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Deligne-Lusztig Induction for Invariant Functions on Finite Lie Algebras of Chevalley's Type

Emmanuel LETELLIER

Sophia University

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Abstract. Let G be a connected reductive algebraic group defined over $\mathbf{F}_q$ with Lie algebra $\mathcal{G}$. We define a Deligne-Lusztig induction for the $\bar{\mathbf{Q}}_\ell$-valued functions on $\mathcal{G}(\mathbf{F}_q)$ which are invariant under the adjoint action of $G(\mathbf{F}_q)$ on $\mathcal{G}(\mathbf{F}_q)$, by making use of the "character formula" where the "two-variable Green functions" are defined via a G-equivariant homeomorphism $\mathcal{G}_{nil} \to G_{uni}$. We verify that it satisfies properties analogous to the group case like transitivity, the Mackey formula or the commutation with duality. The interest of a Deligne-Lusztig induction for invariant functions comes from a conjecture on a commutation formula with Fourier transforms which has no counterpart in the group case. In a forthcoming paper, this conjecture will be proved in almost all cases.

Introduction

Let G be a connected reductive group over an algebraic closure $\mathbf{F}$ of the finite field $\mathbf{F}_q$ with q elements and let p be the characteristic of $\mathbf{F}$. Assume that G is defined over $\mathbf{F}_q$ with associated Frobenius endomorphism F. Then the Lie algebra $\mathcal{G}$ of G and the adjoint action of G on $\mathcal{G}$ are also defined over $\mathbf{F}_q$. We still denote by F the corresponding Frobenius endomorphism on $\mathcal{G}$. We then denote by G^F (resp. $\mathcal{G}^F$) the set of the elements of G (resp. $\mathcal{G}$) which are fixed by F. Let ℓ be a prime $\neq p$ and let $\bar{\mathbf{Q}}_\ell$ be an algebraic closure of the field $\mathbf{Q}_\ell$ of ℓ-adic numbers. We denote by $\mathcal{C}(\mathcal{G}^F)$ the $\bar{\mathbf{Q}}_\ell$-vector space of $\bar{\mathbf{Q}}_\ell$-valued functions on $\mathcal{G}^F$ which are invariant under the adjoint action of G^F on $\mathcal{G}^F$. Let L be an F-stable Levi subgroup of a parabolic subgroup P of G and let $\mathcal{L}$ be the Lie algebra of L. If P is F-stable, then we have the Lie algebra version of Harish-Chandra induction $\mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$. The aim of this paper is to generalize this induction to the case where P is not necessarily F-stable. In the group setting such a generalization, called Deligne-Lusztig induction, has been constructed in [DL76]. In [DM87][Lus86], we have a formula, called "character formula", which expresses the values of the Deligne-Lusztig induction of a class function f on L^F in terms of the values of f and the values of some unipotently supported functions, called "two-variable Green functions" [DM87]. Our definition of Deligne-Lusztig induction in the Lie algebra setting uses the Lie algebra version of the character formula where the two-variable Green functions are transferred to the Lie algebras via a G-equivariant homeomorphism between the nilpotent

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subvariety $\mathcal{G}_{nil}$ of $\mathcal{G}$ and the unipotent subvariety G_{uni} of G. The author was informed that Lusztig already knew this definition (unpublished). In a forthcoming paper, the author will use this definition of Deligne-Lusztig induction to prove for almost all G, a commutation formula between Fourier transforms and Deligne-Lusztig induction. Such a commutation formula was proved by Lehrer [Leh96] for Harish-Chandra induction. It will be also shown that this definition of Deligne-Lusztig induction does not depend on the choice of a G-equivariant homeomorphism $\mathcal{G}_{nil} \to G_{uni}$.

In this paper we start by recalling some well-known facts about the space $\mathcal{C}(\mathcal{G}^F)$ of G^F-invariant functions on $\mathcal{G}^F$. The second part will be devoted to the definition of Deligne-Lusztig induction; we will also verify elementary properties analogous to the group case like transitivity or the fact that it generalizes Harish-Chandra induction. In the fourth part, we will prove the Mackey formula (following Bonnafé's method) and its consequences, like the commutation with the duality map.

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Preliminaries

No assumption on p is required unless otherwise specified.

NOTATION 0.0.1. Let H be a linear algebraic group over $\mathbf{F}$. If $x \in H$, we denote by x_s the semi-simple part of x and by x_u the unipotent part of x. We denote by H^o the neutral component of H and by Z_H the center of H. If $x \in H$, the centralizer of x in H is denoted by $C_H(x)$; it will be more convenient to denote the neutral component of $C_H(x)$ by $C^o_H(x)$ rather than by $C_H(x)^o$. Let $\mathcal{H} = \text{Lie}(H)$ be the Lie algebra of H, for $x \in \mathcal{H}$, we denote by x_s the semi-simple part of x and by x_n the nilpotent part of x. The adjoint action of H on $\mathcal{H}$ is denoted by $\text{Ad} = \text{Ad}_H$ and we put $\text{ad} = \text{ad}_{\mathcal{H}}$ the differential of Ad at $1 \in H$. If K is a subgroup of H, by "K-orbit of $\mathcal{H}$", we shall mean "$\text{Ad}(K)$-orbit of $\mathcal{H}$". If $x \in \mathcal{H}$, then we denote by $C_H(x)$ the centralizer of x in H i.e. $C_H(x) = \{h \in H | \text{Ad}(h)x = x\}$ and by $C_{\mathcal{H}}(x) := \{y \in \mathcal{H} | \text{ad}(x)(y) = 0\}$. If $x \in \mathcal{H}$ is semi-simple, we have $\text{Lie}\big(C_H(x)\big) = C_{\mathcal{H}}(x)$ [Bor, 9.1].

NOTATION 0.0.2. Let now G be a connected reductive algebraic group over $\mathbf{F}$ with Lie algebra $\mathcal{G}$. We assume that G is defined over $\mathbf{F}_q$ and we denote by F the corresponding Frobenius endomorphisms on G and on $\mathcal{G}$. If P is a parabolic subgroup of G, we will denote by U_P the unipotent radical of P and by $\mathcal{U}_P$ the Lie algebra of U_P. Recall that a Levi subgroup L of a parabolic subgroup P of G is a closed subgroup L of P such that $P = L \ltimes U_P$. We will use the shorter expression "Levi subgroup of G" instead of "Levi subgroup of a parabolic subgroup of G". We say that an F-stable Levi subgroup of G is G-split if it is a Levi subgroup of an F-stable parabolic subgroup of G. The letter T will denote a maximal torus of G. The dimension of T is called the *rank* of G and is denoted by $rk(G)$. We denote by Φ the root system of G with respect to T. If $\alpha \in \Phi$, we denote by $\mathcal{G}_\alpha$ the one-dimensional

$\mathbf{F}$-vector space $\{x \in \mathcal{G} | \forall t \in T, \mathrm{Ad}(t)x = \alpha(t)x\}$ and by U_α the unique closed connected one-dimensional unipotent subgroup of G normalized by T such that $\mathrm{Lie}(U_\alpha) = \mathcal{G}_\alpha$. Finally we denote by G_{uni} the subvariety of unipotent elements of G and by $\mathcal{G}_{nil}$ the subvariety of nilpotent elements of $\mathcal{G}$.

REMARK 0.0.3. We will have to consider the Lie algebra of the intersection of closed subgroups of G. This appears for instance in the Mackey formula. Let M and N be two closed subgroups of G, we always have

$$(*)\ \ \mathrm{Lie}(M \cap N) \subset \mathrm{Lie}(M) \cap \mathrm{Lie}(N)\,.$$

In general this inclusion is not an equality; it is an equality exactly when the quotient morphism $\pi : G \to G/N$ induces a separable morphism $M \to \pi(M)$, see [Bor, Proposition 6.12]. However if $M \cap N$ contains a maximal torus of G, then by [Bor, Proposition 13.20], the inclusion (*) is an equality; note that [Bor, Corollary 13.21], which asserts that (*) is an equality whenever M and N are normalized by a maximal torus of G, is not correct since the intersection of two subtori of a maximal torus of G may have finite intersection while their Lie algebras have an intersection of strictly positive dimension. For instance, let $G = SL_3(\mathbf{F})$ and let T be the maximal torus of G consisting of diagonal matrices, then the set Z_G is finite and it is the intersection of the two subtori $T_\alpha = \mathrm{Ker}(\alpha)$ and $T_\beta = \mathrm{Ker}(\beta)$ of T where $\alpha : T \to \mathbf{F}$, $(t_1, t_2, t_1^{-1}t_2^{-1}) \mapsto t_1 t_2^{-1}$ and $\beta : T \to \mathbf{F}$, $(t_1, t_2, t_1^{-1}t_2^{-1}) \mapsto t_2^2 t_1$. The intersection of the Lie algebras of T_α and T_β is of dimension 0 unless $p = 3$, in which case the intersection is of dimension 1.

We will be interested only in the cases where the subgroups M and N in (*) are either equal to $L, L', U_P, U_{P'}, P$ or P' where $P = LU_P$ and $P' = L'U_{P'}$ are two Levi decompositions in G such that $L \cap L'$ contains a maximal torus T of G. In any of these cases, the inclusion (*) is always an equality; the case where $M = U_P$ and N is either $U_{P'}$, L' or P' follows from the fact that the dimension of $M \cap N$ and the dimension of $\mathrm{Lie}(M) \cap \mathrm{Lie}(N)$ are respectively equal to the number of $\alpha \in \Phi$ such that $U_\alpha \subset M \cap N$ and the number of $\alpha \in \Phi$ such that $\mathcal{G}_\alpha \subset \mathrm{Lie}(M) \cap \mathrm{Lie}(N)$.

1. The space of G^F-invariant functions on $\mathcal{G}^F$

We mostly recall here the parts of [Leh96] which will be used in this paper.

NOTATION 1.0.4. Let H be an F-stable closed subgroup of G with Lie algebra $\mathcal{H}$. For any $x \in \mathcal{H}^F$, we denote by $\gamma_x^H \in \mathcal{C}(\mathcal{H}^F)$ the function which takes the value $|C_H(x)^F|$ on the H^F-orbit of x and the value 0 elswhere. We denote by $\eta_o^{\mathcal{H}} \in \mathcal{C}(\mathcal{H}^F)$ the function which takes the value 1 on the set of nilpotent elements of $\mathcal{H}^F$ and the value 0 elsewhere.

NOTATION 1.0.5. Throughout this paper, we choose once for all an automorphism $\bar{\mathbf{Q}}_\ell \to \bar{\mathbf{Q}}_\ell, x \mapsto \bar{x}$ such that $\bar{\zeta} = \zeta^{-1}$ for any root of unity ζ of $\bar{\mathbf{Q}}_\ell$.

DEFINITION 1.0.6. Let H be an F-stable closed subgroup of G with Lie algebra $\mathcal{H}$. For $f, g \in \mathcal{C}(\mathcal{H}^F)$, define the non-degenerate bilinear form $(\ ,\ )_{\mathcal{H}^F}$ by,

$$(f, g)_{\mathcal{H}^F} = |H^F|^{-1} \sum_{x \in \mathcal{H}^F} f(x)\overline{g(x)}\,.$$

Note that for $x \in \mathcal{H}^F$ and $f \in \mathcal{C}(\mathcal{H}^F)$, we have $(f, \gamma_x^H)_{\mathcal{H}^F} = f(x)$ and $(\gamma_x^H, f)_{\mathcal{H}^F} = \overline{f(x)}$.

DEFINITION 1.0.7. Let P be an F-stable parabolic subgroup of G and L be an F-stable Levi subgroup of P. Let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the Lie algebra decomposition corresponding to $P = LU_P$ and let $\pi_{\mathcal{P}} : \mathcal{P} \to \mathcal{L}$ be the canonical projection.

(i) The Harish-Chandra restriction ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{L}^F)$ is defined by the following formula

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x) = |U_P^F|^{-1} \sum_{y \in \mathcal{U}_P^F} f(x + y).$$

(ii) The Harish-Chandra induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathcal{G}^F)$ is defined by

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x) = |P^F|^{-1} \sum_{\{g \in G^F | \mathrm{Ad}(g)x \in \mathcal{P}^F\}} f(\pi_{\mathcal{P}}(\mathrm{Ad}(g)x))\,.$$

We have the following proposition (see [Leh96]).

PROPOSITION 1.0.8. *The maps* ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ *and* $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ *are adjoint with respect to the forms* $(\ ,\ )_{\mathcal{G}^F}$ *and* $(\ ,\ )_{\mathcal{L}^F}$. *Moreover they are independent of* P.

NOTATION 1.0.9. Since the map $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ is independent of P, we write $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ instead of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$.

1.0.10. We define (following Kawanaka [Kaw82] in the Lie algebra case and Lusztig, Curtis and Alvis in the group case) the "duality map" $\mathcal{D}_{\mathcal{G}} : \mathcal{C}(\mathcal{G}^F) \to \mathcal{C}(\mathcal{G}^F)$. For any connected reductive group H defined over $\mathbf{F}_q$, we denote by $r(H)$ the semi-simple $\mathbf{F}_q$-rank of H, i.e. the $\mathbf{F}_q$-rank of H/Z_H^o.

DEFINITION 1.0.11. Let B be an F-stable Borel subgroup of G. For $f \in \mathcal{C}(\mathcal{G}^F)$, we define $\mathcal{D}_{\mathcal{G}}(f)$ by

$$\mathcal{D}_{\mathcal{G}}(f) = \sum_{P \supset B} (-1)^{r(P)} \mathcal{R}^{\mathcal{G}}_{\mathcal{L}_P} \circ {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}_P}(f)$$

where the summation is over the set of the F-stable parabolic subgroups P of G containing B and where $\mathcal{L}_P$ denotes the Lie algebra of an arbitrarily chosen F-stable Levi subgroup of P.

It is known that the map $\mathcal{D}_{\mathcal{G}}$ does not depend on the F-stable Borel subgroup B and on the choice of the $\mathcal{L}_P$.

PROPOSITION 1.0.12. [Kaw82] *We have the following assertions*,
(i) *The duality map $\mathcal{D}_{\mathcal{G}}$ is an isometry with respect to the form $(\ ,\ )_{\mathcal{G}^F}$.*
(ii) *$\mathcal{D}_{\mathcal{G}}$ is an involution, i.e. $\mathcal{D}_{\mathcal{G}} \circ \mathcal{D}_{\mathcal{G}} = Id_{\mathcal{C}(\mathcal{G}^F)}$.*

PROPOSITION 1.0.13. [Leh96, Proposition 3.15] *Let L be an F-stable G-split Levi subgroup of G and let $\mathcal{L} = Lie(L)$. Then*

$$\mathcal{D}_{\mathcal{G}} \circ \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} = \mathcal{R}_{\mathcal{L}}^{\mathcal{G}} \circ \mathcal{D}_{\mathcal{L}}\,.$$

2. Deligne-Lusztig induction: definition and basic properties

2.1. Deligne-Lusztig induction for class functions. If X is a variety over $\mathbf{F}$, then we denote by $H_c^i(X, \bar{\mathbf{Q}}_\ell)$ the i-th group of ℓ-adic cohomology with compact support as in [Del77]. All what we need to know (in this paper) about these groups can be found in [DM91, Chapter 10].

2.1.1. Let L be an F-stable Levi subgroup of G, let $P = LU_P$ be a Levi decomposition of a (possibly non F-stable) parabolic subgroup P of G and let $\mathcal{P} = \mathcal{L} \oplus \mathcal{U}_P$ be the corresponding Lie algebra decomposition. We denote by $\mathcal{L}_G$ the Lang map $G \to G, x \mapsto x^{-1}F(x)$. The variety $\mathcal{L}_G^{-1}(U_P)$ is endowed with an action of G^F on the left and with an action of L^F on the right. By [DM91, Proposition 10.2], these actions induce actions on the cohomology and so make $H_c^i(\mathcal{L}_G^{-1}(U_P), \bar{\mathbf{Q}}_\ell)$ into a G^F-module-L^F. The virtual $\bar{\mathbf{Q}}_\ell$-vector space $H_c^*(\mathcal{L}_G^{-1}(U_P)) := \sum_i (-1)^i H_c^i(\mathcal{L}_G^{-1}(U_P), \bar{\mathbf{Q}}_\ell)$ is thus a virtual G^F-module-L^F.

NOTATION 2.1.2. If $(g, l) \in G^F \times L^F$, define $S_{L\subset P}^G(g, l) := \text{Trace}((g, l^{-1})| H_c^*(\mathcal{L}_G^{-1}(U_P)))$.

To each L^F-module M, corresponds thus a virtual G^F-module $R_{L\subset P}^G(M) := H_c^*(\mathcal{L}_G^{-1}(U_P)) \otimes_{L^F} M$ (see [Lus76]). Hence, using the basis of the $\bar{\mathbf{Q}}_\ell$-vector space of class functions on L^F formed by the irreducible characters of L^F, the map $R_{L\subset P}^G$ gives rise to a natural $\bar{\mathbf{Q}}_\ell$-linear map, so-called *Deligne-Lusztig induction* and still denoted by $R_{L\subset P}^G$, from the $\bar{\mathbf{Q}}_\ell$-vector space of class functions on L^F onto the $\bar{\mathbf{Q}}_\ell$-vector space of class functions on G^F. More precisely if f is a class function on L^F, the class function $R_{L\subset P}^G(f)$ on G^F is given by the following formula:

2.1.3. $R_{L\subset P}^G(f)(g) = |L^F|^{-1} \sum_{h\in L^F} S_{L\subset P}^G(g, h) f(h)$ for any $g \in G^F$.

REMARK 2.1.4. It is conjectured and proved for large enough values of q that $R_{L\subset P}^G$ is independent of the parabolic subgroup P having L as a Levi subgroup (see section 3 for more details).

We now define the two-variable Green functions; they appear naturally in the computation of the values of the Deligne-Lusztig induction of class functions (see 2.1.6 below).

DEFINITION 2.1.5. The function $Q^G_{L\subset P}: G^F\times L^F\to\bar{\mathbf{Q}}_\ell$ defined by

$$Q^G_{L\subset P}(u,v)=\begin{cases}|L^F|^{-1}\mathrm{Trace}\Big((u,v^{-1})|\ H^*_c(\mathcal{L}_G^{-1}(U_P))\Big) & \text{if } (u,v)\in G^F_{uni}\times L^F_{uni}\,,\\ 0 & \text{otherwise}\,.\end{cases}$$

is called a two-variable Green function.

In the case where L is a maximal torus of G, the two-variable Green functions become one-variable functions and are the ordinary Green functions introduced for any reductive groups by Deligne-Lusztig [DL76]. In the case of $G=GL_n(\mathbf{F})$, they were first introduced by Green [Gre55].

The following formula [DM91, 12.2][DM87][Lus86], the so-called character formula for $R^G_{L\subset P}$, expresses the values of the functions $R^G_{L\subset P}(f)$, where f is a class function on L^F, in terms of the values of f and in terms of the values of some two-variable Green functions:

2.1.6. For any $x\in G^F$,

$$R^G_{L\subset P}(f)(x)=$$
$$|L^F|^{-1}|C^o_G(x_s)^F|^{-1}\sum_{\{h\in G^F|x_s\in {}^hL\}}|C^o_{{}^hL}(x_s)^F|\sum_{v\in(C^o_{{}^hL}(x_s)_{uni})^F}Q^{C^o_G(x_s)}_{C^o_{{}^hL}(x_s)}(x_u,v)\,{}^hf(x_sv)$$

where ${}^hL:=hLh^{-1}$ and ${}^hf(y):=f(h^{-1}yh)$.

To simplify the notation, we usually omit the parabolic subgroup ${}^hP\cap C^o_G(x_s)$ from the notation $Q^{C^o_G(x_s)}_{C^o_{{}^hL}(x_s)}$.

2.2. Deligne-Lusztig induction for invariant functions. In the Lie algebra setting, we define the Deligne-Lusztig induction using the Lie algebra version of the character formula where the two-variable Green functions are transfered to the Lie algebra by means of a G-equivariant homeomorphism $\mathcal{G}_{nil}\to G_{uni}$, where G acts by conjugacy on G_{uni} and by the adjoint action on $\mathcal{G}_{nil}$.

ASSUMPTION 2.2.1. From now we assume that p is good for G so that there exists a G-equivariant homeomorphism $\phi:\mathcal{G}_{nil}\to G_{uni}$ defined over $\mathbf{F}_q$ [Spr69].

LEMMA 2.2.2. [Bon02, Lemma 3.2] *For any Levi decomposition $P=LU_P$ in G with corresponding Lie algebra decomposition $\mathcal{P}=\mathcal{L}\oplus\mathcal{U}_P$, we have*:

(i) $\phi(\mathcal{L}_{nil})=L_{uni}$,

(ii) *for any $x\in\mathcal{L}_{nil}$, $\phi(x+\mathcal{U}_P)=\phi(x)U_P$.*

DEFINITION 2.2.3. With the notation of 2.1.1, the two-variable Green function $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} : \mathcal{G}^F \times \mathcal{L}^F \to \mathbf{Z}$ is defined by

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(u,v) = \begin{cases} |L^F|^{-1}\mathrm{Trace}((\phi(u),\phi(v)^{-1})|\ H^*_c(\mathcal{L}_G^{-1}(U_P))) & \text{if } (u,v)\in\mathcal{G}^F_{nil}\times\mathcal{L}^F_{nil}\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

REMARK 2.2.4. Assume that ϕ is the exponential map (which is well-defined if $p > 3(h^G_o - 1)$ where h^G_o is the Coxeter number of G). Let T be an F-stable maximal torus of G contained in a (possibly non F-stable) Borel subgroup B of G. Assume that $\sigma \in \mathcal{T}^F$ satisfies $C^o_G(\sigma) = T$ and let $\mathcal{B} = \mathcal{T} \oplus \mathcal{U}_B$ be the Lie algebra decomposition corresponding to $B = TU_B$. By a result of Kazhdan-Springer [Kaz77][Spr76], for any non-trivial additive character $\Psi : \mathbf{F}_q \to \bar{\mathbf{Q}}_\ell^\times$, any non-degenerate, symmetric, G-invariant bilinear form $\langle , \rangle : \mathcal{G}\times\mathcal{G} \to \mathbf{F}$ defined over $\mathbf{F}_q$, we have, for any $u \in \mathcal{G}^F_{nil}$:

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{T}\subset\mathcal{B}}(u,0) = \varepsilon_G\varepsilon_T q^{\frac{|\Phi|}{2}} \sum_{x\in\mathcal{O}^{G^F}_\sigma} \Psi(\langle x,u\rangle)$$

where $\varepsilon_G = (-1)^{\mathbf{F}_q - rank(G)}$ and where $\mathcal{O}^{G^F}_\sigma$ denotes the G^F-orbit of σ.

DEFINITION 2.2.5. Let L be an F-stable Levi subgroup of G and let $P = LU_P$ be a Levi decomposition of P with corresponding Lie algebra decomposition $\mathcal{P} = \mathcal{L}\oplus\mathcal{U}_P$.

(i) Let $f \in \mathcal{C}(\mathcal{L}^F)$, then the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f) \in \mathcal{C}(\mathcal{G}^F)$ of f is defined by

$$\begin{aligned}&\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(x_s+x_n)\\ &\quad= |L^F|^{-1}|C^o_G(x_s)^F|^{-1} \sum_{\{h\in G^F|x_s\in{}^h\mathcal{L}\}} |C^o_{{}^hL}(x_s)^F| \sum_{v\in C_{{}^h\mathcal{L}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{{}^h\mathcal{L}}(x_s)}(x_n,v)\mathrm{Ad}_h(f)(x_s+v)\end{aligned}$$

where for any $g \in G^F$, ${}^gL := gLg^{-1}$, ${}^g\mathcal{L} = \mathrm{Ad}(g)\mathcal{L}$ and $\mathrm{Ad}_g : \mathcal{C}(\mathcal{L}^F) \to \mathcal{C}(\mathrm{Ad}(g)\mathcal{L}^F)$ is given by, $\mathrm{Ad}_g(f)(x) = f(\mathrm{Ad}(g^{-1})x)$.

(ii) Let $g \in \mathcal{C}(\mathcal{G}^F)$, then the Deligne-Lusztig restriction ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g) \in \mathcal{C}(\mathcal{L}^F)$ of g is defined by

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)(x_s+x_n) = |C^o_L(x_s)^F||C^o_G(x_s)^F|^{-1} \sum_{u\in C_{\mathcal{G}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{L}}(x_s)}(u,x_n)g(x_s+u)\,.$$

The group version of 2.2.5(ii) is due to Digne-Michel [DM87].

REMARK 2.2.6. Since p is good for G, the connected component of the centralizer in G of a semi-simple element of $\mathcal{G}$ is a Levi subgroup of G. Indeed, if $\mathcal{T}$ is the Lie algebra of the maximal torus T of G, then for $x \in \mathcal{T}$, the set $\{\alpha\in\Phi|d\alpha(x)=0\}$, where $d\alpha : \mathcal{T}\to\mathbf{F}$ is the differential of α at 1, is a $\mathbf{Q}$-closed root subsystem of Φ [Slo80, 3.14]. Hence, with the notation of 2.2.5, the map ϕ induces a well-defined map $C_{\mathcal{G}}(x_s)_{nil} \to C^o_G(x_s)_{uni}$.

REMARK 2.2.7. The notation $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ is used both for Deligne-Lusztig induction and Harish-Chandra induction; this is justified by 2.3.7.

OPEN PROBLEM 2.2.8. Define $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ using ℓ-adic cohomology but without using any G-equivariant homeomorphism $\mathcal{G}_{nil}\to G_{uni}$.

REMARK 2.2.9. It follows easily from the formulae of 2.2.5 that

(i) for any $f\in\mathcal{C}(\mathcal{L}^F)$, we have

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f.\eta^{\mathcal{L}}_o)=\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f).\eta^{\mathcal{G}}_o\,,$$

(ii) for any $g\in\mathcal{C}(\mathcal{G}^F)$, we have

$$^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g.\eta^{\mathcal{G}}_o)={}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g).\eta^{\mathcal{L}}_o.$$

2.3. Basic properties of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$

In this section, we prove the transitivity of the $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$. We also verify that $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincides with Harish-Chandra induction if P is F-stable, and that $^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ and $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ are adjoint with respect to $(\ ,\ )_{\mathcal{L}^F}$ and $(\ ,\ )_{\mathcal{G}^F}$. Note that the group version of these properties are proved from the general properties of generalized induction associated to a bi-module [DM91, Chapters 4, 11], and so it is not possible to adapt these proofs to our Lie algebra version of Deligne-Lusztig induction; we will thus come down to problems on two-variable Green functions.

As it can be seen from 2.1.3, the function $S^G_{L\subset P}:G^F\times L^F\to\bar{\mathbf{Q}}_\ell$ plays a fundamental role in Deligne-Lusztig's theory. We would like to have such a function in the Lie algebra case; this is possible thanks to [DM91, Lemma 12.3] which gives an expression of $S^G_{L\subset P}(g,l)$ (where $g\in G^F, l\in L^F$) in terms of the values of some two-variable Green functions. More precisely the function $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}:\mathcal{G}^F\times\mathcal{L}^F\to\bar{\mathbf{Q}}_\ell$ we are looking for is defined as follows:

DEFINITION 2.3.1. For $x\in\mathcal{G}^F$, $y\in\mathcal{L}^F$, we define $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)$ by

$$S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)=\sum_{\{h\in G^F|\mathrm{Ad}(h)y_s=x_s\}}|C^o_L(y_s)^F||C^o_G(y_s)^F|^{-1}\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(\mathrm{Ad}(h^{-1})x_n,y_n).$$

REMARK 2.3.2. Note that $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)=|L^F|\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)$ for any $(x,y)\in\mathcal{G}^F_{nil}\times\mathcal{L}^F_{nil}$.

The following lemma is the Lie algebra version of 2.1.3:

LEMMA 2.3.3. *Let $f\in\mathcal{C}(\mathcal{G}^F)$, $g\in\mathcal{C}(\mathcal{L}^F)$, we have*

$$(1)\quad \mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)(x)=|L^F|^{-1}\sum_{y\in\mathcal{L}^F}S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)g(y)\,,$$

$$(2)\quad {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y) = |G^F|^{-1}\sum_{x\in\mathcal{G}^F} S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(x)\,.$$

PROOF. We first prove (2).

$$\begin{aligned}
&|G^F|^{-1}\sum_{x\in\mathcal{G}^F} S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(x)\\
&\quad= |G^F|^{-1}|C^o_L(y_s)^F||C^o_G(y_s)^F|^{-1}\sum_{x\in\mathcal{G}^F}\ \sum_{\{h\in G^F|\mathrm{Ad}(h)y_s=x_s\}} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(\mathrm{Ad}(h^{-1})x_n, y_n)f(x)\\
&\quad= |G^F|^{-1}|C^o_L(y_s)^F||C^o_G(y_s)^F|^{-1}\\
&\quad\quad\times\sum_{h\in G^F}\ \sum_{x_n\in C_{\mathcal{G}}(\mathrm{Ad}(h)y_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(\mathrm{Ad}(h^{-1})x_n, y_n)f(\mathrm{Ad}(h)y_s + x_n)\\
&\quad= |G^F|^{-1}|C^o_L(y_s)^F||C^o_G(y_s)^F|^{-1}\\
&\quad\quad\times\sum_{h\in G^F}\ \sum_{x_n\in C_{\mathcal{G}}(y_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{L}}(y_s)}(x_n, y_n)f(y_s + x_n) = {}^*\mathcal{R}^{G}_{L\subset P}(f)(y)\,.
\end{aligned}$$

Using the G-equivariance of ϕ, it is straightforward to see that,

$$S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y) = \sum_{h\in G^F|\mathrm{Ad}(h)y_s=x_s} |C^o_{{}^hL}(x_s)^F||C^o_G(x_s)^F|^{-1}\mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{{}^h\mathcal{L}}(x_s)}(x_n, \mathrm{Ad}(h)y_n)\,.$$

It is then not difficult to get (1). □

PROPOSITION 2.3.4. *The maps* $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ *and* ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ *are adjoint with respect to the forms* $(\ ,\ )_{\mathcal{G}^F}$ *and* $(\ ,\ )_{\mathcal{L}^F}$.

PROOF. Let $g\in\mathcal{C}(\mathcal{L}^F)$ and $f\in\mathcal{C}(\mathcal{G}^F)$. We have

$$\begin{aligned}
(f,\mathcal{R}^{G}_{L\subset P}(g))_{\mathcal{G}^F} &= |G^F|^{-1}\sum_{x\in\mathcal{G}^F} f(x)\overline{\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g)(x)}\\
&= |L^F|^{-1}|G^F|^{-1}\sum_{x\in\mathcal{G}^F}\sum_{y\in\mathcal{L}^F} f(x)\overline{S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)g(y)}\quad\text{by 2.3.3(1)}\\
&= |L^F|^{-1}|G^F|^{-1}\sum_{y\in\mathcal{L}^F}\sum_{x\in\mathcal{G}^F} S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)f(x)\overline{g(y)}\,.
\end{aligned}$$

The last equality follows from the fact that $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x,y)\in\mathbf{Q}$. We thus get from 2.3.3 (2) that $(f,\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(g))_{\mathcal{G}^F} = ({}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f), g)_{\mathcal{L}^F}$. □

We now prove the transitivity of Deligne-Lusztig induction, that is, if $M\subset L\subset G$ is an inclusion of F-stable Levi subgroups of G, we have $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}\circ\mathcal{R}^{\mathcal{L}}_{\mathcal{M}\subset\mathcal{L}\cap\mathcal{Q}} = \mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}$

where $\mathcal{L} = \mathrm{Lie}(L)$ and $\mathcal{M} = \mathrm{Lie}(M)$ and where $\mathcal{P} = \mathrm{Lie}(P)$ and $\mathcal{Q} = \mathrm{Lie}(Q)$ with P, Q two parabolic subgroups of G having respectively L and M as Levi subgroup and such that $Q \subset P$. We start by proving a "transitivity formula" for two-variable Green functions:

LEMMA 2.3.5. *With the above notation, for any $(x, z) \in \mathcal{G}^F_{nil} \times \mathcal{M}^F_{nil}$, we have*

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}(x, z) = \sum_{v\in\mathcal{L}^F_{nil}} \mathcal{Q}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, v)\mathcal{Q}^{\mathcal{L}}_{\mathcal{M}\subset\mathcal{L}\cap\mathcal{Q}}(v, z)\,.$$

PROOF. The lemma will follow from its group version. From the proof of [DM91, 11.5], we have

$$S^G_{M\subset Q}(x, z) = |L^F|^{-1} \sum_{y\in L^F} S^G_{L\subset P}(x, y)S^G_{M\subset L\cap Q}(y, z)$$

for any $(x, z) \in G^F_{uni} \times M^F_{uni}$. By [DM91, Lemma 12.3], we have $S^G_{L\subset P}(x, y) = 0$ if x_s and y_s are not G^F-conjugate. Hence for any $(x, z) \in G^F_{uni} \times M^F_{uni}$, we deduce that

$$S^G_{M\subset Q}(x, z) = |L^F|^{-1} \sum_{y\in L^F_{uni}} S^G_{L\subset P}(x, y)S^G_{M\subset L\cap Q}(y, z)\,.$$

It follows that

$$Q^G_{M\subset Q}(x, z) = \sum_{y\in L^F_{uni}} Q^G_{L\subset P}(x, y)Q^L_{M\subset L\cap Q}(y, z)$$

for any $(x, z) \in G^F_{uni} \times M^F_{uni}$. □

PROPOSITION 2.3.6. *We have*

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}} \circ \mathcal{R}^{\mathcal{L}}_{\mathcal{M}\subset\mathcal{L}\cap\mathcal{Q}} = \mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}.$$

PROOF. Thanks to 2.3.3 (1), it is enough to prove the following statement: for any $x \in \mathcal{G}^F$, $z \in \mathcal{M}^F$, we have

$$|L^F|^{-1} \sum_{y\in\mathcal{L}^F} S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, y)S^{\mathcal{L}}_{\mathcal{M}\subset\mathcal{Q}\cap\mathcal{L}}(y, z) = S^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}(x, z)\,.$$

Now a simple calculation shows that this statement reduces to 2.3.5 . □

We have the following proposition:

PROPOSITION 2.3.7. *If the parabolic subgroup P is F-stable, then the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincides with Harish-Chandra induction.*

PROOF. From the adjunction property 2.3.4 it is equivalent to prove that Deligne-Lusztig restriction ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincides with Harish-Chandra restriction. Let $(x, y) \in \mathcal{G}^F \times \mathcal{L}^F$. We first compute the quantity $S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, y)$. Define $L_{y_s} := C^o_L(y_s)$, $G_{y_s} := C^o_G(y_s)$ and $V_{y_s} := G_{y_s} \cap U_P$. Let $\mathcal{L}_{y_s} = \mathrm{Lie}(L_{y_s})$, $\mathcal{G}_{y_s} = \mathrm{Lie}(G_{y_s})$ and $\mathcal{V}_{y_s} = \mathrm{Lie}(V_{y_s})$; then $L_{y_s}V_{y_s}$ is a Levi decomposition of the parabolic subgroup $P \cap G_{y_s}$ of G_{y_s}. We denote by $\mathcal{L}_{G_{y_s}} : G_{y_s} \to G_{y_s}$ the Lang map $x \mapsto x^{-1}F(x)$. Since V_{y_s} is F-stable, by [DM91, p. 81], the bi-module $H^*_c(\mathcal{L}^{-1}_{G_{y_s}}(V_{y_s}))$ is isomorphic as $G^F_{y_s}$-module-$L^F_{y_s}$ to $\bar{\mathbf{Q}}_\ell[G^F_{y_s}/V^F_{y_s}]$. Let $h \in G^F$ be such that $\mathrm{Ad}(h)y_s = x_s$. Then we have

$$\begin{aligned}\mathcal{Q}^{\mathcal{G}_{y_s}}_{\mathcal{L}_{y_s}}(\mathrm{Ad}(h^{-1})x_n, y_n) &= |L^F_{y_s}|^{-1}\mathrm{Trace}((h^{-1}\phi(x_n)h, \phi(y_n)^{-1})|\bar{\mathbf{Q}}_\ell[G^F_{y_s}/V^F_{y_s}])\\ &= |L^F_{y_s}|^{-1}\sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ (hz)^{-1}\phi(x_n)hz \in \phi(y_n)V^F_{y_s}\}\,.\end{aligned}$$

From the G-equivariance of ϕ, we get that

$$\begin{aligned}\mathcal{Q}^{\mathcal{G}_{y_s}}_{\mathcal{L}_{y_s}}(\mathrm{Ad}(h^{-1})x_n, y_n) &= |L^F_{y_s}|^{-1}\sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \phi(\mathrm{Ad}((hz)^{-1})x_n) \in \phi(y_n)V^F_{y_s}\}\\ &= |L^F_{y_s}|^{-1}\sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \mathrm{Ad}((hz)^{-1})x_n \in y_n + \mathcal{V}^F_{y_s}\} \quad \text{by 2.2.2}\\ &= |L^F_{y_s}|^{-1}\sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \mathrm{Ad}((hz)^{-1})x \in y + \mathcal{V}^F_{y_s}\}\,.\end{aligned}$$

We deduce that

$$S^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(x, y) = |G^F_{y_s}|^{-1} \sum_{\{h\in G^F|\mathrm{Ad}(h)y_s=x_s\}} \sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \mathrm{Ad}((hz)^{-1})x \in y + \mathcal{V}^F_{y_s}\}\,.$$

Thus for any $f \in \mathcal{C}(\mathcal{G}^F)$, we have:

$$\begin{aligned}{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y) &= |G^F|^{-1}|G^F_{y_s}|^{-1}\\ &\times \sum_{x\in\mathcal{G}^F}\ \sum_{\{h\in G^F|\mathrm{Ad}(h)y_s=x_s\}} \sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \mathrm{Ad}((hz)^{-1})x \in y + \mathcal{V}^F_{y_s}\}f(x)\,.\end{aligned}$$

By interchanging the sums we get that

$$\begin{aligned}{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y) &= |G^F|^{-1}|G^F_{y_s}|^{-1}\\ &\times \sum_{h\in G^F}\ \sum_{\{x\in\mathcal{G}^F|x_s=\mathrm{Ad}(h)y_s\}} \sharp\{zV^F_{y_s} \in (G^F_{y_s}/V^F_{y_s})|\ \mathrm{Ad}((hz)^{-1})x \in y + \mathcal{V}^F_{y_s}\}f(x)\,.\end{aligned}$$

We deduce that,

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y) = |G^F|^{-1}|G^F_{y_s}|^{-1} \sum_{h\in G^F}\ \sum_{\substack{x\in\mathcal{G}^F\\ x_s=\mathrm{Ad}(h)y_s}}\ \sum_{\substack{zV^F_{y_s}\in(G^F_{y_s}/V^F_{y_s})\\ \mathrm{Ad}((hz)^{-1})x\in y+\mathcal{V}^F_{y_s}}} f(x)\,.$$

By interchanging the second with the third sum, we get that

$$^{*}\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y)=|G^F|^{-1}|G^F_{y_s}|^{-1}\sum_{h\in G^F}\ \sum_{\{zV^F_{y_s}\in(G^F_{y_s}/V^F_{y_s})\}}\ \sum_{\substack{x\in\mathcal{G}^F\\ x\in \mathrm{Ad}(hz)(y+\mathcal{V}^F_{y_s})}} f(x)\,.$$

Since the function f is G^F-invariant, we deduce that

$$^{*}\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y)=|G^F|^{-1}|G^F_{y_s}|^{-1}\sum_{h\in G^F}\ \sum_{\{zV^F_{y_s}\in(G^F_{y_s}/V^F_{y_s})\}}\ \sum_{v\in\mathcal{V}^F_{y_s}} f(y+v)\,.$$

Hence

$$^{*}\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y)=|V^F_{y_s}|^{-1}\sum_{v\in\mathcal{V}^F_{y_s}} f(y+v)\,. \tag{1}$$

To complete the proof, we need the following proposition (which is the Lie algebra version of [DM91, Proposition 7.1]):

PROPOSITION 2.3.8. *With the above notation, let $h : U^F_P\times\mathcal{V}^F_{y_s}\to y+\mathcal{U}^F_P$ be the morphism given by $h(u,v)=Ad(u)(y+v)$. Then h is surjective and the cardinality of its fibers is equal to $|V^F_{y_s}|$.*

PROOF OF 2.3.8. Since $y\in\mathcal{L}$, the map h is well-defined. To prove the surjectivity of h it is enough to prove that $|\mathrm{Im}(h)|=|U^F_P|$. Let $X=\mathrm{Im}(h)$ and $z\in X$. There exists $\delta\in U^F_P$ and $v\in\mathcal{V}^F_{y_s}$ such that $z=\mathrm{Ad}(\delta)(y+v)$. Now the map $h^{-1}(z)\to h^{-1}(y)$ which sends (γ,w) onto $(\delta^{-1}\gamma, w-\mathrm{Ad}(\gamma^{-1}\delta)v)$ is a bijection whose inverse is given by $(a,x)\mapsto(\delta a, x+\mathrm{Ad}(a^{-1})v)$. Hence the fibers of the map $h : U^F_P\times\mathcal{V}^F_{y_s}\to X$ are all of same cardinality equal to $|h^{-1}(y)|$. We deduce that $|X|=\frac{|U^F_P||\mathcal{V}^F_{y_s}|}{|h^{-1}(y)|}$. Thus we need to prove that $|h^{-1}(y)|=|\mathcal{V}^F_{y_s}|$. Since $y\in\mathcal{L}_{y_s}$, we have $\mathrm{Ad}(u)y-y\in\mathcal{V}_{y_s}$ for any $u\in V_{y_s}$. We thus have an injective map $\psi : V^F_{y_s}\to h^{-1}(y)$ mapping u onto $(u,\mathrm{Ad}(u^{-1})y-y)$. It remains to prove the surjectivity of ψ. Let $(\delta,v)\in h^{-1}(y)$; we have $\mathrm{Ad}(\delta)(y+v)=y$. Since $y\in\mathcal{L}_{y_s}$, by [Leh96, 3.7], there exists $\zeta\in V_{y_s}$ such that $\mathrm{Ad}(\zeta)y_s=(y+v)_s$. We thus have $\mathrm{Ad}(\delta\zeta)y_s=y_s$ from which we deduce that $\delta\in G_{y_s}\cap U_P=V_{y_s}$ which proves the surjectivity of ψ since $\psi(\delta)=(\delta,v)$. □

From 2.3.8 and (1) we deduce that

$$^{*}\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(f)(y)=|U^F_P|^{-1}\sum_{v\in\mathcal{U}^F_P} f(y+v)\,.$$

Hence $^{*}\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ coincides with Harish-Chandra restriction. □

PROPOSITION 2.3.9. *Let L be an F-stable Levi subgroup of G and P be a parabolic subgroup of G having L as a Levi subgroup. Let $\mathcal{L} := Lie(L)$ and $\mathcal{P} := Lie(P)$. Let $x \in \mathcal{L}^F$ be such that $C^o_G(x_s) \subseteq L$, then $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x) = \gamma^G_x$.*

PROOF. We compute the values of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)$. Let $y \in \mathcal{G}^F$, then

$$(\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x), \gamma^G_y)_{\mathcal{G}^F} = \mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)(y)\,.$$

From 2.3.4 we have

$$(\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x), \gamma^G_y)_{\mathcal{G}^F} = (\gamma^L_x, {}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y))_{\mathcal{L}^F}\,.$$

Combining the above two equations we get that

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x)(y) = \overline{{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x)}\,. \tag{1}$$

Now, by definition we have

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = |C^o_L(x_s)^F||C^o_G(x_s)^F|^{-1} \sum_{n\in C_{\mathcal{G}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{L}}(x_s)}(n, x_n)\gamma^G_y(x_s + n)\,.$$

Since by assumption $C^o_G(x_s) \subseteq L$, we have $C^o_G(x_s) = C^o_L(x_s)$, and so we get that

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = \sum_{n\in C_{\mathcal{G}}(x_s)^F_{nil}} \mathcal{Q}^{C_{\mathcal{G}}(x_s)}_{C_{\mathcal{G}}(x_s)}(n, x_n)\gamma^G_y(x_s + n)\,.$$

This formula shows that if x_s is not G^F-conjugate to y_s, then ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = 0$. Hence we may assume that $y_s = x_s$, and we have

$${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = |C_G(y)^F| \sum_{n\in \mathcal{O}^{C_G(y_s)^F}_{y_n}} \mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n)\,. \tag{2}$$

We now compute the quantity $\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n)$. By definition of Green functions, we have

$$\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n) = |C^o_G(y_s)^F|^{-1}\mathrm{Trace}((\phi(n), \phi(x_n)^{-1})|H^*_c(C^o_G(y_s)^F))\,.$$

From [DM91, Proposition 10.8], we deduce that

$$\begin{aligned}\mathcal{Q}^{C_{\mathcal{G}}(y_s)}_{C_{\mathcal{G}}(y_s)}(n, x_n) &= |C^o_G(y_s)^F|^{-1}\mathrm{Trace}((\phi(n), \phi(x_n)^{-1})|\bar{\mathbf{Q}}_\ell[C^o_G(y_s)^F]) \\ &= |C^o_G(y_s)^F|^{-1}\sharp\{g \in C^o_G(y_s)^F|\phi(n)g\phi(x_n)^{-1} = g\} \\ &= |C^o_G(y_s)^F|^{-1}\sharp\{g \in C^o_G(y_s)^F|\mathrm{Ad}(g)x_n = n\}\,.\end{aligned}$$

From the last formula and (2), we deduce that ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = |C_G(y)^F|$ if x is G^F-conjugate to y and ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^G_y)(x) = 0$ otherwise. From (1), it follows that $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}(\gamma^L_x) = \gamma^G_x$. □

3. The Mackey formula and its applications

In this section, we first discuss the validity of the Mackey formula for $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$; in the group case, this has been discussed by many authors including Deligne-Lusztig [DL83, Theorem 7], and Bonnafé [Bon98] [Bon00]. Here, we prove that the Mackey formula holds in the Lie algebra case whenever it does in the group case (assuming that p is good for G so that $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ exists). To prove this, we follow [Bon98] where it is shown (in the group case) that the Mackey formula is equivalent to a formula on two-variable Green-functions. In a second part, we will see some consequences of the Mackey formula (well-known in the group case) such as the independence of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ from the parabolic subgroup P or the commutation of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ with the duality map.

3.1. The Mackey formula: definition. The Mackey formula describes the composition ${}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}\circ\mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}:\mathcal{C}(\mathcal{M}^F)\rightarrow\mathcal{C}(\mathcal{L}^F)$. More precisely, the *Mackey formula with respect to* (G, L, M, P, Q), is the following equality:

3.1.1.

$$
{}^*\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}\circ\mathcal{R}^{\mathcal{G}}_{\mathcal{M}\subset\mathcal{Q}}=\sum_{x\in L^F\backslash\mathcal{S}_G(L,M)^F/M^F}\mathcal{R}^{\mathcal{L}}_{\mathcal{L}\cap{}^x\mathcal{M}\subset\mathcal{L}\cap{}^x\mathcal{Q}}\circ{}^*\mathcal{R}^{{}^x\mathcal{M}}_{\mathcal{L}\cap{}^x\mathcal{M}\subset\mathcal{P}\cap{}^x\mathcal{M}}\circ\mathrm{Ad}_x
$$

where $\mathcal{S}_G(L, M)$ denotes the set of $x\in G$ such that $L\cap{}^xM$ contains a maximal torus of G.

3.2. The main result of this section. Here we state the Lie algebra analogue of a result of Bonnafé reducing the proof of the Mackey formula to the proof of a formula on two-variable Green functions so called the "Mackey formula for Green functions".

NOTATION 3.2.1. If H is a reductive group, we denote by $rk_{ss}(H)$ the semi-simple rank of H, i.e the rank of H/Z^o_H.

Let L and M be two F-stable Levi subgroups of G and let P and Q be two parabolic subgroups of G having respectively L and M as Levi subgroup. Then we denote by $\mathcal{T}(G, L, M)$ the set of triples (G', L', M') such that:

(i) G' is an F-stable connected reductive subgroup of G such that G' contains a maximal torus of G and $rk_{ss}(G') < rk_{ss}(G)$,

(ii) L' and M' are two F-stable Levi subgroups of G' which are respectively G^F-conjugate to a subgroup of L and a subgroup of M,

For each $(G', L', M')\in\mathcal{T}(G, L, M)$, we choose two parabolic subgroups P' and Q' of G' such that L' and M' are Levi factors respectively of P' and Q', and such that there exists $x, y\in G^F$ verifying $L'\subset{}^xL$, $M'\subset{}^yM$ and $P'\subset{}^xP$, $Q'\subset{}^yQ$.

NOTATION 3.2.2. If $(G', L', M')\in\mathcal{T}(G, L, M)\cup\{(G, L, M)\}$, we write $(G', L', M')\leq(G, L, M)$. This defines a partial order on $\mathcal{T}(G, L, M)\cup\{(G, L, M)\}$.

NOTATION 3.2.3. If $(G', L', M') \leq (G, L, M)$, we denote by $\mathcal{G}'$, $\mathcal{L}'$, $\mathcal{M}'$, $\mathcal{P}'$ and $\mathcal{Q}'$ the respective Lie algebras of G', L', M', P' and Q'; with this notation, we write $\mathcal{R}_{\mathcal{L}'}^{\mathcal{G}'}$ instead of $\mathcal{R}_{\mathcal{L}'\subset\mathcal{P}'}^{\mathcal{G}'}$.

NOTATION 3.2.4. If $(G', L', M') \leq (G, L, M)$, we denote by $\mathcal{M}(G', L', M')$ the equality 3.1.1 (with (G', L', M', P', Q') instead of (G, L, M, P, Q)) and by $M(G', L', M')$ the corresponding equality in the group case (see [Bon98]).

REMARK 3.2.5. The Mackey formula $\mathcal{M}(G, L, M)$ holds if and only if

(*) $\quad \forall f \in \mathcal{C}(\mathcal{L}^F), \forall g \in \mathcal{C}(\mathcal{M}^F)$,

$$(\mathcal{R}_{\mathcal{L}}^{\mathcal{G}}(f), \mathcal{R}_{\mathcal{M}}^{\mathcal{G}}(g))_{\mathcal{G}^F} = \sum_{x \in L^F \backslash \mathcal{S}_G(L,M)^F / M^F} ({}^*\mathcal{R}_{\mathcal{L}\cap {}^x\mathcal{M}}^{\mathcal{L}}(f), {}^*\mathcal{R}_{\mathcal{L}\cap {}^x\mathcal{M}}^{{}^x\mathcal{M}} \circ \mathrm{Ad}_x(g))_{\mathcal{L}^F \cap \mathrm{Ad}(x)\mathcal{M}^F}.$$

The next formula is somehow the analogue for Green functions of 3.2.5(*):

DEFINITION 3.2.6 (The Mackey formula for Green functions). For $u \in \mathcal{G}^F$ and $v \in \mathcal{L}^F$, we denote by $\mathcal{Q}_{\mathcal{L}}^{\mathcal{G}}(u, .)$ (resp. $\mathcal{Q}_{\mathcal{L}}^{\mathcal{G}}(., v)$) the invariant function on $\mathcal{L}^F$ (resp. on $\mathcal{G}^F$) that takes the value 0 at non-nilpotent elements and that takes the value $\mathcal{Q}_{\mathcal{L}}^{\mathcal{G}}(u, v)$ at v (resp. u). We call the Mackey formula for Green functions with respect to (G, L, M) the following formula:

(*) $\forall u \in \mathcal{L}_{nil}^F$, $\forall v \in \mathcal{M}_{nil}^F$,

$$(\mathcal{Q}_{\mathcal{L}}^{\mathcal{G}}(., u), \mathcal{Q}_{\mathcal{M}}^{\mathcal{G}}(., v))_{\mathcal{G}^F} = \sum_{x \in L^F \backslash \mathcal{S}_G(L,M)^F / M^F} (\mathcal{Q}_{\mathcal{L}\cap {}^x\mathcal{M}}^{\mathcal{L}}(u, .), \mathcal{Q}_{\mathcal{L}\cap {}^x\mathcal{M}}^{{}^x\mathcal{M}}({}^x v, .))_{\mathcal{L}^F \cap \mathrm{Ad}(x)\mathcal{M}^F}.$$

NOTATION 3.2.7. We denote by $\mathcal{Q}(G, L, M)$ the formula 3.2.6(*) and by $Q(G, L, M)$ the corresponding formula in the group case [Bon98, 2.2].

REMARK 3.2.8. It is clear from our definition of the two-variable Green functions that the formula $\mathcal{Q}(G, L, M)$ holds exactly when $Q(G, L, M)$ does.

The following proposition is the main result of this section (see [Bon98, Proposition 2.3.6] for the group case).

PROPOSITION 3.2.9. *The following assertions are equivalent,*

(i) *For any* $(G', L', M') \leq (G, L, M)$, *the Mackey formula for Green functions* $\mathcal{Q}(G', L', M')$ *holds.*

(ii) *For any* $(G', L', M') \leq (G, L, M)$, *the Mackey formula* $\mathcal{M}(G', L', M')$ *holds.*

COROLLARY 3.2.10. *The following assertions are equivalent,*

(i) *For any* $(G', L', M') \leq (G, L, M)$, *the Mackey formula* $\mathcal{M}(G', L', M')$ *holds.*

(ii) *For any* $(G', L', M') \leq (G, L, M)$, *the Mackey formula* $M(G', L', M')$ *holds.*

The corollary is a straightforward consequence of 3.2.9, 3.2.8 and the group version of 3.2.9 (which is [Bon98, Proposition 2.3.6]).

3.3. Proof of 3.2.9. The proof of 3.2.9 is entirely similar to that of its group version [Bon98]. We sketch it for the convenience of the reader.

For $f \in \mathcal{C}(\mathcal{L}^F)$ and $g \in \mathcal{C}(\mathcal{M}^F)$, define

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}(f,g) = (\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}(f), \mathcal{R}^{\mathcal{G}}_{\mathcal{M}}(g))_{\mathcal{G}^F} - \sum_{x \in L^F \backslash \mathcal{S}_G(L,M)^F/M^F} ({}^*\mathcal{R}^{\mathcal{L}}_{\mathcal{L}\cap^x\mathcal{M}}(f), {}^*\mathcal{R}^{^x\mathcal{M}}_{\mathcal{L}\cap^x\mathcal{M}} \circ \mathrm{Ad}_x(g))_{\mathcal{L}^F \cap \mathrm{Ad}(x)\mathcal{M}^F},$$

and for $u \in \mathcal{L}^F_{nil}$ and $v \in \mathcal{M}^F_{nil}$, define

$$\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}(u,v) = (\mathcal{Q}^{\mathcal{G}}_{\mathcal{L}}(.,u), \mathcal{Q}^{\mathcal{G}}_{\mathcal{M}}(.,v))_{\mathcal{G}^F} - \sum_{x \in L^F \backslash \mathcal{S}_G(L,M)^F/M^F} (\mathcal{Q}^{\mathcal{L}}_{\mathcal{L}\cap^x\mathcal{M}}(u,.), \mathcal{Q}^{^x\mathcal{M}}_{\mathcal{L}\cap^x\mathcal{M}}({}^xv,.))_{\mathcal{L}^F \cap \mathrm{Ad}(x)\mathcal{M}^F}.$$

The following result gives an expression of the $\mathcal{R}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}$ in terms of $\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}$, see [Bon98, Corollary 2.3.5] for the group case.

LEMMA 3.3.1. *We assume that $\mathcal{M}(G', L', M')$ holds for all triples (G', L', M') of $\mathcal{T}(G, L, M)$. Then for any $f \in \mathcal{C}(\mathcal{L}^F)$ and $g \in \mathcal{C}(\mathcal{M}^F)$ we have*

$$\mathcal{R}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}(f,g) = \sum_{z \in z(\mathcal{G})^F} \sum_{v \in \mathcal{L}^F_{nil}} \sum_{w \in \mathcal{M}^F_{nil}} f(z+v)\overline{g(z+w)}\mathcal{Q}^{\mathcal{G}}_{\mathcal{L},\mathcal{M}}(v,w).$$

The proof of [Bon98, Corollary 2.3.5] can be adapted without difficulties to the Lie algebra case.

PROOF OF 3.2.9. Assuming (i) and using 3.3.1, we can prove (ii) easily by induction on $\dim G' + \dim L' + \dim M'$ where (G', L', M') runs over the set of triples $\leq (G, L, M)$.

Assume that (ii) is true. Let $(G', L', M') \leq (G, L, M)$. We want to prove that for any $u \in \mathcal{L}'^F_{nil}$ and $v \in \mathcal{M}'^F_{nil}$, we have $\mathcal{Q}^{\mathcal{G}'}_{\mathcal{L}',\mathcal{M}'}(u, v) = 0$. Since the Mackey formula holds for any triple $\leq (G, L, M)$, it does for any triple $\leq (G', L', M')$ and so by 3.3.1, for any $u \in \mathcal{L}'^F_{nil}$ and $v \in \mathcal{M}'^F_{nil}$, we get that (see notation 1.0.4):

$$\mathcal{R}^{\mathcal{G}'}_{\mathcal{L}',\mathcal{M}'}(\gamma^{L'}_u, \gamma^{M'}_v) = |L'^F||M'^F|\mathcal{Q}^{\mathcal{G}'}_{\mathcal{L}',\mathcal{M}'}(u,v).$$

But by assumption, the left hand side of the above equation is equal to 0, so $\mathcal{Q}^{\mathcal{G}'}_{\mathcal{L}',\mathcal{M}'}(u, v) = 0$. □

3.4. Consequences. By [Bon98], there exists an integer q_o, depending only on G, such that if $q > q_o$, then for any F-stable Levi subgroups L and M of G, the Mackey formula $M(G', L', M')$ holds for any triple $(G', L', M') \leq (G, L, M)$. Hence by 3.2.10 we have:

THEOREM 3.4.1. *If $q > q_o$, the Mackey formula $\mathcal{M}(G, L, M)$ holds for any F-stable Levi subgroups L and M.*

REMARK 3.4.2. In some cases, we can prove that the Mackey formula $\mathcal{M}(G, L, M)$ holds without assumption on q. This is the case for instance if G is of type A_n, or if L or M is a maximal torus. These results follow from their group versions (see [Bon00] if G is of type A_n, and see [DL83] if L or M is a maximal torus) together with 3.2.10.

PROPOSITION 3.4.3. *If $q > q_o$, the Deligne-Lusztig induction $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$ does not depend on the choice of the parabolic subgroup P of G having L as a Levi subgroup.*

PROOF. The proof is entirely similar to that of [DM91, Proposition 6.8]. □

NOTATION 3.4.4. We denote $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ instead of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}\subset\mathcal{P}}$; this is justified in view of 3.4.3.

Now we are interested in the relationship between duality maps and Deligne-Lusztig induction. This relationship is known in the group case, see [DM91, p.66]; the corresponding formula for Lie algebras is given in the following theorem:

THEOREM 3.4.5. *Assume $q > q_o$. Let L be an F-stable Levi subgroup of G and let $\mathcal{L}$ be its Lie algebra. Then*

$$\mathcal{D}_{\mathcal{G}} \circ \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} = \varepsilon_G \varepsilon_L \mathcal{R}^{\mathcal{G}}_{\mathcal{L}} \circ \mathcal{D}_{\mathcal{L}}$$

where $\varepsilon_G = (-1)^{\mathbf{F}_q - rank(G)}$.

PROOF. The proof is entirely similar to that of [DM91, Theorem 8.11] since the only properties of $\mathcal{R}^{\mathcal{G}}_{\mathcal{L}}$ it uses are transitivity (see 2.3.6), the Mackey formula for (G, M, L) with M a G-split Levi subgroup of G and the following formula (which is easy to verify)

$$\mathrm{Ad}_{x^{-1}} \circ \mathcal{R}^{{}^x\mathcal{L}}_{{}^x\mathcal{M}} \circ \mathrm{Ad}_x = \mathcal{R}^{\mathcal{L}}_{\mathcal{M}}$$

for any $x \in G^F$ and any $\mathcal{L}$, $\mathcal{M}$ such that $\mathcal{M} \subset \mathcal{L}$. □

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Present Address:
DEPARTMENT OF MATHEMATICS, SOPHIA UNIVERSITY,
KIOICHO, CHIYODA-KU, TOKYO 102–8554, JAPAN.