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Boundary Theta Curves in S^3

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Abstract. It is introduced a split extension of groups $1 \to P_2 \to C_{1,2} \to \Theta \to 1$, where P_2 is the group of pure braids in 2 strings, $C_{1,2}$ is the group of cobordism classes of (pure) 2-string links and Θ is the group of cobordism classes of theta curves. The concept of boundary theta curve is introduced and it is proved that the group of boundary cobordism classes of boundary theta curves is isomorphic to the group of boundary cobordism classes of boundary string links in 2 strings.

1. Introduction

One way of trying to detect how intertwined are the different components of a link is the concept of boundary link (that is, a link whose components bound disjoint Seifert surfaces). This paper introduces a similar concept for theta curves.

It is divided in two parts. In the first part we relate the group of cobordism of theta curves with the group of cobordism classes of 2-string links. String links were introduced in [2] and there it was observed that since the group of cobordism classes of (pure) n-string links contains the group of (pure) n-braids, it is non abelian if $n \geq 3$, on the other side, if $n = 1$, it coincides with the group of cobordism classes of knots and therefore is abelian. The case $n = 2$ was open, but it follows from Proposition 2 below that it is non abelian.

In the second part we introduce the group of boundary cobordism classes of boundary theta curves and prove that it is isomorphic to the group BSL(2) of boundary cobordism classes of boundary 2-string links, introduced in [1].

2. Theta curves in S^3

The group of cobordism of theta curves was introduced in [5]. We recall it. We work in the piecewise linear category.

A *labelled theta curve*, or simply *theta curve*, is a graph θ with two vertices v_1 and v_2 and three edges e_1, e_2 and e_3 each of which joins v_1 and v_2. We give an orientation from v_1 to v_2 to each edge.

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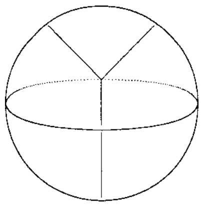

FIGURE 1.

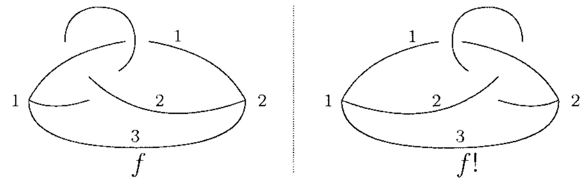

FIGURE 2.

Let $f : \theta \to S^3$ be an embedding of θ into the three-sphere S^3. Then f is called a *spatial embedding* and the image $f(\theta)$ is called a *spatial theta curve*. Then $f(e_2 \cup e_3)$, $f(e_3 \cup e_1)$ and $f(e_1 \cup e_2)$ together with the orientation of e_2, e_3 and e_1 respectively are oriented knots in S^3 which are called *constituent knots* of f and denoted by $k_1(f)$, $k_2(f)$ and $k_3(f)$ respectively.

Two embedding $f, g : \theta \to S^3$ are said to be *cobordant* if there is a 'locally flat' embedding $\Phi : \theta \times [0, 1] \to S^3 \times [0, 1]$ such that $\Phi\big|_{\theta \times \{0\}} : \theta \times \{0\} \to S^3 \times \{0\}$ equals to f and $\Phi\big|_{\theta \times \{1\}} : \theta \times \{1\} \to S^3 \times \{1\}$ equals to g, where 'locally flat' means that the image of Φ in $S^3 \times [0, 1]$ is locally homeomorphic to the standard disk pair (D^4, D^2) or $(D^3, Y) \times D^1$ where (D^3, Y) is shown in Figure 1.

The cobordism class of f is denoted by $[f]$. For two spatial embeddings $f : \theta \to S_1^3$, $g : \theta \to S_2^3$, remove small balls centered at $f(v_2)$ and $g(v_1)$ from S_1^3 and S_2^3 respectively, then identify the boundaries such that the images of the i-th edge are joined for each i. Then we obtain a new embedding of θ into $S^3 = S_1^3 \# S_2^3$ which is called the *vertex connected sum* of f and g and denoted by $f \# g$. The vertex connected sum is well defined up to ambient isotopy [6].

PROPOSITION 1 (Taniyama). *The cobordism classes of embeddings of θ into S^3 form a group under the vertex connected sum.*

The inverse of $[f]$ is $[f!]$ where $f!$ is the reflected inverse of f. See Figure 2.

We denote this group by Θ.

For an embedding $f : \theta \to S^3$, choose a regular neighbourhood N of $f(e_i)$ in S^3, $i \in \{1, 2, 3\}$. Then the pair $(N, N \cap f(\theta))$ is homeomorphic to (D^3, A) of Figure 3.

Let $h_i : (D^3, A) \to (N, N \cap f(\theta))$ be a homeomorphism such that $(f(\theta) \setminus N) \cup h_i(B)$ is a 2-component link with linking number zero, where B is a pair of strings in D^3 as illustrated in Figure 4.

h_i is called a 0-*framing* of $f(e_i)$ and the link is called the *i-th parallel link* of f and denoted by $\ell_i(f)$.

The following result was suggested to us by J. Levine.

PROPOSITION 2. *There is a split extension of groups*

$$1 \to P_2 \to C_{1,2} \overset{\alpha}{\to} \Theta \to 1$$

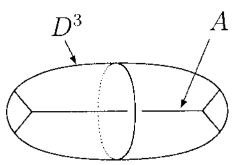

FIGURE 3.

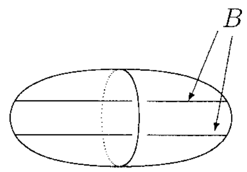

FIGURE 4.

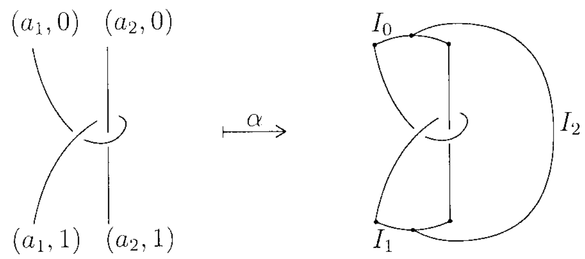

FIGURE 5.

where P_2 *is the group of pure braids in 2 strings and* $C_{1,2}$ *is the group of cobordism classes of 2-string links.* (see [1] for definitions and notation)

PROOF. Considering $D \times I \subseteq \mathbb{R}^3 \subseteq S^3$ and three intervals I_0, I_1, I_2 embedded in S^3, I_0 connecting $(a_1, 0)$ to $(a_2, 0)$, I_1 connecting $(a_1, 1)$ to $(a_2, 1)$ and I_2 connecting the midpoints of I_0 and I_1, we can associate to each 2-string link a theta curve (see Figure 5).

A cobordism between 2-string links can clearly be extended to a cobordism between the respective theta curves. Besides this association provides a homomorphism $\alpha : C_{1,2} \to \Theta$. α is surjective since given a theta curve one can, without changing its cobordism class, push one of its vertices along one of its edges until obtain one unknotted edge that does not undercross nor overcross the others.

According to [2], the inclusion map $j_k : P_k \to C_{1,k}$ is a monomorphism, that is, two pure braids in k strings are ambient isotopic if and only if they are cobordant as k-string links.

Let $b(f)$ be the unique pure braid in 2 strings whose linking number between the strings is the same as the linking number between the 2 strings of a given 2-string link f.

By [5] (Theorem 5), the class of cobordism of f is in the kernel of α if and only if the closure of $b(f^{-1}) \cdot f$ is slice but, by [3], this happens if and only if $b(f^{-1}) \cdot f$ is slice (i.e., cobordant to the trivial string link). Since P_2 is in the center of $C_{1,2}$, the map $\beta : C_{1,2} \to C_{1,2}$ that sends the cobordism class of f to the cobordism class of $b(f^{-1}) \cdot f$ is also a homomorphism. The cobordism class of f belongs to $\ker \beta$ if and only if f is cobordant to $b(f^{-1})^{-1} = b(f)$, that is, if and only if $f \in \operatorname{im} j_2$. Therefore $\ker \alpha = \ker \beta = \operatorname{im} j_2 \cong P_2$ and we have an extension of groups

$$1 \to P_2 \to C_{1,2} \overset{\alpha}{\to} \Theta \to 1\,.$$

It is clear that $\alpha \circ \beta = \alpha$. This equation together with the fact that $\ker \alpha = \ker \beta$ implies that α maps the image of β isomorphically onto Θ. Thus the extension splits as stated. $\square$

Le Dimet has observed that $C_{1,1}$ is commutative (the group of cobordism of knots) while $C_{1,k}$ for $k \geq 3$ is not (since there is a monomorphism $j_k : P_k \to C_{1,k}$). From Proposition 2 it follows that $C_{1,2}$ is not commutative since, by [4], the theta curve cobordism group Θ is not commutative.

3. Boundary theta curves

DEFINITION 1. A **Seifert surface** for a θ-curve f is a pair (S_1, S_2) of oriented connected surfaces in S^3 such that ∂S_1 and ∂S_2 are two different constituent knots of f, $S_i \cap f(\theta) = \partial S_i$ $(i = 1, 2)$ and $(S_1 \setminus \partial S_1) \cap (S_2 \setminus \partial S_2) = \emptyset$. A θ-curve that has a Seifert surface is a **boundary θ-curve**[1].

If f is a boundary θ-curve, given any two constituent knots of f there are Seifert surfaces as in the definition. This can be seen geometrically but we shall follow a different approach.

Let us represent a general θ-curve f as in Figure 6.

Let us call **top meridians** of f the following elements x_1, x_2 and x_3 of $\pi_1(f)$ (where $\pi_1(f)$ stands for the fundamental group of the complement of $f(\theta)$.)

They are obtained by taking the homotopy class of the following loops: beginning at the eye of the reader the loop goes straight to the beginning of one arrow, follows that arrow until its end and goes back to the eye of the reader.

Similarly one defines **bottom meridians** y_1, y_2 and y_3 (see Figure 8).

DEFINITION 2. Let $F(2)$ be the free group in 2 generators α_1 and α_2 . An epimorphism $\eta : \pi_1(f) \to F(2)$ that sends two top meridians $x_i \neq x_j$, $i, j \in \{1, 2, 3\}$, to α_1 and α_2 respectively and send the corresponding bottom meridians y_i to α_1 and y_j to α_2 is called a **splitting** for the θ-curve f .

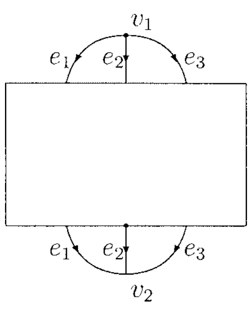

FIGURE 6.

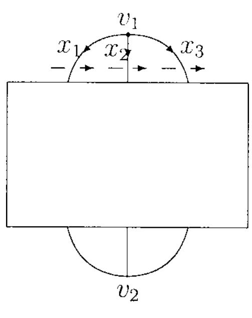

FIGURE 7.

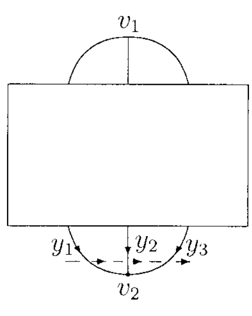

FIGURE 8.

[1] At the time of revision it was brought to our attention that R. Shinjo and R. Nikkuni are also working with this concept, in a different direction.

It follows from Thom-Pontryagin construction that a θ-curve has a splitting if and only if it is a boundary θ-curve.

By composing η with automorphisms of $F(2)$ we see that the existence of the splitting does not depend upon which meridians have index i and j.

From this it follows easily that the vertex connected sum of boundary θ-curves is a boundary θ-curve.

If F is a cobordism between θ-curves f_1 and f_2, there are homomorphism $i_j : \pi_1(f_j) \to \pi_1(F)$ induced by the inclusion maps. If f_1 and f_2 are boundary θ-curves with splittings η_1 and η_2 respectively, an epimorphism $\eta : \pi_1(F) \to F(2)$ such that $\eta \circ i_j = \eta_j$, $j = 1, 2$, is called a **splitting for the cobordism** F. A cobordism F that has a splitting η is called a **boundary cobordism** between f_1 and f_2 and in this case, f_1 and f_2 are said to be **boundary cobordant** (boundary) θ-curves.

Of course the reflected inverse of a boundary θ-curve is a boundary θ-curve and we have a group $B\Theta$ of boundary cobordism classes of boundary θ-curves.

If f is a boundary string-link in two strings, its associated θ-curve is a boundary θ-curve. In fact, there is a splitting $\eta : \pi_1(f) \to F(2)$ sending x_1 and y_1 to α_1 and x_2 and y_2 to α_2. If $\widetilde{f}$ is the associated θ-curve, then $\pi_1(\widetilde{f}) \cong \pi_1(f)$ and we have the correspondent splitting $\widetilde{\eta} : \pi_1(\widetilde{f}) \to F(2)$.

Besides if f_1 and f_2 are boundary cobordant (boundary) string links with splittings η_1 and η_2 and if F is a boundary cobordism between them with splitting η, we have that F induces a boundary cobordism $\widetilde{F}$ between the θ-curves $\widetilde{f_1}$ and $\widetilde{f_2}$ with a splitting $\widetilde{\eta}$ that extends $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$.

Therefore there is an epimorphism $\beta : \mathrm{BSL}(2) \to B\Theta$, $\beta(f) = \widetilde{f}$, where BSL(2) is the group of boundary cobordism classes of boundary string links in 2 strings.

Let f be a boundary string-link in two strings and $\widetilde{f}$ its associated θ-curve. Since f is a boundary string link, the linking number between its strings is zero, so one of the parallel links of $\widetilde{f}$, let us call it $\ell_1(\widetilde{f})$, is just the closure $\widehat{f}$ of f (see [1] for definition.)

Clearly $\widetilde{f}$ is boundary slice (that is, represents the unit element of $B\Theta$) if and only if $\ell_1(\widetilde{f})$ is a boundary slice link, but $\ell_1(\widetilde{f}) = \widehat{f}$ and, by [1], Theorem 14, $\widehat{f}$ is boundary slice if and only if f is. Therefore we have

PROPOSITION 3. *$\beta : BSL(2) \to B\Theta$ is an isomorphism.* □

A question that still remains is if $B\Theta$ is abelian.

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