then the first sum in (6.2). equals

$$\frac{(-1)^p}{h^{p-1}p!}\left\{s_p(\alpha;h,k) - \frac{\frac{1}{2}(k-1)}{k^p}\bar{B}_p(\alpha)\right\}$$

and the second sum in (6.2) equals

$$\frac{(-1)^p}{k^{p-1}p!}\left\{s_p(\alpha;k,h) - \frac{\frac{1}{2}(h-1)}{h^p}\bar{B}_p(\alpha)\right\}.$$

A straightforward calculation, as in the proof of the Theorem, shows that our formula holds.

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