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A Proof of the Paley-Wiener Theorem for Hyperfunctions with a Convex Compact Support by the Heat Kernel Method

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Abstract. In this paper we shall give a proof of the Paley-Wiener theorem for hyperfunctions supported by a convex compact set by the heat kernel method.

1. Introduction

In 1987, T. Matsuzawa gave a new proof of the Paley-Wiener theorem for hyperfunctions supported by *a ball* by the heat kernel method [4]. S. Lee and S.-Y. Chung gave a proof of the Paley-Wiener-Schwartz theorem for distributions supported by a convex compact set by the heat kernel method [3]. M. Suwa and K. Yoshino treated the case of tempered distributions supported by a proper convex cone [6].

In this paper we shall treat the Paley-Wiener theorem for hyperfunctions supported by *a convex compact set* by the heat kernel method (Theorem 4.2).

2. Preliminaries

DEFINITION 2.1. We define some notations:

$$x = (x_1, \cdots, x_n) \in \mathbf{R}^n, \quad x^2 = x_1^2 + \cdots + x_n^2 .$$

$$\langle x, \xi \rangle = \sum_{j=1}^{n} x_j \xi_j \quad \text{for } x, \xi \in \mathbf{R}^n .$$

$$\overline{B(0,\delta)} = \{x \in \mathbf{R}^n : |x| \le \delta\} .$$

$$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}^n, \quad |\alpha| = \alpha_1 + \cdots + \alpha_n ,$$

$$\alpha! = \alpha_1! \cdots \alpha_n! .$$

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$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}, \quad \triangle = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}.$$

$$E(x,t) = (4\pi t)^{-\frac{n}{2}} \exp(-x^2/4t), \quad (t > 0, \quad x \in \mathbf{R}^n).$$

For $\zeta \in \mathbf{C}^n$, $\zeta = (\zeta_1, \cdots, \zeta_n)$, we put $|\zeta| = \sqrt{|\zeta_1|^2 + \cdots + |\zeta_n|^2}$.

DEFINITION 2.2 ([1]). If $K \subset \mathbf{C}^n$ is a compact set, then $\mathcal{A}'(K)$, the space of analytic functionals carried by K, is the space of linear forms u on the space $\mathcal{A}$ of entire functions in $\mathbf{C}^n$ such that for every neighborhood ω of K

$$|u(\varphi)| \le C_\omega \sup_\omega |\varphi|, \quad \varphi \in \mathcal{A}.$$

DEFINITION 2.3. Let $K \subset \mathbf{R}^n$ be a compact set. Then we call the element of $\mathcal{A}'(K)$ hyperfunctions supported by K.

DEFINITION 2.4. $\mathcal{D}(\mathbf{R}^n)$ is the space of $\mathcal{C}^\infty$ functions with compact support. $\mathcal{S}(\mathbf{R}^n)$ is the space of rapidly decreasing $\mathcal{C}^\infty$ functions.

DEFINITION 2.5. Let K be a convex compact set in $\mathbf{R}^n$. Then for $\delta > 0$ we set $K_\delta = K + \overline{B(0,\delta)}$ and we define supporting function of K by $h_K(x) = \sup_{\xi \in K} \langle \xi, x \rangle$.

Let $K \subset \mathbf{R}^n$. Then the following proposition is known for between $\mathcal{A}(K)$ and $\mathcal{A}$. For the details of the proof we refer the reader to [1]:

PROPOSITION 2.6 ([1]). *Let $K \subset \mathbf{R}^n$ be a compact set, and set for $\varepsilon > 0$*

$$K_{(\varepsilon)} = \{z \in \mathbf{C}^n ; |\mathrm{Re} z - x| + 2|\mathrm{Im} z| \le \varepsilon \ for \ some \ x \in K\}.$$

For every φ which is analytic in a neighborhood V of $K_{(\varepsilon)}$ one can then find a sequence $\varphi_j \in \mathcal{A}$ such that

$$\sup_{K_{(\varepsilon)}} |\varphi_j - \varphi| \to 0, \quad j \to \infty.$$

3. A characterization of hyperfunctions by using the heat kernel

In this section, we shall introduce a characterization of hyperfunctions by the heat kernel method. For the details, we refer the reader to [4], [5].

THEOREM 3.1 ([4], [5]). *Let K be a compact set in $\mathbf{R}^n$, $u \in \mathcal{A}'(K)$ and $U(x,t) = \langle u_y, E(x-y,t) \rangle$. Then $U(x,t) \in \mathcal{C}^\infty(\mathbf{R}^n \times (0,\infty))$ and $U(\cdot,t) \in \mathcal{A}$ for each $t > 0$. Furthermore U satisfies the heat equation*:

$$\left(\frac{\partial}{\partial t} - \triangle\right) U(x,t) = 0 \quad in \ \mathbf{R}^n \times (0,\infty). \tag{1}$$

For every $\varepsilon > 0$ *we have*

(2) $$|U(x,t)| \leq C_\varepsilon e^{\frac{\varepsilon}{t}} \quad in \ \mathbf{R}^n \times (0,\infty)\,.$$

We have for any $\delta > 0$

(3) $$U(\cdot,t) \to 0 \quad uniformly \ \ in \ \ \{x \in \mathbf{R}^n; dis(x,K) \geq \delta\} \ \ as \ \ t \to 0_+\,.$$

(4) $$U(\cdot,t) \to u \quad in \ \ \mathcal{A}'(K) \ \ as \ \ t \to 0_+,$$

i.e.

(5) $$\langle u, \varphi\rangle = \lim_{t\to 0_+} \int_{\mathbf{R}^n} U(x,t)\chi(x)\varphi(x)dx\,, \quad \varphi \in \mathcal{A}\,.$$

for any $\chi(x) \in \mathcal{D}$ *such that* $\chi(x) = 1$ *in a neighborhood of* K.

Conversely, every $U(x,t) \in \mathcal{C}^\infty(\mathbf{R}^n \times (0,\infty))$ *satisfying the condition* (1), (2) *and* (3) *can be expressed in the form* $U(x,t) = \langle u_y, E(x-y,t)\rangle$ *with unique element* $u \in \mathcal{A}'(K)$.

4. A proof of the Paley-Wiener theorem by the heat kernel method

In this section, we shall give a proof of the Paley-Wiener theorem for hyperfunctions with a convex compact support by the heat kernel method given in section 3.

DEFINITION 4.1. Let $u \in \mathcal{A}'(K)$, K is a compact set in $\mathbf{R}^n$. Then we denote the Fourier-Laplace transform $\tilde{u}(\zeta)$ by

$$\tilde{u}(\zeta) = \frac{1}{(2\pi)^{\frac{n}{2}}}\langle u_x, e^{-\imath\zeta x}\rangle\,.$$

Then the following Paley-Wiener type theorem is known [2]:

THEOREM 4.2. *Let* K *be a convex compact set in* $\mathbf{R}^n$ *and* $u \in \mathcal{A}'(K)$. *Then* $\tilde{u}(\zeta)$ *is an entire function such that for every* $\varepsilon > 0$ *there exists a constant* $C_\varepsilon \geq 0$ *such that*

(6) $$|\tilde{u}(\zeta)| \leq C_\varepsilon e^{h_K(\eta)+\varepsilon|\zeta|}\,, \quad \zeta = \xi + \imath\eta \in \mathbf{C}^n\,.$$

Conversely, if $F(\zeta)$ *is an entire function satisfying the estimate* (6) , *then there exists a unique* $u \in \mathcal{A}'(K)$ *such that* $F(\zeta) = \tilde{u}(\zeta)$.

PROOF. By the continuity, we have the necessity. Now we shall prove the sufficiency by the heat kernel method.

Let $\delta > 0$ and $x \in \mathbf{R}^n \backslash K_{2\delta}$. Since $x \notin K_\delta$, there exist $\eta_0 \in \mathbf{R}^n$, $|\eta_0| = 1$, and $c_0 \in \mathbf{R}$ such that

$$\langle x, \eta_0\rangle > \ c_0\,, \quad \langle y, \eta_0\rangle < c_0\,, \quad \forall y \in K_\delta\,.$$

So we have

$$\sup_{y\in K_\delta}\langle y, \eta_0\rangle \leq c_0 \Leftrightarrow h_{K_\delta}(\eta_0) \leq c_0$$

$$\begin{aligned}&\Leftrightarrow h_K(\eta_0) + h_{\overline{B(0,\delta)}}(\eta_0) \le c_0\\ &\quad\Leftrightarrow h_K(\eta_0) + \delta|\eta_0| \le c_0\\ &\qquad\Leftrightarrow h_K(\eta_0) + \delta \le c_0 < \langle x, \eta_0\rangle\,.\end{aligned}$$

Therefore

$$h_K(\eta_0) - \langle x, \eta_0\rangle < -\delta\,. \tag{7}$$

Now we set $U(x,t)$ by

$$U(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbf{R}^n} F(\xi)e^{-t\xi^2}e^{\imath\xi x}d\xi\,,\quad t>0\,. \tag{8}$$

Then for $U(x,t)$, we have the following conditions [4]:

$$\left(\frac{\partial}{\partial t} - \triangle\right)U(x,t) = 0\,,$$

for every $\varepsilon > 0$, there exists a constant $C_\varepsilon \ge 0$ such that

$$|U(x,t)| \le C_\varepsilon e^{\frac{\varepsilon}{t}}\,,\quad \text{in } \mathbf{R}^n \times (0,\infty)\,.$$

Now we shift the integration in (8) into the complex domain:

$$U(x,t) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbf{R}^n} F(\xi + \imath\eta')e^{-t(\xi+\imath\eta')^2}e^{\imath(\xi+\imath\eta')x}d\xi\,,$$

where $\eta' = \frac{\delta}{2t}\eta_0$. Estimating this integral by using (7), we have

$$\begin{aligned}|U(x,t)| &\le Ce^{h_K(\eta')+\varepsilon|\eta'|+t\eta'^2-\eta' x}\int_{\mathbf{R}^n} e^{-t\xi^2+\varepsilon|\xi|}d\xi\\ &= Ce^{h_K(\eta')+\varepsilon|\eta'|+t\eta'^2-\eta' x+\frac{\varepsilon^2}{4t}}\int_{\mathbf{R}^n} e^{-t(|\xi|-\frac{\varepsilon}{2t})^2}d\xi\\ &\le Ce^{h_K(\eta')+\varepsilon|\eta'|+t\eta'^2-\eta' x+\frac{\varepsilon^2}{4t}}\int_{\mathbf{R}^n} e^{-\frac{t}{2}|\xi|^2+\frac{\varepsilon^2}{4t}}d\xi\\ &= Ce^{h_K(\eta')+\varepsilon|\eta'|+t\eta'^2-\eta' x+\frac{\varepsilon^2}{2t}}\int_{\mathbf{R}^n} e^{-\frac{t}{2}|\xi|^2}d\xi\\ &= C(2\pi)^{\frac{n}{2}}t^{-\frac{n}{2}}e^{h_K(\eta')+\varepsilon|\eta'|+t\eta'^2-\eta' x+\frac{\varepsilon^2}{2t}}\\ &= C't^{-\frac{n}{2}}e^{\frac{\delta}{2t}h_K(\eta_0)+\frac{\varepsilon\delta}{2t}+\frac{\delta^2}{4t}-\frac{\delta}{2t}\eta_0 x+\frac{\varepsilon^2}{2t}}\\ &\le C't^{-\frac{n}{2}}e^{-\frac{\delta^2}{2t}+\frac{\varepsilon\delta}{2t}+\frac{\delta^2}{4t}+\frac{\varepsilon^2}{2t}}\,.\end{aligned}$$

If we put $\varepsilon = \frac{\delta}{4}$, then

$$|U(x,t)| \le C't^{-\frac{n}{2}}e^{-\frac{\delta^2}{2t}+\frac{\delta^2}{8t}+\frac{\delta^2}{4t}+\frac{\delta^2}{32t}}$$

$$= C' t^{-\frac{n}{2}} e^{-\frac{3\delta^2}{32t}} .$$

So we have

$$U(x,t) \to 0 \quad (t \to 0_+) ,$$

uniformly in $\mathbf{R}^n \backslash K_{2\delta}$. By Theorem 3.1, there exists $u \in \mathcal{A}'(K)$ such that

$$U(x,t) = \langle u_y, E(x-y,t) \rangle .$$

Since $F(\xi) e^{-t\xi^2} \in \mathcal{S}$,

$$\begin{aligned} F(\xi) e^{-t\xi^2} &= \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbf{R}^n} U(x,t) e^{-\imath \xi x} dx \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \langle U(x,t), e^{-\imath \xi x} \rangle . \end{aligned} \tag{9}$$

LEMMA 4.3.

$$\lim_{t \to 0_+} \int_{\mathbf{R}^n} U(x,t) \chi(x) e^{-\imath \xi x} dx = \lim_{t \to 0_+} \int_{\mathbf{R}^n} U(x,t) e^{-\imath \xi x} dx, \tag{10}$$

where $\chi(x) \in \mathcal{D}$ *and* $\chi(x) = 1$ *in a neighborhood of* K.

PROOF OF LEMMA. Since for $\varepsilon > 0$ there exists $C_\varepsilon \geq 0$ such that

$$|U(x,t)| \leq C_\varepsilon e^{\frac{\varepsilon}{t} - \frac{dis(x,K)^2}{4t}}$$

(see [5]), for

$$\chi(x) = \begin{cases} 1 , & x \in K_\delta , \\ 0 , & x \in \mathbf{R}^n \backslash K_{2\delta} , \end{cases}$$

we have

$$\begin{aligned} \int_{\mathbf{R}^n} |U(x,t)(\chi(x) - 1) e^{\imath \xi x}| dx &= \int_{\mathbf{R}^n \backslash K_\delta} |U(x,t)(\chi(x) - 1) e^{\imath \xi x}| dx \\ &\leq C_\varepsilon e^{\frac{\varepsilon}{t} - \frac{\delta^2}{8t}} \int_{\mathbf{R}^n \backslash K_\delta} e^{-\frac{dis(x,K)^2}{8t}} dx \\ &\leq C'_\varepsilon e^{\frac{\varepsilon}{t} - \frac{\delta^2}{8t}} . \end{aligned}$$

When we put $\varepsilon = \frac{\delta^2}{16}$, we have

$$\lim_{t \to 0_+} \int_{\mathbf{R}^n} |U(x,t)(\chi(x) - 1) e^{\imath \xi x}| dx \leq C'_\varepsilon \lim_{t \to 0_+} e^{-\frac{\delta^2}{16t}} = 0 .$$

The proof is complete. □

Now we resume the proof of Theorem 4.2.

By (4), (9) and (10),

$$\begin{aligned} F(\xi) &= \lim_{t\to 0_+} F(\xi)e^{-\imath\xi^2} \\ &= \lim_{t\to 0_+} \frac{1}{(2\pi)^{\frac{n}{2}}}\langle U(x,t), e^{-\imath\xi x}\rangle \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}}\langle u_x, e^{-\imath\xi x}\rangle = \tilde{u}(\xi)\,. \end{aligned}$$

Since $F(\zeta)$ and $\tilde{u}(\zeta)$ are entire functions, we have $F(\zeta) = \dfrac{1}{(2\pi)^{\frac{n}{2}}}\langle u_x, e^{-\imath\zeta x}\rangle,\quad \zeta = \xi + \imath\eta \in \mathbf{C}^n$. If $\tilde{u}(\zeta) = 0$, then $\langle u_x, x^m\rangle = 0$ for $\forall m \in \mathbf{N}^n$. By Proposition 2.6, for any $\varphi(z) \in \mathcal{A}(K)$, there exists $\varphi_j(z) \in \mathcal{A}$ such that

$$\sup_{z\in K_{(\varepsilon)}} |\varphi - \varphi_j| \to 0\,, \quad j \to 0\,.$$

So we have $\langle u, \varphi\rangle = 0$ for $\forall\varphi(z) \in \mathcal{A}(K)$. This means that u is unique. □

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