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A Comparison Theorem on Crescents for Kähler Magnetic Fields

Toshiaki ADACHI

Nagoya Institute of Technology

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Abstract. For a non-trivial Kähler magnetic field on a Kähler manifold, we consider bow-shapes as substitutions for triangles on a Riemannian manifold. We give a comparison theorem for bow-shapes on a manifold whose sectional curvature is bounded from above.

1. Introduction

In papers [1, 2, 3] the author studied Kähler manifolds from Riemannian geometric point of view by using Kähler magnetic fields. On a Kähler manifold $(M, J, \langle\ ,\ \rangle)$ a closed 2-form $\mathbf{B}_\kappa = \kappa\mathbf{B}_J$ which is a constant multiple of the Kähler form $\mathbf{B}_J$ on M is said to be a Kähler magnetic field. A smooth curve γ parameterized by its arc-length is called a trajectory for $\mathbf{B}_\kappa$ if it satisfies the equation $\nabla_{\dot\gamma}\dot\gamma = \kappa J\dot\gamma$. As it is a geodesic when $\kappa = 0$, the author would like to investigate Kähler manifolds by use of some properties of trajectories for Kähler magnetic fields.

In Riemannian geometry it is a basic idea to compare the geometry of an arbitrary Riemannian manifold with geometry of a space of constant curvature. Powerful results were first obtained by Rauch, Alexandrov, Toponogov and Bishop, and active development was done by many geometers. In [2] the author studied a comparison theorem on Kähler magnetic Jacobi fields, which was generalized by N. Gouda[8] for general magnetic fields. He gave interesting results on geometry of general manifolds with uniform magnetic fields in [8, 9]. In this paper, in order to give another light on the study of non-trivial Kähler magnetic fields on general Kähler manifolds, we consider "bow-shapes" which are consisted of trajectories and a kind of geodesics and study a theorem of comparison type. Through out of this paper we suppose $\kappa \neq 0$.

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2. Jacobi fields associated with a trajectory

Let γ be a trajectory for a non-trivial Kähler magnetic field $\mathbf{B}_\kappa$ on a complete Kähler manifold M. We shall call a map $\alpha : \mathbf{R}^2 \to M$ a variation of geodesics associated with γ if it satisfies the following conditions;

i) $\gamma(s) = \alpha(s, 0)$,

ii) for each s the map $\sigma_s(\cdot) = \alpha(s, \cdot)$ is a geodesic,

iii) $\frac{\partial\alpha}{\partial t}(s, 0)$ is parallel to $J\dot\gamma(s)$ and satisfies $\kappa\langle\frac{\partial\alpha}{\partial t}(s, 0), J\dot\gamma(s)\rangle > 0$.

LEMMA 1. *For a variation α of geodesics associated with a trajectory γ for $\mathbf{B}_\kappa$, we consider a Jacobi field $Y = \frac{\partial\alpha}{\partial s}(s_0, \cdot)$ along a geodesic $\sigma = \alpha(s_0, \cdot)$. Then it satisfies*

1) $Y(0) = \dot\gamma(s_0)$,

2) $\frac{1}{2}\frac{d}{dt}\|Y(t)\|^2\big|_{t=0} = \langle\nabla_{\sigma'}Y(0), \dot\gamma(s_0)\rangle = -\kappa\langle\sigma'(0), J\dot\gamma(s_0)\rangle < 0$,

3) $\nabla_{\sigma'}Y(0)$ *is contained in the complex vector subspace spanned by* $\dot\gamma(s_0)$.

If α is a variation of normal geodesics, that is $\left\|\frac{\partial\alpha}{\partial t}(s, t)\right\| = 1$ for every s, then Y also satisfies $\langle\nabla_{\sigma'}Y(0), J\dot\gamma(s_0)\rangle = 0$.

PROOF. Since $\frac{\partial\alpha}{\partial t}(s, 0)$ is parallel to $J\dot\gamma(s)$, we see

$$0 = \frac{d}{ds}\left\langle\frac{\partial\alpha}{\partial t}(s, 0), \dot\gamma(s)\right\rangle = \left\langle\frac{\partial}{\partial s}\left(\frac{\partial\alpha}{\partial t}\right)(s, 0), \dot\gamma(s)\right\rangle + \left\langle\frac{\partial\alpha}{\partial t}(s, 0), \kappa J\dot\gamma(s)\right\rangle,$$

which shows the second assertion.

If a (local) vector field V along γ is orthogonal to both $\dot\gamma(s)$ and $J\dot\gamma(s)$ at each s, we find $\nabla_{\dot\gamma}V$ is also orthogonal to both $\dot\gamma(s)$ and $J\dot\gamma(s)$ by differentiating both sides of the equalities $\langle V(s), \dot\gamma(s)\rangle = 0$ and $\langle V(s), J\dot\gamma(s)\rangle = 0$. Differentiating both sides of $\langle\frac{\partial\alpha}{\partial t}(s, 0), V(s)\rangle = 0$, we see

$$\left\langle\nabla_{\dot\gamma}\frac{\partial\alpha}{\partial t}(s, 0), V(s)\right\rangle = -\left\langle\frac{\partial\alpha}{\partial t}(s, 0), \nabla_{\dot\gamma}V(s)\right\rangle = 0\,,$$

and obtain the third assertion.

We find the last assertion by differentiating both sides of $\left\|\frac{\partial\alpha}{\partial t}(s, t)\right\|^2 = 1$ by s. □

Following Lemma 1, we shall say that a Jacobi field Y along a geodesic σ is associated with a trajectory for $\mathbf{B}_\kappa$ if it satisfies

i) $Y(0) = -\mathrm{sgn}(\kappa)J\sigma'(0)/\|\sigma'(0)\|$,

ii) $\nabla_{\sigma'}Y(0)$ is contained in the complex vector subspace spanned by $\sigma'(0)$ and satisfies $\langle\nabla_{\sigma'}Y(0), J\sigma'(0)\rangle = \kappa\|\sigma'(0)\|^2$.

Here $\mathrm{sgn}(a)$ is the signature of a real number a. It is clear that every Jacobi field associated with a trajectory for $\mathbf{B}_\kappa$ can be obtained by some variation of geodesics associated with this trajectory. When σ is a normal geodesic, that is a geodesic of unit speed, we find a Jacobi field Y associated with a trajectory for $\mathbf{B}_\kappa$ is of the form $Y = at\dot\sigma(t) - \mathrm{sgn}(\kappa)g(t)J\dot\sigma(t) + Y^\perp$ with a constant a, a function g and a vector field $Y^\perp$ along σ which is orthogonal to both

$\dot{\sigma}(t)$ and $J\dot{\sigma}(t)$ at every t. The function g and the vector field $Y^{\perp}$ satisfy $g(0) = 1$, $g'(0) = -|\kappa|$, $Y^{\perp}(0) = 0$, $\nabla_{\dot{\sigma}} Y^{\perp}(0) = 0$.

LEMMA 2. *We consider a surface formed by a variation α of geodesics associated with a trajectory for $\mathbf{B}_\kappa$, which may have singularities. This surface is a complex line if and only if the vector $R\big(J\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\big)\frac{\partial\alpha}{\partial t}$ is parallel to $J\frac{\partial\alpha}{\partial t}$ at each point. In this case it is totally geodesic.*

PROOF. The variation α forms a complex line if and only if the corresponding Jacobi field Y satisfies $Y^{\perp} \equiv 0$. Such case occurs if and only if $R\big(J\frac{\partial\alpha}{\partial t}, \frac{\partial\alpha}{\partial t}\big)\frac{\partial\alpha}{\partial t}$ is parallel to $J\frac{\partial\alpha}{\partial t}$. We show α forms a totally geodesic surface in this case. For the sake of simplicity, we may only treat the case that α is a variation of normal geodesics. In this case we have $\frac{\partial\alpha}{\partial s}(s,t) = -\mathrm{sgn}(\kappa)g(s,t)J\frac{\partial\alpha}{\partial t}(s,t)$ with a function g satisfying

$$\frac{\partial^2 g}{\partial t^2}(s,t) + g(s,t)\mathrm{HR}\bigg(\frac{\partial\alpha}{\partial t}(s,t)\bigg) \equiv 0\,, \quad g(s,0) = 1\,, \quad \frac{\partial g}{\partial t}(s,0) = -|\kappa|\,,$$

where $\mathrm{HR}(v)$ denotes the holomorphic sectional curvature of the line spanned by a unit vector v. Thus we have

$$\begin{aligned}
&\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t} = 0\,,\\
&\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t} = \nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial s} = -\mathrm{sgn}(\kappa)\nabla_{\frac{\partial\alpha}{\partial t}}\bigg(gJ\frac{\partial\alpha}{\partial t}\bigg) = -\mathrm{sgn}(\kappa)\frac{\partial g}{\partial t}J\frac{\partial\alpha}{\partial t}\,,\\
&\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial s} = -\mathrm{sgn}(\kappa)\nabla_{\frac{\partial\alpha}{\partial s}}\bigg(gJ\frac{\partial\alpha}{\partial t}\bigg) = -\mathrm{sgn}(\kappa)\bigg(\frac{\partial g}{\partial s}J\frac{\partial\alpha}{\partial t} + gJ\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}\bigg).
\end{aligned}$$

Hence these vectors $\nabla_{\frac{\partial\alpha}{\partial t}}\frac{\partial\alpha}{\partial t}$, $\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial t}$, $\nabla_{\frac{\partial\alpha}{\partial s}}\frac{\partial\alpha}{\partial s}$ are contained in the tangent space of a surface formed by α, and the surface is totally geodesic. □

We call a point $\sigma(t_0)$ a $\mathbf{B}_\kappa$-trajectory focal point of $\sigma(0)$ if there is a Jacobi field along σ associated with a $\mathbf{B}_\kappa$-trajectory which vanishes at t_0, and call the value $t_0/\|\sigma'\|$ a $\mathbf{B}_\kappa$-trajectory focal value of $\sigma(0)$. The minimum positive $\mathbf{B}_\kappa$-trajectory focal value is said to be the first $\mathbf{B}_\kappa$-trajectory focal value of $\sigma(0)$, and is denoted by $t_f(\sigma(0);\sigma,\kappa)$ or $t_f(\sigma(0);\kappa)$. In case every point $\sigma(t)$, $t > 0$ is not a $\mathbf{B}_\kappa$-trajectory focal point of $\sigma(0)$ we put $t_f(\sigma(0);\sigma,\kappa) = \infty$. We denote the maximum negative $\mathbf{B}_\kappa$-trajectory focal value by $-t_n(\sigma(0);\sigma,\kappa)$. We also put $t_n(\sigma(0);\sigma,\kappa) = \infty$ if there are no negative $\mathbf{B}_\kappa$-trajectory focal values.

We here study $\mathbf{B}_\kappa$-trajectory focal values for a complex space form $\mathbf{C}M^n(c)$, which is a complex projective space $\mathbf{C}P^n(c)$ of constant holomorphic sectional curvature c, a complex Euclidean space $\mathbf{C}^n$ or a complex hyperbolic space $\mathbf{C}H^n(c)$ of constant holomorphic sectional curvature c according c is positive, null or negative. On a complex Euclidean space $\mathbf{C}^n$, a trajectory for $\mathbf{B}_\kappa$ is a circle of radius $1/|\kappa|$ in the sense of Euclidean geometry. Thus geodesics associated with a trajectory meet at its center, hence the first $\mathbf{B}_\kappa$-trajectory focal value is $1/|\kappa|$ and has no negative $\mathbf{B}_\kappa$-trajectory focal values.

On a complex projective space $\mathbf{C}P^n(c)$, every trajectory lies on some totally geodesic sphere $\mathbf{C}P^1(c)$ and is a small circle. If we regard this as a latitude line, geodesics associated with this trajectory meet at poles of $\mathbf{C}P^1(c)$ which are the center of this small circle and its anti-podal point. Hence the first $\mathbf{B}_\kappa$-trajectory focal value is $(1/\sqrt{c})\tan^{-1}(\sqrt{c}/|\kappa|)$ and the maximum negative $\mathbf{B}_\kappa$-trajectory focal value is $-(1/\sqrt{c})\{\pi - \tan^{-1}(\sqrt{c}/|\kappa|)\}$. A Jacobi field along a normal geodesic σ associated with a trajectory for $\mathbf{B}_\kappa$ on $\mathbf{C}P^n(c)$ is of the form

$$Y_c(t) = at\dot{\sigma}(t) + \left\{ -\operatorname{sgn}(\kappa)\cos\sqrt{c}t + \frac{\kappa}{\sqrt{c}}\sin\sqrt{c}t \right\} J\dot{\sigma}(t)$$

with a constant a.

On a complex hyperbolic space $\mathbf{C}H^n(c)$, every trajectory lies on some totally geodesic real hyperbolic plane $\mathbf{C}H^1(c)$. A Jacobi field along a normal geodesic σ associated with a trajectory for $\mathbf{B}_\kappa$ on $\mathbf{C}H^n(c)$ is of the form

$$Y_c(t) = at\dot{\sigma}(t) + \left\{ -\operatorname{sgn}(\kappa)\cosh\sqrt{|c|}t + \frac{\kappa}{\sqrt{|c|}}\sinh\sqrt{|c|}t \right\} J\dot{\sigma}(t)$$

with a constant a. Therefore, if $|\kappa| \le \sqrt{|c|}$, there are no $\mathbf{B}_\kappa$-trajectory focal points, and if $|\kappa| > \sqrt{|c|}$, the first $\mathbf{B}_\kappa$-trajectory focal value is $(1/2\sqrt{|c|})\log(\sqrt{|c|}+|\kappa|)/(|\kappa|-\sqrt{|c|})$ and there is no negative $\mathbf{B}_\kappa$-trajectory focal values.

We denote by $t_f(c;\kappa)$ and $-t_n(c;\kappa)$ the first $\mathbf{B}_\kappa$-trajectory focal value and the maximum negative $\mathbf{B}_\kappa$-trajectory focal value on a complex space form of constant holomorphic sectional curvature c. Then we see

$$t_f(c;\kappa) = \begin{cases} \frac{1}{\sqrt{c}}\tan^{-1}\frac{\sqrt{c}}{|\kappa|}, & \text{if } c > 0, \\ \frac{1}{|\kappa|}, & \text{if } c = 0, \\ \frac{1}{2\sqrt{|c|}}\log\frac{\sqrt{|c|}+|\kappa|}{|\kappa|-\sqrt{|c|}}, & \text{if } c < 0 \text{ and } |\kappa| > \sqrt{|c|}, \\ \infty, & \text{if } c < 0 \text{ and } |\kappa| \le \sqrt{|c|}, \end{cases}$$

$$t_n(c;\kappa) = \begin{cases} \frac{1}{\sqrt{c}}\left(\pi - \tan^{-1}\frac{\sqrt{c}}{|\kappa|}\right), & \text{if } c > 0, \\ \infty, & \text{if } c \le 0. \end{cases}$$

We here give comparison results on first $\mathbf{B}_\kappa$-focal values.

PROPOSITION 1. *Let M be a Kähler manifold and σ be a normal geodesic on M. If the sectional curvatures of 2-planes spanned by $\dot{\sigma}(t)$ and a vector orthogonal to $\dot{\sigma}(t)$ are not greater than c for $0 \le t \le t_f(\sigma(0);\sigma,\kappa)$, then we have the following*:

(1) $t_f(\sigma(0);\sigma,\kappa) \ge t_f(c;\kappa)$.

(2) *Let Y be a Jacobi field along a geodesic σ which is associated with a trajectory for $\mathbf{B}_\kappa$, and $\hat{Y}$ be a Jacobi field along a normal geodesic $\hat{\sigma}$ which is associated with a trajectory for $\mathbf{B}_\kappa$ on a simply connected surface $\hat{M} = \mathbf{C}M^1(c)$ of constant sectional curvature c.*

If Y is orthogonal to $\dot\sigma$ and $\hat Y$ is orthogonal to $\dot{\hat\sigma}$, then $\|Y(t)\| \geq \|\hat Y(t)\|$ for every t with $0 \leq t \leq t_f(c;\kappa)$. The equality $\|Y(t_0)\| = \|\hat Y(t_0)\|$ $(0 < t_0 \leq t_f(c;\kappa))$ holds if and only if $Y(t)$ is parallel to $J\dot\sigma(t)$ and the holomorphic sectional curvature of the line spanned by $\dot\sigma(t)$ is equal to c for $0 \leq t \leq t_0$.

PROOF. For the sake of simplicity, we suppose $\kappa > 0$. We denote Y and $\hat Y$ by $Y = hE$ and $\hat Y = -gJ\dot{\hat\sigma}$ with functions g, h and a vector field E along σ satisfying $h(0) = g(0) = 1$, $\|E\| = 1$ and $\langle E, \dot\sigma\rangle = 0$. As $\langle \nabla_{\dot\sigma} E, E\rangle = 0$ and $\langle \nabla_{\dot\sigma}\nabla_{\dot\sigma} E, E\rangle = -\|\nabla_{\dot\sigma} E\|^2$, we see

$$h'' + h(\langle R(E,\dot\sigma)\dot\sigma, E\rangle - \|\nabla_{\dot\sigma} E\|^2) = 0\,.$$

Therefore, for $0 \leq t < \min\{t_f(\sigma(0);\sigma,\kappa), t_f(c;\kappa)\}$ we have

$$\begin{aligned}(h'g - hg')' &= hg(\|\nabla_{\dot\sigma} E\|^2 - \langle R(E,\dot\sigma)\dot\sigma, E\rangle + c)\\ &\geq hg(c - \langle R(E,\dot\sigma)\dot\sigma, E\rangle) \geq 0\,.\end{aligned}$$

By the definition of Jacobi fields associated with trajectories for $\mathbf{B}_\kappa$, we see $h'(0) = g'(0) = -\kappa$ and $(h'g - hg')(0) = 0$. Therefore we find $(h/g)' \geq 0$, hence $h \geq g$.

The equality $h(t_0) = g(t_0)$ at some point $0 < t_0 < t_f(c;\kappa)$ holds if and only if $\nabla_{\dot\sigma} E \equiv 0$ and $\langle R(E,\dot\sigma)\dot\sigma, E\rangle \equiv c$ for $0 \leq t \leq t_0$. This is the case that $E = -J\dot\sigma$ and the holomorphic sectional curvature of the complex line spanned by $\dot\sigma(t)$ is c for $0 \leq t \leq t_0$. □

The proof of Proposition 1 also guarantees the following.

PROPOSITION 2. *Let M be a Kähler manifold and σ be a normal geodesic on M. If the sectional curvatures of 2-planes spanned by $\dot\sigma(t)$ and a vector orthogonal to $\dot\sigma(t)$ are not greater than c for $t < 0$, then $t_n(\sigma(0);\sigma,\kappa) \geq t_n(c;\kappa)$.*

For two unit tangent vectors $v, w \in UM$ $(v \neq \pm w)$, we denote by $\mathrm{Riem}(v,w)$ the sectional curvature of the plane spanned by v, w. In view of the values $t_f(c;\kappa)$ of the first $\mathbf{B}_\kappa$-focal value on a complex space form and $t_n(c;\kappa)$, these propositions guarantee the following.

COROLLARY 1. *On a Kähler manifold M whose sectional curvature satisfies* $\mathrm{Riem} \leq c$ *with some nonpositive constant c, every variation of geodesics associated with a trajectory for $\mathbf{B}_\kappa$ forms an immersed surface without singularities if $|\kappa| \leq \sqrt{|c|}$.*

3. Crescents and Bow-shapes

In Riemannian geometry, a comparison theorem for triangles plays quite an important role. This Toponogov theorem says that triangles on a manifold of large sectional curvature are fatter than triangles on a manifold of small sectional curvature. We here prepare a corresponding result for trajectories. As a substitute for a triangle we consider a bow-shape, which is consisted of a trajectory segment and a kind of geodesic segment.

A *crescent* for $\mathbf{B}_\kappa$ on a Kähler manifold M is a pair $\mathcal{C} = (\gamma, \tau)$ of a trajectory segment $\gamma : [0, L] \to M$ for $\mathbf{B}_\kappa$ and a nonnegative function $\tau : [0, L] \to [0, \infty)$ satisfying $\tau(0) =$

$\tau(L) = 0$ and $0 \le \tau(s) < t_f(\gamma(s); \kappa)$ for every s. For a crescent $\mathcal{C} = (\gamma, \tau)$ we call γ the arc of $\mathcal{C}$. If $\alpha : [0, L] \times \mathbf{R} \to M$ is the variation of normal geodesics associated with γ, we call the set $\mathrm{Rep}(\mathcal{C}) = \{\alpha(s, t) \mid 0 \le s \le L, 0 \le t \le \tau(s)\}$ the represented shape of $\mathcal{C}$. We denote by $\rho_{\mathcal{C}}$ the curve $[0, L] \ni s \mapsto \alpha(s, \tau(s)) \in M$. If a crescent $\mathcal{B}$ with arc γ has the minimum length of periphery among crescents $\mathcal{C}$ with arc γ, that is, $\mathrm{length}(\rho_{\mathcal{C}}) \ge \mathrm{length}(\rho_{\mathcal{B}})$, we shall call it a *bow-shape* with arc γ, and call the curve $\rho_{\mathcal{B}}$ the bow-string of $\mathcal{B}$. As a matter of course, a bow-shape does not necessarily exists for every trajectory segment. Roughly speaking, if a bow-string $\rho_{\mathcal{B}}$ exists for a trajectory segment γ its image is an image of minimal geodesic joining the origin and terminus of γ on a part of a surface formed by a variation of geodesics associated with γ. We here make mention of bow-shapes on a complex space form $\mathbf{C}M^n(c)$ of constant holomorphic sectional curvature c. For bow-strings we sometimes call their images also bow-strings.

EXAMPLE 1. On a complex Euclidean space $\mathbf{C}^n$, for a trajectory segment γ for $\mathbf{B}_\kappa$ with $\mathrm{length}(\gamma) < \pi/|\kappa|$, we have a unique bow-shape $\mathcal{B}$ whose bow-string $\rho_{\mathcal{B}}$ is an image of a geodesic segment and satisfies

$$\mathrm{length}(\rho_{\mathcal{B}}) = \frac{2}{|\kappa|}\sin\left(\frac{1}{2}|\kappa|\mathrm{length}(\gamma)\right).$$

The image of this bow-shape lies on a totally geodesic $\mathbf{C}^1$. But if $\mathrm{length}(\gamma) \ge \pi/|\kappa|$, there does not exist bow-shapes with arc γ.

EXAMPLE 2. On a complex projective space $\mathbf{C}P^n(c)$, for a trajectory segment γ for $\mathbf{B}_\kappa$ with $\mathrm{length}(\gamma) < \pi/\sqrt{\kappa^2 + c}$, we have a unique bow-shape $\mathcal{B}$ whose bow-string $\rho_{\mathcal{B}}$ is an image of a geodesic segment and satisfies

$$\sqrt{c}\sin\left(\frac{1}{2}\sqrt{\kappa^2 + c}\ \mathrm{length}(\gamma)\right) = \sqrt{\kappa^2 + c}\sin\left(\frac{1}{2}\sqrt{c}\ \mathrm{length}(\rho_{\mathcal{B}})\right).$$

In particular, we see $\mathrm{length}(\rho_{\mathcal{B}}) \le (2/\sqrt{c})\sin^{-1}\sqrt{c/(\kappa^2 + c)}$. The image of this bow-shape lies on some totally geodesic standard sphere $\mathbf{C}P^1(c)$, and its periphery consists of a part of a small circle and a part of a great circle.

For a trajectory segment γ with $\mathrm{length}(\gamma) \ge \pi/\sqrt{\kappa^2 + c}$, there does not exist bow-shapes with arc γ.

EXAMPLE 3. Let γ be a trajectory segment for $\mathbf{B}_\kappa$ on a complex hyperbolic space $\mathbf{C}H^n(c)$.

(1) When $|\kappa| \le \sqrt{|c|}$, we have a unique bow-shape $\mathcal{B}$ whose bow-string $\rho_{\mathcal{B}}$ is an image of a geodesic segment and satisfies

$$\begin{cases} \sqrt{|c|}\sinh\left(\sqrt{|c| - \kappa^2}\ \mathrm{length}(\gamma)/2\right) \\ \qquad\qquad = \sqrt{|c| - \kappa^2}\sinh(\sqrt{|c|}\ \mathrm{length}(\rho_{\mathcal{B}})/2), & \text{if } |\kappa| < \sqrt{|c|}, \\ \sqrt{|c|}\ \mathrm{length}(\gamma) = 2\sinh(\sqrt{|c|}\ \mathrm{length}(\rho_{\mathcal{B}})/2), & \text{if } \kappa = \pm\sqrt{|c|}. \end{cases}$$

(2) When $|\kappa| > \sqrt{|c|}$, if length$(\gamma) < \pi/\sqrt{\kappa^2+c}$, we have a unique bow-shape $\mathcal{B}$ whose bow-string $\rho_{\mathcal{B}}$ is an image of a geodesic segment and satisfies

$$\sqrt{|c|}\sin\left(\frac{1}{2}\sqrt{\kappa^2+c}\ \mathrm{length}(\gamma)\right) = \sqrt{\kappa^2+c}\sinh\left(\frac{1}{2}\sqrt{|c|}\ \mathrm{length}(\rho_{\mathcal{B}})\right).$$

In particular, we see length$(\rho_{\mathcal{B}}) \leq (2/\sqrt{|c|})\log\big((|\kappa|+\sqrt{|c|})/\sqrt{\kappa^2+c}\big)$. But if length$(\gamma) \geq \pi/\sqrt{\kappa^2+c}$, there does not exist bow-shapes with arc γ.

(3) Every image of above bow-shapes is contained in some totally geodesic real hyperbolic plane $\mathbf{C}H^1(c)$.

Needless to say that on $\mathbf{C}M^n(c)$ the represented shape of each bow-shape is simply connected. As a matter of fact, it is an image of a simply connected subset of the tangent space through the exponential map. For example, when $\mathcal{B}$ is a bow-shape on $\mathbf{C}^n$ whose arc $\gamma : [0, \ell] \to \mathbf{C}^n$ is a trajectory for $\mathbf{B}_\kappa$, then we see

$$\mathrm{Rep}(\mathcal{B}) = \exp_{\gamma(0)}\left(\left\{\ v(u,\theta) \in T_{\gamma(0)}\mathbf{C}^n\ \middle|\ \begin{array}{l} 0 \leqq \theta \leqq |\kappa|\ell/2, \\ 0 \leqq u \leqq (2/|\kappa|)\sin\theta \end{array}\right\}\right),$$

where $v(u,\theta) = u\cos\theta\ \dot\gamma(0) + \mathrm{sgn}(\kappa)u\sin\theta\ J\dot\gamma(0)$.

We now give a comparison theorem on bow-shapes.

THEOREM 1. *Let M be a Kähler manifold satisfying* Riem $\leq c$ *with a constant c. If a $\mathbf{B}_\kappa$-crescent $\mathcal{C} = (\gamma, \tau)$ satisfies*

i) length$(\gamma) < \pi/\sqrt{\kappa^2+c}$ *when* $\kappa^2 + c > 0$,

ii) $\tau(s) < t_f(c;\kappa)$ *for every s,*

then length$(\rho_{\mathcal{C}})$ *is not smaller than the length* length$(\rho_{\mathcal{B}})$ *of a bow-string of a $\mathbf{B}_\kappa$-bow-shape $\mathcal{B}$ on a complex space form $\mathbf{C}M^n(c)$ whose length of arc is* length(γ).

The equality length$(\rho_{\mathcal{C}})$ = length$(\rho_{\mathcal{B}})$ *holds if and only if the represented shape of $\mathcal{C}$ is complex analytically isometrically immersed image of the represented shape of $\mathcal{B}$ and is totally geodesic. In this case, it is a bow-shape with arc γ.*

PROOF. Put L = length(γ). We take a trajectory segment $\hat\gamma$ on $\hat M = \mathbf{C}M^n(c)$ satisfying length$(\hat\gamma) = L$, and consider a crescent $\hat{\mathcal{C}} = (\hat\gamma, \tau)$. Let $\alpha : [0, L] \times \mathbf{R} \to M$ and $\hat\alpha : [0, L] \times \mathbf{R} \to \hat M$ be variations of normal geodesics associated with γ and $\hat\gamma$ respectively. As we have $\langle \frac{\partial\alpha}{\partial s}, \frac{\partial\alpha}{\partial t}\rangle = \langle \frac{\partial\hat\alpha}{\partial s}, \frac{\partial\hat\alpha}{\partial t}\rangle = 0$, we find by Proposition 1 that

$$\begin{aligned} \mathrm{length}(\rho_{\mathcal{C}}) &= \int_0^L \|\rho_{\mathcal{C}}'(s)\|ds = \int_0^L \sqrt{\left\|\frac{\partial\alpha}{\partial s}(s,\tau(s))\right\|^2 + \tau'(s)^2}\ ds \\ &\geq \int_0^L \sqrt{\left\|\frac{\partial\hat\alpha}{\partial s}(s,\tau(s))\right\|^2 + \tau'(s)^2}\ ds = \mathrm{length}(\rho_{\hat{\mathcal{C}}}) \\ &\geq \mathrm{length}(\rho_{\mathcal{B}}). \end{aligned}$$

The equality holds if and only if $\left\|\frac{\partial\alpha}{\partial s}(s,\tau(s))\right\| = \left\|\frac{\partial\hat{\alpha}}{\partial s}(s,\tau(s))\right\|$ holds for every s and $\hat{\mathcal{C}} = \mathcal{B}$. Again by Proposition 1, we see it is the case that the holomorphic sectional curvature of the line spanned by $\frac{\partial\alpha}{\partial s}(s,t)$ is c for every (s,t) with $0 \le t \le \tau(s)$. □

REMARK. The above proof also shows that the length of smooth curve on a surface $A = \{\alpha(s,t) \mid 0 \le s \le L,\ 0 \le t \le t_f(c;\kappa)\}$ joining $\gamma(0)$ and $\gamma(L)$ is not smaller than the length of bow-string length$(\rho_{\mathcal{B}})$ of a bow-shape $\mathcal{B}$ on $\mathbf{C}M^n(c)$.

If a crescent $\widetilde{\mathcal{C}} = (\gamma, \tilde{\tau})$ on M does not lie on A, as we have s_0 with $\tau(s_0) > t_f(c;\kappa)$, we find length$(\rho_{\widetilde{\mathcal{C}}}) \ge 2t_f(c;\kappa) >$ length$(\rho_{\mathcal{B}})$. In particular, if there is a crescent $\widetilde{\mathcal{C}}$ on M with length$(\rho_{\widetilde{\mathcal{C}}}) =$ length$(\rho_{\mathcal{B}})$, then it should lie on A.

When $\kappa^2 + c \le 0$, or when $\kappa^2 + c > 0$ and $L < \pi/\sqrt{\kappa^2 + c}$, we denote by $\ell(\kappa, L; c)$ the length of a bow-string of a bow-shape for $\mathbf{B}_\kappa$ on a complex space form $\mathbf{C}M^n(c)$ whose length of arc is L. We set $\delta(\kappa, c) = \pi/\sqrt{\kappa^2 + c}$ when $\kappa^2 + c > 0$ and $\delta(\kappa, c) = \infty$ when $\kappa^2 + c \le 0$. By standing another point of view, we can conclude the following.

PROPOSITION 3. *Let M be a Kähler manifold satisfying* Riem $\le c$ *with a constant c. If a bow-shape $\mathcal{C}$ for $\mathbf{B}_\kappa$ on M satisfies* length$(\rho_{\mathcal{C}}) = \ell(\kappa, L; c)$ *for some positive L satisfying $L < \delta(\kappa, c)$, then the length of the arc of $\mathcal{C}$ is not longer than L.*

If a trajectory segment γ for $\mathbf{B}_\kappa$ on a Kähler manifold satisfying Riem $\le c$ has a bow-shape $\mathcal{B}$ with arc γ and length$(\rho_{\mathcal{B}}) = \ell(\kappa, \text{length}(\gamma); c)$, then the holomorphic sectional curvature of the complex line spaned by $\dot{\gamma}$ is c. Taking account of this we shall say that a trajectory γ for $\mathbf{B}_\kappa$ ($\kappa \ne 0$) on a Kähler manifold is of c-space type if there exists a sequence $\{s_j\}_{j=-\infty}^{\infty}$ satisfying the following conditions:

i) $\lim_{j\to\infty} s_j = \delta(\kappa, c)$ and $\lim_{j\to-\infty} s_j = -\delta(\kappa, c)$,

ii) for each s_j ($s_j \ne 0$), the trajectory segment $\gamma|_{I_j}$, which is a restriction of γ on the interval I_j, has a bow-shape $\mathcal{B}_j = (\gamma|_{I_j}, \tau_j)$ with length$(\rho_{\mathcal{B}_j}) = \ell(\kappa, |s_j|; c)$, where $I_j = [0, s_j]$ for $s_j > 0$ and $I_j = [s_j, 0]$ for $s_j < 0$.

It is needless to say that every trajectory on a complex space form $\mathbf{C}M^n(c)$ is of c-space type. A trajectory γ for $\mathbf{B}_\kappa$ on $\mathbf{C}M^n(c)$ is closed if and only if $\kappa^2 + c > 0$. In this case its minimal period length(γ) is $2\pi/\sqrt{\kappa^2 + c} = 2\delta(\kappa, c)$ and the geodesic with initial vector sgn$(\kappa)J\dot{\gamma}(0)$ goes through the point $\gamma(\delta(\kappa, c))$ (see [1]).

As a direct consequence of Theorem 1 we have

COROLLARY 2. *Let M be a Kähler manifold satisfying* Riem $\le c$.

(1) *If $b > c$, there does not exist a trajectory of b-space type.*

(2) *If $\kappa^2 + c > 0$, every trajectory γ of c-space type for $\mathbf{B}_\kappa$ is closed and* length$(\gamma) = 2\pi/\sqrt{\kappa^2 + c}$. *For a variation α of normal geodesics associated with γ, the interior*

$$\mathcal{F}_\alpha = \{\alpha(s,t) \mid |s| \le \delta(\kappa; c),\ 0 \le t \le t_f(c;\kappa)\}$$

is totally geodesic, complex and of constant curvature c.

PROOF. (1) Since $\ell(\kappa, L; b) < \ell(\kappa, L; c)$ when $b > c$, the first assertion is trivial by Theorem 1.

(2) If there is a crescent $\mathcal{C} = (\beta, \tau)$ on M with length$(\beta) < \delta(\kappa, c)$ and length$(\rho_{\mathcal{C}}) = \ell(\kappa, \text{length}(\beta); c)$, then by Theorem 1 and Remark we see the represented shape of $\mathcal{C}$ is totally geodesic and of holomorphic sectional curvature c. We put $v_j = \dot{\rho}_{\mathcal{C}_j}(0)/\|\dot{\rho}_{\mathcal{C}_j}(0)\| \in U_{\gamma(s_j)}M$. Since the represented shape of $\mathcal{B}_j$ is complex analytically isometrically immersed image of the represented shape of a bow-shape on $\mathbf{C}M^n(c)$ whose length of arc is $|s_j|$, we find that $\lim_{j\to\infty} v_j = \lim_{j\to-\infty} v_j = \text{sgn}(\kappa)J\dot{\gamma}(0)$. This shows $\gamma(\delta(\kappa, c)) = \gamma(-\delta(\kappa, c))$, hence γ is closed and length$(\gamma) = 2\delta(\kappa, c)$. □

In the last stage we make mention of bow-shapes on a product of complex space forms. On a product $M = M_1 \times M_2$ of Kähler manifolds M_i, every trajectory γ is of the form $\gamma(t) = (\gamma_1(\lambda_1 t), \gamma_2(\lambda_2 t))$. Here, λ_1, λ_2 are nonnegative constants with $\lambda_1^2 + \lambda_2^2 = 1$, and γ_i is a trajectory for $\mathbf{B}_{\kappa/\lambda_i}$ on M_i when $\lambda_i > 0$ and is a point curve on M_i when $\lambda_i = 0$ (see [3, 4]). One can easily compute the length of bow-string on a product of complex space forms. For example, on a product $\mathbf{C}P^{n_1}(c_1) \times \cdots \times \mathbf{C}P^{n_p}(c_p)$ of complex projective spaces, a trajectory segment γ of the form $\gamma(t) = (\gamma_1(\lambda_1 t), \cdots, \gamma_p(\lambda_p t))$ for $\mathbf{B}_\kappa$ with nonnegative constants $\lambda_1, \cdots, \lambda_p$ satisfying $\sum_{i=1}^p \lambda_i^2 = 1$ has a bow-shape if

$$\text{length}(\gamma) < \min\left\{ \pi \Big/ \sqrt{\kappa^2 + c_i\lambda_i^2} \;\Big|\; \lambda_i \neq 0,\ 1 \le i \le p \right\}.$$

The length of its bow-string is given by $\sqrt{\sum_{i=1}^p d_i^2}$ with d_i satisfying

$$\lambda_i\sqrt{c_i}\sin\left(\sqrt{\kappa^2 + c_i\lambda_i^2}\ \text{length}(\gamma)\Big/2\right) = \sqrt{\kappa^2 + c_i\lambda_i^2}\,\sin(\sqrt{c_i}\ d_i/2)\,.$$

We here consider a subset $\mathcal{S}_x(c)$ of the unit tangent space U_xM given by

$$\mathcal{S}_x(c) = \left\{ v \in U_xM \;\middle|\; \begin{array}{c} \text{there is a positive } \varepsilon \text{ such that for every} \\ \kappa \text{ with } 0 < |\kappa| \le \varepsilon \text{ the trajectory for } \mathbf{B}_\kappa \\ \text{with initial vector } v \text{ is of } c\text{-space type} \end{array} \right\}.$$

For a complex space form $\mathbf{C}M^n(c)$ we see $U_x\mathbf{C}M^n(c) = \mathcal{S}_x(c)$ at each point, and for a product of complex space forms $M = \mathbf{C}M^{n_1}(c_1) \times \cdots \times \mathbf{C}M^{n_p}(c_p)$, we see that $\mathcal{S}_x(c)$ is either an empty set or a disjoint sum of spheres; $\mathcal{S}_x(c) = S^{2n_{i_1}-1} + \cdots + S^{2n_{i_q}-1}$, where $c = c_{i_j}$ for $1 \le j \le q$, $c_i \neq c$ for $i \neq i_j$. Here, if we denote $x \in M$ by $(x_1, \cdots, x_p)$, the set $S^{2n_{i_j}}$ corresponds to $U_{x_{i_j}}\mathbf{C}M^{n_{i_j}}(c)$. For a Hermitian symmetric space M of rank r, it was pointed out by Ikawa[10] that every trajectory lies on a totally geodesic r-product $\mathbf{C}M^1(c) \times \cdots \times \mathbf{C}M^1(c)$, where c is the maximum sectional curvature when M is of compact type and is the minimum sectional curvature when M is of noncompact type. We hence see that for this c the set $S_x(c)$ contains a r-sum $S^1 + \cdots + S^1$ of circles S^1.

COROLLARY 3. *Let M be a simply connected Kähler manifold of* Riem $\leq c$ *for some nonnegative c. If $\mathcal{S}_x(c) \neq \emptyset$ at some point x, then M contains a totally geodesic $\mathbf{C}M^1(c)$.*

PROOF. For $v \in \mathcal{S}_x(c)$ we denote by γ_κ $(0 < |\kappa| < \varepsilon)$ a trajectory of c-space type for $\mathbf{B}_\kappa$ with $\dot{\gamma}_\kappa(0) = v$, and by α_κ the variation of normal geodesics associated with γ_κ. Since M satisfies Riem $\leq c$, we see $\mathcal{F}_{\alpha_\kappa}$ is totally geodesic and of constant curvature c. Moreover, as M is simply connected, the condition that γ_κ is of c-space type guarantees that $\mathcal{F}_{\alpha_\kappa}$ is contained in the inside of the geodesic ball centered at $\gamma(0)$ whose radius is the injectivity radius at $\gamma(0)$. Thus $\mathcal{F}_{\alpha_\kappa}$ is an image of a simply connected subset of $\{a\dot{\gamma}(0) + bJ\dot{\gamma}(0) \mid a \in \mathbf{R}, b > 0\}$ through the exponential map $\exp_{\gamma(0)}$ when $\kappa > 0$ and is an image of a simply connected subset of $\{a\dot{\gamma}(0)+bJ\dot{\gamma}(0) \mid a \in \mathbf{R}, b<0\}$ through this exponential map when $\kappa < 0$. We hence find that $\mathcal{F}_{\alpha_\kappa}$ is simply connected and that $\mathcal{F}_{\alpha_{\kappa_1}} \supset \mathcal{F}_{\alpha_{\kappa_2}}$ if $0 < \kappa_1 < \kappa_2 < \varepsilon$ or $0 > \kappa_1 > \kappa_2 > -\varepsilon$. Therefore we see $\mathcal{F} = \bigcup_{0<|\kappa|<\varepsilon} \mathcal{F}_{\alpha_\kappa}$ is totally geodesic and is complex analytically isometric to $\mathbf{C}M^1(c) \setminus \{$image of a geodesic on $\mathbf{C}M^1(c)\}$. Since the topological closure

$$\overline{\mathcal{F}} = \mathcal{F} \cup \{\text{the image of the geodesic with initial vector } v\}$$

of $\mathcal{F}$ is of constant curvature c, we see it is complex analytically isometric to $\mathbf{C}M^1(c)$. □

If we restrict ourselves on Hermitian symmetric spaces, as every trajectory lies on a totally geodesic r-product of $\mathbf{C}M^1$'s, the following is trivial.

COROLLARY 4. *If a Hermitian symmetric space M satisfies $\mathcal{S}_x(c) = U_xM$ for some c, then M is $\mathbf{C}M^n(c)$.*

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Present Address:
DEPARTMENT OF MATHEMATICS, NAGOYA INSTITUTE OF TECHNOLOGY,
NAGOYA 466–8555, JAPAN.
e-mail: adachi@nitech.ac.jp