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Complex Equifocal Submanifolds and Infinite Dimensional Anti-Kaehlerian Isoparametric Submanifolds

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Abstract. In a symmetric space of non-compact type, we have recently defined the notion of a complex equifocal submanifold. In this paper, we introduce the notion of an infinite dimensional anti-Kaehlerian isoparametric submanifold. We show that the investigation of complete real analytic complex equifocal submanifolds is reduced to that of infinite dimensional anti-Kaehlerian isoparametric submanifolds. Also, we show that an infinite dimensional anti-Kaehlerian isoparametric submanifold is multi-foliated by complex spheres (or complex affine subspaces) and that the main part of the focal set of the submanifold at each point consists of some complex hyperplanes in the normal space.

1. Introduction

In 1995, C. L. Terng and G. Thorbergsson [49] defined the notion of an equifocal submanifold in a symmetric space as a submanifold with globally flat and abelian normal bundle such that the focal radii for each parallel normal vector field are constant. This notion is a general one of isoparametric submanifolds in the Euclidean space and isoparametric hypersurfaces in a sphere or a hyperbolic space. They showed that the investigation of equifocal submanifolds in a symmetric space of compact type is reduced to that of isoparametric submanifolds in a (separable) Hilbert space through a Riemannian submersion of a Hilbert space onto the symmetric space. Here isoparametric submanifolds in the Hilbert space are proper Fredholm submanifolds with globally flat normal bundle such that, for each parallel normal vector field v, the spectrum of the shape operator of direction v is constant. The following problem is one of open problems in [49].

Is there the similar argument for equifocal submanifolds in a symmetric space of non-compact type?

Recently we tackled this problem. Concretely we defined the notion of a real isoparametric submanifold in a pseudo-Hilbert space and showed that the investigation of equifocal submanifolds in a symmetric space of non-compact type is reduced to that of real isoparametric submanifolds in a pseudo-Hilbert space. However, the following example indicates that the equifocality is a rather weak (non-rigid) condition for submanifolds in a symmetric space of non-compact type.

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EXAMPLE 1. Let $\mathfrak{M}$ be the set of all submanifolds in the m-dimensional hyperbolic space $H^m(c)$ of constant curvature $c\,(<0)$ such that, for each unit normal vector v of M, all the absolute values of principal curvatures of direction v are smaller than $\frac{\sqrt{-c}}{2}$. Then, for an arbitrary $M \in \mathfrak{M}$ and a sufficiently small positive constant ε, the ε-tube $t_\varepsilon(M)$ of M has the only focal radius ε at each point. Hence $t_\varepsilon(M)$ is equifocal. Thus the equifocality in a symmetric space of non-compact type is a rather weak condition than that in a symmetric space of compact type.

In the case where the ambient symmetric space is of non-compact type, we considered that the notion of the focal radius should be defined in the complex number field and recently defined the notion of a complex focal radius in [25]. We [25] recently defined the notion of a complex equifocal submanifold as a submanifold with globally flat and abelian normal bundle such that complex focal radii for each parallel normal vector field are constant. Note that the ε-tube $t_\varepsilon(M)$ is not complex equifocal for almost all $M \in \mathfrak{M}$ and a sufficiently small positive number ε, where $\mathfrak{M}$ is the class of submanifolds in $H^m(c)$ as in Example 1. It is shown that isoparametric submanifolds with flat section in the sense of Heintze-Liu-Olmos [16] are complex equifocal and that the converse also holds under certain condition (see Theorem 15). Also, we [25] recently defined the notion of a complex isoparametric submanifold in the pseudo-Hilbert space as a Fredholm submanifold with globally flat normal bundle such that complex principal curvatures for each parallel normal vector field are constant. Here we note that the inverse numbers of the complex principal curvatures give complex focal radii of the submanifold and hence complex isoparametric submanifolds are interpreted as complex equifocal submanifolds in the pseudo-Hilbert space. Further, we [25] defined the notion of a proper complex isoparametric submanifold in the space and the notion of the complex reflection group associated with the submanifold. We [25] showed that the investigation of curvature adapted and complex equifocal submanifolds in a symmetric space of non-compact type is reduced to that of complex isoparametric submanifolds in a pseudo-Hilbert space through a pseudo-Riemannian submersion of the pseudo-Hilbert space onto the symmetric space. Here a curvature adapted submanifold is a submanifold such that for each normal vector v, $R(\cdot, v)v$ preserves the tangent space of the submanifold and A_v and $R(\cdot, v)v$ are commutative, where A is the shape tensor of the submanifold and R is the curvature tensor of the symmetric space. (Non-real) complex focal radii are imaginary notions because the focal points corresponding to them do not exist. So we need to catch the geometrical essence of complex focal radii. For its purpose, we should define the complexifications of the ambient symmetric space and the ambient pseudo-Hilbert space and the extrinsic complexification of a submanifold. Let G/K be a symmetric space of non-compact type, where, without loss of generality, G can be assumed to be a connected semi-simple Lie group and have its complexification, and K can be assumed to be a maximal compact subgroup of G. Since G admits a faithful linear representation, we can define the complexification $G^{\mathbf{c}}$ (resp. $K^{\mathbf{c}}$) of G (resp. K) and the compact dual $G^*\,(\subset G^{\mathbf{c}})$ of G. In the sequel, we assume that the compact dual G^* is simply connected. Hence, since $G^{\mathbf{c}} = G^{*\mathbf{c}}$ and $G^{*\mathbf{c}}$ is regarded as the tangent bundle of G^*, $G^{\mathbf{c}}$ is simply connected. Therefore, $(G^{\mathbf{c}}, K^{\mathbf{c}})$ is a symmetric pair and $G^{\mathbf{c}}/K^{\mathbf{c}}$ is a simply connected (pseudo-Riemannian) symmetric space. Also, $G^{\mathbf{c}}/K^{\mathbf{c}}$ is an anti-Kaehlerian manifold in a natural manner. We call this anti-Kaehlerian manifold $G^{\mathbf{c}}/K^{\mathbf{c}}$ the *anti-Kaehlerian symmetric space associated with* G/K. We regard $G^{\mathbf{c}}/K^{\mathbf{c}}$ as the complexification of G/K. Also, we consider the infinite dimensional anti-Kaehlerian space $(V^{\mathbf{c}}, \mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}})$ as the complexification of a pseudo-Hilbert space $(V, \langle\ ,\ \rangle)$. Further we define the extrinsic complexifications of submanifolds in G/K as anti-Kaehlerian submanifolds in $G^{\mathbf{c}}/K^{\mathbf{c}}$, where we need to assume that the submanifolds are complete and real analytic. Let M be a complete real analytic submanifold in G/K and $M^{\mathbf{c}}$ be its extrinsic complexification. Note that $M^{\mathbf{c}}$ is an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$. For an anti-Kaehlerian submanifold in a general anti-Kaehlerian manifold, we define complex focal radii as the notion one-to-one corresponding to the focal points of the submanifold. We show that complex focal radii of M coincide with those of $M^{\mathbf{c}}$ along M $(\subset M^{\mathbf{c}})$. Thus we can catch the geometrical essence of complex focal radii of M as focal points of $M^{\mathbf{c}}$. By using the complex focal radii, we introduce the notion of an anti-Kaehlerian equifocal submanifold in an anti-Kaehlerian symmetric space. Also, we introduce the notion of an anti-Kaehlerian isoparametric (and proper anti-Kaehlerian isoparametric) submanifold in the infinite dimensional anti-Kaehlerian space. We show that a complete real analytic submanifold M is complex equifocal in G/K if and only if $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal in $G^{\mathbf{c}}/K^{\mathbf{c}}$ (see Theorem 5). In general, the submanifold theories in Riemannian manifolds with negative curvature and pseudo-Riemannian manifolds seem to be closely connected with the anti-Kaehlerian submanifold theory. Let $\pi^{\mathbf{c}} : G^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ be the natural projection and $\phi^{\mathbf{c}} : H^0([0,1], \mathfrak{g}^{\mathbf{c}}) \to G^{\mathbf{c}}$ be the parallel transport map for $G^{\mathbf{c}}$. See §6 about the definition of $\phi^{\mathbf{c}}$. The complex Lie group $G^{\mathbf{c}}$ becomes an anti-Kaehlerian manifold with respect to the bi-invariant pseudo-Riemannian metric inducing the metric of $G^{\mathbf{c}}/K^{\mathbf{c}}$ and the natural complex structure. Also, the space $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ becomes an infinite dimensional anti-Kaehlerian space with respect to the non-degenerate inner product defined from the $\mathrm{Ad}(G^{\mathbf{c}})$-invariant non-degenerate inner product of $\mathfrak{g}^{\mathbf{c}}$ inducing the bi-invariant pseudo-Riemannian metric of $G^{\mathbf{c}}$. It is shown that $\phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion. The main theorem of this paper is as follows.

THEOREM 1. *Let M be a complete real analytic submanifold with globally flat and abelian normal bundle in a symmetric space G/K of non-compact type. Then the following statements* (i) *and* (ii) *hold*:

(i) *M is complex equifocal if and only if each component of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ is anti-Kaehlerian isoparametric in $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. In detail, for each unit normal vector v of M, complex focal radii along the geodesic γ_v coincide with the inverse numbers of complex principal curvatures of the horizontal lift $(\iota_* v)^L$-direction, where ι is the natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$.*

(ii) *Assume that M is curvature adapted. Then M is complex equifocal in G/K and for each $w \in (T^{\perp}M)^{\mathbf{c}}$ and each $\alpha \in \triangle_+$ with $\alpha^{\mathbf{c}}(g_*^{-1}w) \neq 0$, $\pm\alpha^{\mathbf{c}}(g_*^{-1}w)$ is not eigenvalues of*

$A^{\mathbf{c}}_w|_{g_*\mathfrak{p}^{\mathbf{c}}_\alpha}$ *if and only if each component of* $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ *is proper anti-Kaehlerian isoparametric in* $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$, *where* g *is a representative element of the base point of* w *and* $\triangle_+$ *is the positive root system with respect to a maximal abelian subspace (equipped with some lexicographical ordering) whose complexification contains* $g_*^{-1}w$ *and* $\mathfrak{p}^{\mathbf{c}}_\alpha$ *is the complexification of the root space for* $\alpha\in\triangle_+$. *Further, each component of* $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ *then extends to a complete proper anti-Kaehlerian isoparametric submanifold.*

REMARK 1. (i) All isoparametric submanifolds in a Euclidean space are catched as the level sets of an isoparametric map (which is a polynomial map) and hence they are real analytic submanifolds. Also, all isoparametric hypersurfaces in a sphere or a hyperbolic space are catched as the level sets of an isoparametric function (which is a polynomial function) and hence they are real analytic. Also, all known examples of equifocal submanifolds and all examples of complex equifocal submanifolds given in this paper are real analytic. Thus the assumption that submanifolds are real analytic seems to be admissible.

(ii) According to this theorem, the investigation of complete real analytic complex equifocal submanifolds in a symmetric space of non-compact type is replaced by that of anti-Kaehlerian isoparametric submanifolds in an infinite dimensional anti-Kaehlerian space. Anti-Kaehlerian isoparametric submanifolds in the anti-Kaehlerian space seems be easier to treat than complete real analytic complex equifocal submanifolds in the symmetric space because the complex focal radii of the complex equifocal submanifold are imaginary but those of the anti-Kaehlerian isoparametric submanifold correspond to its focal points and further the ambient space is a linear space.

(iii) In the statement (ii) of this theorem, the condition for the eigenvalues of $A^{\mathbf{c}}_w$ implies that M has no imaginary focal point on the ideal boundary of G/K. Hence, it is conjectured that each component of $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ is a proper anti-Kaehlerian isoparametric submanifold for each (not necessarily curvature adapted) complex equifocal submanifold M having no imaginary focal point on the ideal boundary of G/K.

(iv) When M is immersed by f and hence $M^{\mathbf{c}}$ is immersed by the complexification $f^{\mathbf{c}}$ of f, $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ means a submanifold $\widetilde{M}^{\mathbf{c}}:=\{(x,u)\in M^{\mathbf{c}}\times H^0([0,1],\mathfrak{g}^{\mathbf{c}})\,|\,f^{\mathbf{c}}(x)=(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})(u)\}$ immersed by $\tilde{f}^{\mathbf{c}}:(x,u)\in\widetilde{M}^{\mathbf{c}}\hookrightarrow u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}})$.

For proper anti-Kaehlerian isoparametric submanifolds, we prove the following result.

THEOREM 2. *Let* $(M,\langle\ ,\ \rangle,J)$ *be a proper anti-Kaehlerian isoparametric submanifold in the infinite dimensional anti-Kaehlerian space* $(V,\langle\ ,\ \rangle,\widetilde{J})$, $\{\lambda_i\,|\,i\in I\}$ (*resp.* $\{v_i\,|\,i\in I\}$) *be the set of all complex principal curvatures* (*resp. the set of all complex curvature normals*) *of* $(M,\langle\ ,\ \rangle,J)$ *and* E_i $(i\in I)$ *be the complex curvature distribution for* λ_i. *Then the following statements* (i) *and* (ii) *hold*:

(i) *The focal set of* (M,x) *coincides with the sum* $\bigcup_{i\in I}\lambda_i(x)^{-1}(1)$ *of the complex hyperplanes* $\lambda_i(x)^{-1}(1)$ $(i\in I)$.

(ii) *E_i ($i \in I$) is totally geodesic on M. If $\lambda_i \neq 0$, then the leaves of E_i are open potions of complex spheres of radius $\frac{\sqrt{\lambda_i(v_i)}}{|\lambda_i(v_i)|}$ (this quantity is constant over M) and the mean curvature vector of leaves of E_i is equal to v_i. Also, if $\lambda_i = 0$, then the leaves of E_i are open potions of complex affine subspaces.*

REMARK 2. According to the fact (i), a complex reflection group associated with each proper anti-Kaehlerian isoparametric submanifold $(M, \langle\ ,\ \rangle, J)$ is defined as the group generated by the reflections (of angle π) of the normal space $T_x^{\perp}M$ with respect to the complex hyperplane $\lambda_i(x)^{-1}(1)$ ($i \in I$). We conjecture that this group is discrete if M is properly immersed.

For a complete real analytic complex equifocal submanifold M, the focal set of $(M^{\mathbf{c}}, x)$ coincides with that of $((\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}}), u)$ ($u \in (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(x)$) because $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion. Hence we shall call the above complex reflection group associated with the proper anti-Kaehlerian isoparametric submanifold $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ the *complex reflection group associated with M*. Here we state a plan of research of complex equifocal submanifolds for the future.

A plan of research for the future. *We plan to research complex equifocal submanifolds in terms of the associated complex reflectioin group. For example, we plan to investigate if a splitting theorem of Ewert-type (see* [9]) *holds for complex equifocal submanifolds and the associated complex reflection groups.*

It is very worth to find systematic constructions of homogeneous complex (or anti-Kaehlerian) equifocal submanifolds and homogeneous complex (or anti-Kaehlerian) isoparametric ones. We can find the following systematic constructions of those homogeneous submanifolds.

THEOREM 3. *Let G/K be a symmetric space of non-compact type and H be the group of all fixed points of an involution σ ($\neq \mathrm{id}_G$) of G or a closed subgroup of G whose action on G/K is of cohomogeneity one. Then the following statements* (i)–(vi) *hold:*

(i) *All principal orbits of the H-action on G/K are complex equifocal.*

(ii) *All principal orbits of the $H \times K$-action on G are complex equifocal, where we give G the bi-invariant pseudo-Riemannian metric inducing the Riemannian metric of G/K.*

(iii) *All principal orbits of the $P(G, H \times K)$-action on $H^0([0,1], \mathfrak{g})$ are complex isoparametric.*

(iv) *All principal orbits of the $H^{\mathbf{c}}$-action on $G^{\mathbf{c}}/K^{\mathbf{c}}$ are anti-Kaehlerian equifocal, where $H^{\mathbf{c}} = \exp \mathfrak{g}_H^{\mathbf{c}}$.*

(v) *All principal orbits of the $H^{\mathbf{c}} \times K^{\mathbf{c}}$-action on $G^{\mathbf{c}}$ are anti-Kaehlerian equifocal.*

(vi) *All principal orbits of the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action on $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ are anti-Kaehlerian isoparametric.*

We introduce the notions of complex hyperpolar actions on a symmetric space of non-compact type, a semi-simple Lie group equipped with a bi-invariant pseudo-Riemannian metric and a pseudo-Hilbert space, and that of an anti-Kaehlerian hyperpolar action on an infinite dimensional anti-Kaehlerian space (see §10). Principal orbits of a complex hyperpolar action on a symmetric space of non-compact type (resp. a pseudo-Hilbert space) are complex equifocal (resp. complex isoparametric) (see Theorem 12) and principal orbits of an anti-Kaehlerian hyperpolar action on an infinite dimensional anti-Kaehlerian space are anti-Kaehlerian isoparametric (see Theorem 14). For these complex and anti-Kaehlerian hyperpolar actions, we prove the following fact.

THEOREM 4. *Let G/K be a symmetric space of non-compact type and H be a closed subgroup of G. Then the following statements* (i)–(iii) *are equivalent*:

(i) *the H-action* (*on G/K*) *is complex hyperpolar*,

(ii) *the $P(G, H \times K)$-action* (*on $H^0([0,1], \mathfrak{g})$*) *is complex hyperpolar*,

(iii) *the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action* (*on $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$*) *is anti-Kaehlerian hyperpolar*, *where $P(G, H \times K) := \{g \in H^1([0,1], G) \mid (g(0), g(1)) \in H \times K\}$ and $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}}) := \{g \in H^1([0,1], G^{\mathbf{c}}) \mid (g(0), g(1)) \in H^{\mathbf{c}} \times K^{\mathbf{c}}\}$.*

In §2, we recall basic notions and facts. In §3, we first introduce the notion of a complex focal radius for an anti-Kaehlerian submanifold. Next we introduce the notion of the associated anti-Kaehlerian symmetric space as a complexification of a symmetric space of non-compact type and that of an anti-Kaehlerian equifocal submanifold in the anti-Kaehlerian symmetric space. In §4, we define the extrinsic complexification of a complete real analytic submanifold in a symmetric space of non-compact type. In §5, we introduce the notion of an infinite dimensional anti-Kaehlerian space, and introduce the notions of an anti-Kaehlerian isoparametric submanifold and a proper anti-Kaehlerian isoparametric one in the space. For an anti-Kaehlerian isoparametric submanifold, we define the notions of its complex principal curvatures, its complex curvature distributions and its complex curvature normals. In §6, we define the notion of the parallel transport map for the complexification of a semi-simple Lie group. In §7, we prove Theorem 1. In §§8 and 9, we prove Theorems 2 and 3, respectively. In §10, we introduce the notions of a complex hyperpolar action and an anti-Kaehlerian hyperpolar one, and prove Theorem 4. In §11, we first show that isoparametric submanifolds with flat section in the sense of Heintze-Liu-Olmos ([16]) are complex equifocal and that the converse also holds under certain condition.

We would like to thank Professor Ernst Heintze for his valuable advice in discussion with him among staying at Universität Augsburg with respect to the equivalence of the complex equifocality and the isoparametricness with flat section in the sense of [16] (see Theorem 15). Also, we would like to thank Professor Yoshihiro Ohnita for introducing [40–44].

2. Complex equifocal submanifolds

In this section, we recall the notion of a complex equifocal submanifold introduced in [25]. Let $N = G/K$ be a symmetric space, $(\mathfrak{g}, \sigma)$ be its orthogonal symmetric Lie algebra and $\mathfrak{p}$ be the eigenspace for -1 of σ. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Let M be an immersed submanifold in N and $T^{\perp}M$ be its normal bundle. If, for each $x(= gK) \in M$, $g_*^{-1}T_x^{\perp}M$ is an abelian subspace in $\mathfrak{p}$, then M is said to *have abelian normal bundle*. Also, if the normal connection of M is flat and has trivial holonomy, then M is said to *have globally flat normal bundle*. Let M be an immersed submanifold with globally flat and abelian normal bundle in a symmetric space N. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, where $\gamma_{\tilde{v}_x}$ is the maximal geodesic such that the velocity vector $\dot{\gamma}_{\tilde{v}_x}(0)$ of $\gamma_{\tilde{v}_x}$ at 0 is equal to $\widetilde{v}_x$. Note that the number is infinite in the case where N is of compact type. Further, assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots\}$ ($|r_{i,x}| < |r_{i+1,x}|$ or $r_{i,x} = -r_{i+1,x} > 0$) be the set of all focal radii along $\gamma_{\tilde{v}_x}$ and r_i $(i = 1, 2 \cdots)$ be functions on M defined by assigning $r_{i,x}$ to each $x \in M$. These functions r_i $(i = 1, 2, \cdots)$ are called *focal radius functions for* $\tilde{v}$. The normal vector field $r_i\tilde{v}$ is called a *focal normal vector field for* $\tilde{v}$. If M is compact and, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$ and further each focal radius function for $\tilde{v}$ is constant on M (in the case where the number is not equal to 0), then M is called an *equifocal submanifold*. Here we note that each focal radius function has automatically constant multiplicity. This notion was introduced in [49]. We use the terminology without assuming the compactness of M.

For a submanifold in a hyperbolic space $H^m(c)$ of constant curvature c, there does not exist the focal radius corresponding to a principal curvature whose absolute value is smaller than or equal to $\sqrt{-c}$. This fact indicates that imaginary focal radius should be defined for submanifolds in a complete Riemannian manifold of negative sectional curvature. In [25], we defined the notion of complex focal radii as imaginary focal radii of submanifolds in a symmetric space of non-compact type as follows. Let M be an immersed submanifold with abelian normal bundle in a symmetric space $N = G/K$ of non-compact type. Denote by A the shape tensor of M. Let $v \in T_x^{\perp}M$ and $X \in T_xM$ $(x = gK)$. Denote by γ_v the geodesic in N with $\dot{\gamma}_v(0) = v$. The Jacobi field Y along γ_v with $Y(0) = X$ and $Y'(0) = -A_vX$ is given by

$$Y(s) = (P_{\gamma_v|_{[0,s]}} \circ (D^{co}_{sv} - sD^{si}_{sv} \circ A_v))(X)\,,$$

where $Y'(0) = \widetilde{\nabla}_v Y$, $P_{\gamma_v|_{[0,s]}}$ is the parallel translation along $\gamma_v|_{[0,s]}$,

$$D^{co}_{sv} = g_* \circ \cos(\sqrt{-1}\mathrm{ad}(sg_*^{-1}v)) \circ g_*^{-1}$$

and

$$D^{si}_{sv} = g_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}(sg_*^{-1}v))}{\sqrt{-1}\mathrm{ad}(sg_*^{-1}v)} \circ g_*^{-1}$$

(see [49] or [23] in detail). Here ad is the adjoint representation of the Lie algebra $\mathfrak{g}$ of G. Since M has abelian normal bundle, all focal radii (other than conjugate radii) of M are strong focal radii in the sense of [26] (see the proof of Theorem A in [26]). Hence all focal radii (other than conjugate radii) of M along γ_v are catched as real numbers s_0 with $\mathrm{Ker}(D^{co}_{s_0v} - s_0D^{si}_{s_0v} \circ A_v) \neq \{0\}$. So, we call a complex number z_0 with $\mathrm{Ker}(D^{co}_{z_0v} - z_0D^{si}_{z_0v} \circ A^{\mathbf{c}}_v) \neq \{0\}$ a *complex focal radius of M along γ_v* and call $\dim \mathrm{Ker}(D^{co}_{z_0v} - z_0D^{si}_{z_0v} \circ A^{\mathbf{c}}_v)$ the *multiplicity* of the complex focal radius z_0, where $D^{co}_{z_0v}$ (resp. $D^{si}_{z_0v}$) implies the complexification of a map $(g_* \circ \cos(\sqrt{-1}z_0\mathrm{ad}(g_*^{-1}v)) \circ g_*^{-1})|_{T_xM}$ $\left(\text{resp. } \left(g_* \circ \frac{\sin(\sqrt{-1}z_0\mathrm{ad}(g_*^{-1}v))}{\sqrt{-1}z_0\mathrm{ad}(g_*^{-1}v)} \circ g_*^{-1}\right)\Big|_{T_xM}\right)$ from T_xM to $T_xN^{\mathbf{c}}$. Also, for a complex focal radius z_0 of M along γ_v, we call z_0v ($\in T^{\perp}_xM^{\mathbf{c}}$) a *complex focal normal vector of M at x*. Further, assume that M has globally flat normal bundle. Let $\tilde{v}$ be a parallel unit normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$. Further assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots\}$ be the set of all complex focal radii along $\gamma_{\tilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} > 0$". Let r_i $(i = 1, 2, \cdots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call these functions r_i $(i = 1, 2, \cdots)$ *complex focal radius functions for $\tilde{v}$*. We call $r_i\tilde{v}$ a *complex focal normal vector field for $\tilde{v}$*. If, for each parallel unit normal vector field $\tilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\tilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\tilde{v}$ is constant on M and it has constant multiplicity, then we call M a *complex equifocal submanifold*.

REMARK 3. In 1998, H. Ewert [10] defined the notion of a strongly equifocal hypersurface in a symmetric space of non-compact type. Easily we can show that all strongly equifocal hypersurfaces are complex equifocal. However, the converse does not hold.

In a pseudo-Riemannian symmetric space, we can define the notion of a complex equifocal submanifold similarly. We will use the notion in a semi-simple Lie group equipped with a bi-invariant pseudo-Riemannian metric.

3. Anti-Kaehlerian equifocal submanifolds

In this section, we first introduce the notions of an anti-Kaehlerian submanifold and its complex focal radius. Let J be a parallel complex structure on an even dimensional pseudo-Riemannian manifold $(M, \langle\ ,\ \rangle)$ of half index. If $\langle JX, JY\rangle = -\langle X, Y\rangle$ holds for every $X,\ Y \in TM$, then $(M, \langle\ ,\ \rangle, J)$ is called an *anti-Kaehlerian manifold*. Let R be the curvature tensor of $(M, \langle\ ,\ \rangle, J)$. We have the following relations:

$$R(X,Y)J = JR(X,Y)\,, \tag{3.1}$$

$$R(JX,JY) = -R(X,Y)\,, \tag{3.2}$$

$$\langle R(X,JY)JY, X\rangle = -\langle R(X,Y)Y,X\rangle\,, \tag{3.3}$$

where $X, Y \in TM$. Let f be an isometric immersion of an anti-Kaehlerian manifold $(M, \langle\ ,\ \rangle, J)$ into an anti-Kaehlerian manifold $(\widetilde{M}, \langle\ ,\ \rangle, \widetilde{J})$. If $\widetilde{J}\circ f_* = f_*\circ J$, then we call $(M, \langle\ ,\ \rangle, J)$ an *anti-Kaehlerian submanifold* in $(\widetilde{M}, \langle\ ,\ \rangle, \widetilde{J})$ immersed by f. Let A (resp. $\nabla^{\perp}$) be the shape tensor (resp. the normal connection) of $(M, \langle\ ,\ \rangle, J)$. We have the following relations:

$$A_{\tilde{J}v}X = A_v(JX) = J(A_vX)\,, \tag{3.4}$$

where $X \in TM$ and $v \in T^{\perp}M$. Denote by $\exp^{\perp}$ the normal exponential map of $(M, \langle\ ,\ \rangle, J)$. Let v be a unit normal vector of $(M, \langle\ ,\ \rangle, J)$ at x. If $\exp^{\perp}(av + bJv)$ is a focal point of (M, x), then we call the complex number $a + b\sqrt{-1}$ a *complex focal radius along the geodesic* γ_v.

Let $N = G/K$ be a symmetric space of non-compact type and $(\mathfrak{g}, \sigma)$ be its orthogonal symmetric Lie algebra, where G can be assumed to be a connected semi-simple Lie group and have its complexification, K can be assumed to be a maximal compact subgroup of G and the compact dual G^* of G is assumed to be simply connected as stated in Introduction. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition. Note that $\mathfrak{k}$ is the Lie algebra of K and $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$, where e is the identity element of G. Let $\langle\ ,\ \rangle$ be the $\mathrm{Ad}(G)$-invariant non-degenerate inner product of $\mathfrak{g}$ inducing the Riemannian metric of N. Let $\mathfrak{g}^{\mathbf{c}}$, $\mathfrak{k}^{\mathbf{c}}$, $\mathfrak{p}^{\mathbf{c}}$ and $\langle\ ,\ \rangle^{\mathbf{c}}$ be the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ and $\langle\ ,\ \rangle$, respectively. Let $\mathfrak{h}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{p} = \mathfrak{h} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$ be the root space decomposition with respect to $\mathfrak{h}$, that is, $\mathfrak{p}_\alpha = \{X \in \mathfrak{p} \mid \mathrm{ad}(a)^2(X) = \alpha(a)^2X \text{ for all } a \in \mathfrak{h}\}$. Then we have $\mathfrak{p}^{\mathbf{c}} = \mathfrak{h}^{\mathbf{c}} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_\alpha^{\mathbf{c}}$ and $\mathfrak{p}_\alpha^{\mathbf{c}} = \{X \in \mathfrak{p}^{\mathbf{c}} \mid \mathrm{ad}(a)^2(X) = \alpha^{\mathbf{c}}(a)^2X \text{ for all } a \in \mathfrak{h}^{\mathbf{c}}\}$ $(\alpha \in \triangle_+)$, where $\mathfrak{h}^{\mathbf{c}}$, $\mathfrak{p}_\alpha^{\mathbf{c}}$ and $\alpha^{\mathbf{c}}$ are the complexifications of $\mathfrak{h}$, $\mathfrak{p}_\alpha$ and α, respectively. We denote $\{\alpha^{\mathbf{c}} \mid \alpha \in \triangle_+\}$ by $\triangle_+^{\mathbf{c}}$ and express $\mathfrak{p}_\alpha^{\mathbf{c}}$ as $\mathfrak{p}_{\alpha^{\mathbf{c}}}$. We call $\alpha^{\mathbf{c}}$ $(\in \triangle_+^{\mathbf{c}})$ a *positive root for* $\mathfrak{h}^{\mathbf{c}}$ (under some lexicographical ordering of $\mathfrak{h}^{\mathbf{c}}$) and call $\mathfrak{p}^{\mathbf{c}} = \mathfrak{h}^{\mathbf{c}} + \sum_{\alpha^{\mathbf{c}}\in\triangle_+^{\mathbf{c}}}\mathfrak{p}_{\alpha^{\mathbf{c}}}$ the *root space decomposition with respect to* $\mathfrak{h}^{\mathbf{c}}$. Let $G^{\mathbf{c}}$ (resp. $K^{\mathbf{c}}$) be the complexification of G (resp. K). The real part $\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}$ of $\langle\ ,\ \rangle^{\mathbf{c}}$ is an $\mathrm{Ad}(G^{\mathbf{c}})$-invariant non-degenerate inner product of $\mathfrak{g}^{\mathbf{c}}$. The restriction $\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}}\times\mathfrak{p}^{\mathbf{c}}}$ is an $\mathrm{Ad}(K^{\mathbf{c}})$- invariant non-degenerate inner product of $\mathfrak{p}^{\mathbf{c}}$ $(= T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}))$. Denote by $\langle\ ,\ \rangle'$ the $G^{\mathbf{c}}$-invariant pseudo-Riemannian metric on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from $\mathrm{Re}\langle\ ,\ \rangle^{\mathbf{c}}|_{\mathfrak{p}^{\mathbf{c}}\times\mathfrak{p}^{\mathbf{c}}}$. Define an almost complex structure J_0 of $\mathfrak{p}^{\mathbf{c}}$ by $J_0(X + \sqrt{-1}Y) = -Y + \sqrt{-1}X$ $(X, Y \in \mathfrak{p})$. It is clear that J_0 is $\mathrm{Ad}(K^{\mathbf{c}})$-invariant. Denote by $\widetilde{J}$ the $G^{\mathbf{c}}$-invariant almost complex structure on $G^{\mathbf{c}}/K^{\mathbf{c}}$ induced from J_0. It is shown that $(G^{\mathbf{c}}/K^{\mathbf{c}}, \langle\ ,\ \rangle', \widetilde{J})$ is an anti-Kaehlerian manifold and a pseudo-Riemannian symmetric space. We call this anti-Kaehlerian manifold an *anti-Kaehlerian symmetric space associated with G/K* and simply denote it by $G^{\mathbf{c}}/K^{\mathbf{c}}$.

EXAMPLE 2. We consider the n-dimensional hyperbolic space

$$H^n(-1) = \{(x_1, \cdots, x_{n+1}) \in \mathbf{R}_1^{n+1} \mid -x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = -1\} \\ (= SO^0(n,1)/SO(n))$$

of constant curvature -1, where $\mathbf{R}_1^{n+1}$ is the $(n+1)$-dimensional Lorentzian space equipped with the Lorentzian inner product $\langle\ ,\ \rangle_1$ defined by $\left\langle \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_i} \right\rangle_1 = -\delta_{1i}$ $(i = 1, \cdots, n+1)$ and $\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_1 = \delta_{ij}$ $(i, j = 2, \cdots, n+1)$. The anti-Kaehlerian symmetric space associated with $H^n(-1)$ is the complex quadric

$$\{(z_1, \cdots, z_{n+1}) \in \mathbf{C}_1^{n+1} \mid -z_1^2 + z_2^2 + \cdots + z_{n+1}^2 = -1\} \\ (= SO(n+1, \mathbf{C})/SO(n, \mathbf{C}))\ ,$$

where $\mathbf{C}_1^{n+1}$ is the $(n+1)$-dimensional anti-Kaehlerian space equipped with the non-degenerate inner product $\mathrm{Re}\langle\ ,\ \rangle_1^{\mathbf{c}}$.

Let M be an immersed anti-Kaehlerian submanifold with globally flat and abelian normal bundle in the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$. Let $\widetilde{v}$ be a parallel normal vector field of M. Assume that the number (which may be 0 and ∞) of distinct complex focal radii along the geodesic $\gamma_{\widetilde{v}_x}$ is independent of the choice of $x \in M$. Further assume that the number is not equal to 0. Let $\{r_{i,x} \mid i = 1, 2, \cdots\}$ be the set of all complex focal radii along $\gamma_{\widetilde{v}_x}$, where $|r_{i,x}| < |r_{i+1,x}|$ or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} > \mathrm{Re}\, r_{i+1,x}$" or "$|r_{i,x}| = |r_{i+1,x}|$ & $\mathrm{Re}\, r_{i,x} = \mathrm{Re}\, r_{i+1,x}$ & $\mathrm{Im}\, r_{i,x} = -\mathrm{Im}\, r_{i+1,x} > 0$". Let r_i $(i = 1, 2, \cdots)$ be complex valued functions on M defined by assigning $r_{i,x}$ to each $x \in M$. We call this function r_i the *i-th complex focal radius function for $\widetilde{v}$*.

LEMMA 1. *Assume that the multiplicity of the complex focal radius $r_i(x)$ $(i = 1, 2, \cdots)$ is independent of the choice of $x \in M$, where the multiplicity of $r_i(x)$ implies that of the focal point corresponding to $r_i(x)$. Then the functions r_i $(i = 1, 2, \cdots)$ are holomorphic.*

PROOF. Define a function $Q_x : \mathbf{C} \to \mathbf{C}$ $(x \in M)$ by

$$Q_x(z) := \det(D^{co}_{(\mathrm{Re} z)\widetilde{v}_x + (\mathrm{Im} z)J\widetilde{v}_x} - D^{si}_{(\mathrm{Re} z)\widetilde{v}_x + (\mathrm{Im} z)J\widetilde{v}_x} \circ \hat{A}_{(\mathrm{Re} z)\widetilde{v}_x + (\mathrm{Im} z)J\widetilde{v}_x})\ .$$

It is clear that Q_x is holomorphic. Complex focal radii along $\gamma_{\widetilde{v}_x}$ are catched as zero points of Q_x (see the proof of Theorem 5), which is discrete by the holomorphicity of Q_x. Fix a number i_0 and $x_0 \in M$. Set $I_0 := \{i \mid |r_i(x_0)| = |r_{i_0}(x_0)|\}$, $i_1 := \min I_0$ and $i_2 := \max I_0$. Further, set $I_1 := \{i \mid |r_i(x_0)| = |r_{i_1-1}(x_0)|\}$ and $I_2 := \{i \mid |r_i(x_0)| = |r_{i_2+1}(x_0)|\}$. Note that I_0, I_1 and I_2 are finite. Take a simple closed curve C in the domain $D := \left\{ z \in \mathbf{C} \,\middle|\, \frac{|r_{i_1-1}(x_0)| + |r_{i_0}(x_0)|}{2} < |z| < \frac{|r_{i_0}(x_0)| + |r_{i_2+1}(x_0)|}{2} \right\}$ which surrounds $r_{i_0}(x_0)$ and does not surround $r_i(x_0)$ $(i \in I_0 \setminus \{i_0\})$.

The functions r_i $(i = 1, 2, \cdots)$ are continuous by the constancy of the multiplicities of r_i. Hence we can take a neighborhood U of x_0 such that, for every $x \in U$, $r_{i_0}(x)$ positions inside C, $r_i(x)$'s $(i \in I_0 \setminus \{i_0\})$ position outside C and $r_i(x)$'s $(i \in I_1 \cup I_2)$ do not belong to D. Then it is clear that other complex focal radii $r_i(x)$'s $(i \notin I_0 \cup I_1 \cup I_2,\ x \in U)$ position outside C. Let m_x be the order of the zero point $r_{i_0}(x)$ of Q_x. Then we have

$$r_{i_0}(x) = \frac{1}{2m_x\pi\sqrt{-1}}\int_C z\frac{Q_x'(z)}{Q_x(z)}dz \quad (x \in U)\,.$$

It follows from the continuity of r_{i_0} that m_x is independent of the choice of $x \in U$. Define a complex function F_z $(z \in \mathbf{C})$ on M by $F_z(x) := Q_x(z)$ $(x \in M)$. It is clear that this function F_z is a holomorphic function on M. Hence, it follows from the above integral representation that r_{i_0} is holomorphic over U. Further, it follows from the arbitrariness of x_0 that r_{i_0} is holomorphic (over M). q.e.d.

If, for each parallel unit normal vector field $\widetilde{v}$ of M, the number of distinct complex focal radii along $\gamma_{\widetilde{v}_x}$ is independent of the choice of $x \in M$, each complex focal radius function for $\widetilde{v}$ is constant on M and it has constant multiplicity, then we call M an *anti-Kaehlerian equifocal submanifold*.

4. The extrinsic complexifications of complete real analytic submanifolds in non-compact symmetric spaces

In this section, we introduce the new notion of the extrinsic complexifications of complete real analytic submanifolds in symmetric spaces of non-compact type. First we recall the complexifications of complete real analytic Riemannian manifolds. Let N be a complete real analytic Riemannian manifold. The notion of the adapted complex structure on a neighborhood U of the 0-section of the tangent bundle TN is defined as the complex structure (on U) such that, for each geodesic $\gamma : \mathbf{R} \to N$, the restriction of its differential $\gamma_* : T\mathbf{R} = \mathbf{C} \to TN$ to $\gamma_*^{-1}(U)$ is holomorphic. We take U as largely as possible under the condition that $U \cap T_xN$ is a star-shaped neighborhood of 0_x for each $x \in N$, where 0_x is the zero vector of T_xN. If N is of non-negative curvature, then we have $U = TN$. Also, if all sectional curvatures of N are bigger than or equal to c $(c < 0)$, then U contains the ball bundle $T^rN := \{X \in TN \mid ||X|| < r\}$ of radius $r := \frac{\pi}{2\sqrt{-c}}$. In detail, see [40$\sim$44]. Denote by J_A the adapted complex structure on U. The complex manifold (U, J_A) is interpreted as the complexification of N. We denote (U, J_A) by $N^{\mathbf{c}}$ and call it the complexification of N, where we note that $N^{\mathbf{c}}$ is given no Riemannian metric. In particular, in case of $N = \mathbf{R}^m$ (the Euclidean space), we have $(U, J_A) = \mathbf{C}^m$. Also, in the case where N is a symmetric space G/K of non-compact type, there exists the holomorphic diffeomorphism δ of (U, J_A) onto an open subset of $G^{\mathbf{c}}/K^{\mathbf{c}}$ satisfying the following commutative diagram:

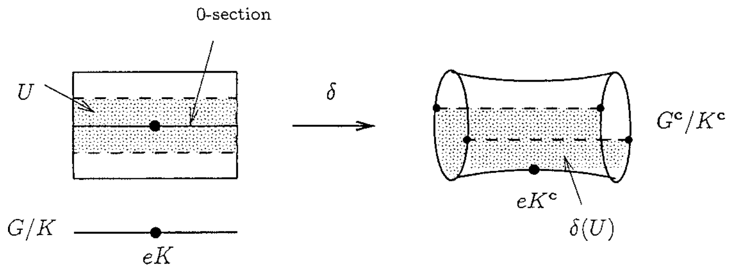

FIGURE 1.

$$
\begin{array}{ccc}
i(\gamma_*^{-1}(U)) & \xrightarrow{\gamma^{\mathbf{c}}} & G^{\mathbf{c}}/K^{\mathbf{c}} \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle \delta} \\
\gamma_*^{-1}(U) & \xrightarrow[\gamma_*]{} & (U, J_A)
\end{array}
$$

for an arbitrary geodesic $\gamma : \mathbf{R} \to N = G/K$. Here i is the natural bijection of $T\mathbf{R}$ onto $\mathbf{C}$, γ_* is the differential of γ and $\gamma^{\mathbf{c}}$ is defined by $\gamma^{\mathbf{c}}(z) := g_0 \exp(z g_{0*}^{-1} \dot{\gamma}(0)) K^{\mathbf{c}}$, where g_0 is an element of G with $\gamma(0) = g_0 K$ and $g_{0*}^{-1}\dot{\gamma}(0)$ is regarded as an element of $\mathfrak{p}$ under the identification of $T_{eK}N$ and $\mathfrak{p}$ ($\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$: the Cartan decomposition of G/K). In this case, the above value r means the half of the minimal conjugate radius of the compact dual G^*/K of G/K.

Now we shall define the extrinsic complexifications of complete real analytic submanifolds in a symmetric space G/K of non-compact type. Let M be an immersed complete real analytic submanifold in G/K. Denote by f its immersion. Let $M^{\mathbf{c}}$ be the complexification of M (defined as above). We want to define the complexification $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of f, where we shrink $M^{\mathbf{c}}$ to a neighborhood of the 0-section of TM if necessary. For its purpose, we first define the complexification of a real analytic curve $\alpha : \mathbf{R} \to G/K$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition associated with G/K and $W : \mathbf{R} \to \mathfrak{p}$ be the curve in $\mathfrak{p}$ with $(\exp W(t))K = \alpha(t)$ ($t \in \mathbf{R}$), where we note that W is uniquely determined because G/K is of non-compact type. Since α is real analytic, so is also W. Let $W^{\mathbf{c}} : D \to \mathfrak{p}^{\mathbf{c}}$ (D : a neighborhood of $\mathbf{R}$ in $\mathbf{C}$) be the holomorphic extension of W. We define the complexification $\alpha^{\mathbf{c}} : D \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of α by $\alpha^{\mathbf{c}}(z) = (\exp W^{\mathbf{c}}(z))K^{\mathbf{c}}$. By using this complexification of a real analytic curve in G/K, we define the complexification $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ of f by $f^{\mathbf{c}}(X) := (f \circ \gamma_X^M)^{\mathbf{c}}(\sqrt{-1})$ ($X \in M^{\mathbf{c}}\,(\subset TM)$), where γ_X^M is the geodesic in M with $\dot{\gamma}_X^M(0) = X$. Here we shrink $M^{\mathbf{c}}$ to a neighborhood of the 0-section of TM if necessary in order to assure that $\sqrt{-1}$ belongs to the domain of $(f \circ \gamma_X^M)^{\mathbf{c}}$ for each $X \in M^{\mathbf{c}}$.

PROPOSITION 1. (i) *The map $f^{\mathbf{c}} : M^{\mathbf{c}} \to G^{\mathbf{c}}/K^{\mathbf{c}}$ is holomorphic.*

(ii) *The restriction of $f^{\mathbf{c}}$ to a neighborhood of the* 0*-section of TM is an immersion.*

PROOF. First we shall show the statement (i). According to Theorem 3.4 of [42], we have only to show that, for each geodesic $\gamma : \mathbf{R} \to M$, $f^{\mathbf{c}} \circ \gamma_* : \gamma_*^{-1}(M^{\mathbf{c}})$ $(\subset T\mathbf{R} = \mathbf{C}) \to G^{\mathbf{c}}/K^{\mathbf{c}}$ is holomorphic. Denote by τ the natural coordinate of $\mathbf{R}$. From the definition of $f^{\mathbf{c}}$, we have

$$(f^{\mathbf{c}} \circ \gamma_*)(s + \sqrt{-1}t) = (f^{\mathbf{c}} \circ \gamma_*)\left(t\left(\frac{\partial}{\partial \tau}\right)_s\right) = f^{\mathbf{c}}(t\dot{\gamma}(s)) = (f \circ \gamma^M_{t\dot{\gamma}(s)})^{\mathbf{c}}(\sqrt{-1})\,.$$

Let $W : \mathbf{R} \to \mathfrak{p}$ be the real analytic curve in $\mathfrak{p}$ with $(f \circ \gamma)(t) = \exp W(t)K$ $(t \in \mathbf{R})$. Then we have $(f \circ \gamma^M_{t\dot{\gamma}(s)})(u) = \exp W(s + tu)K$ $(u \in \mathbf{R})$ and hence $(f \circ \gamma^M_{t\dot{\gamma}(s)})^{\mathbf{c}}(\sqrt{-1}) = \exp(W^{\mathbf{c}}(s + \sqrt{-1}t))K^{\mathbf{c}}$. Thus we obtain $(f^{\mathbf{c}} \circ \gamma_*)(s + \sqrt{-1}t) = \exp(W^{\mathbf{c}}(s + \sqrt{-1}t))K^{\mathbf{c}}$. It is clear that the complex curve of the right-hand side is holomorphic. Therefore, so is also $f^{\mathbf{c}} \circ \gamma_*$. Next we shall show the statement (ii). Denote by UM (resp. $U(TM)$) the unit tangent bundle of M (resp. TM). Take $Z \in UM$ and $\eta \in U_Z(TM)$. Let $X : (-\varepsilon, \varepsilon) \to TM$ be a real analytic curve with $\dot{X}(0) = \eta$, where ε is a sufficiently small positive number. Let a be a positive number with $aZ \in M^{\mathbf{c}}$. Set $X_a(s) := aX(s)$ and $\eta_a := \dot{X}_a(0)$ $(\in T_{aZ}(TM))$. Also, let $W(s) : \mathbf{R} \to \mathfrak{p}$ $(s \in (-\varepsilon, \varepsilon))$ be the curve in $\mathfrak{p}$ with $(f \circ \gamma^M_{X(s)})(t) = (\exp W(s)(t))K$ $(t \in \mathbf{R})$. Set $\widehat{W}(s, t) := W(s)(t)$ $((s, t) \in (-\varepsilon, \varepsilon) \times \mathbf{R})$ and $\widehat{W}^{\mathbf{c}}(s, z) := W(s)^{\mathbf{c}}(z)$ $((s, z) \in (-\varepsilon, \varepsilon) \times \mathbf{C})$. It is clear that $\widehat{W} : (-\varepsilon, \varepsilon) \times \mathbf{R} \to \mathfrak{p}$ is real analytic. Denote by $\pi^{\mathbf{c}}$ the natural projection of $G^{\mathbf{c}}$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$. Easily we have $(f_*)_*(\eta_a) = ((\pi \circ \exp)_*)_*\big(a\frac{\partial^2 \widehat{W}}{\partial s \partial t}(0, 0)\big)$ and $f^{\mathbf{c}}_*(\eta_a) = (\pi^{\mathbf{c}} \circ \exp)_*\big(\frac{d}{ds} W(s)^{\mathbf{c}}(a\sqrt{-1})\big|_{s=0}\big)$. Since $\widehat{W}^{\mathbf{c}}(s, z)$ is real analytic with respect to s, it is expanded as $\widehat{W}^{\mathbf{c}}(s, z) = \sum_{k=0}^{\infty} s^k w_k(z)$ $(w_k(z) \in \mathfrak{p}^{\mathbf{c}})$ on $(-\varepsilon, \varepsilon)$. Then we have $\widehat{W}(s, t) = \sum_{k=0}^{\infty} s^k w_k(t)$. It follows from the holomorphicity of $\widehat{W}^{\mathbf{c}}(s, z)$ with respect to z that $w_k : \mathbf{C} \to \mathfrak{p}^{\mathbf{c}}$ is holomorphic. Easily we have $\frac{\partial^2 \widehat{W}}{\partial s \partial t}(0, 0) = (w_1|_{\mathbf{R}})'(0)$. Since w_1 is holomorphic and $w_1(\mathbf{R}) \subset \mathfrak{p}$, we have $(w_1|_{\mathbf{R}})'(0) = ((w_1)_{\sqrt{-1}\mathfrak{p}}|_{\sqrt{-1}\mathbf{R}})'(0)$ by the theorem of Cauchy-Riemann, where $(w_1)_{\sqrt{-1}\mathfrak{p}}$ is the $\sqrt{-1}\mathfrak{p}$-component of w_1. Thus we have

$$(f_*)_*(\eta_a) = ((\pi \circ \exp)_*)_*(a(w_1)_{\sqrt{-1}\mathfrak{p}}|_{\sqrt{-1}\mathbf{R}})'(0))\,. \tag{4.1}$$

Also we have

$$f^{\mathbf{c}}_*(\eta_a) = (\pi^{\mathbf{c}} \circ \exp)_*(w_1(a\sqrt{-1}))\,. \tag{4.2}$$

Let B be the largest connected neighborhood of $0 \in \mathfrak{p}^{\mathbf{c}}$ where $\pi^{\mathbf{c}} \circ \exp$ is a diffeomorphism. Note that B is a tubular neighborhood of $\mathfrak{p}$ $(\subset \mathfrak{p}^{\mathbf{c}})$ because G/K is of non-compact type. Set

$$\bar{\varepsilon}_\eta := \sup\{\varepsilon > 0 \mid (w_1)_{\sqrt{-1}\mathfrak{p}}|_{\sqrt{-1}(0,\varepsilon]} \text{ has no zero point } \& \ w_0(\sqrt{-1}[0, \varepsilon]) \subset B\}\,.$$

Since f is an immersion and $\eta_a \neq 0$, we have $((w_1)_{\sqrt{-1}\mathfrak{p}}|_{\sqrt{-1}\mathbf{R}})'(0) \neq 0$ by (4.1). On the other hand, it follows from $w_1(0) \in \mathfrak{p}$ that $(w_1)_{\sqrt{-1}\mathfrak{p}}(0) = 0$. Also, we have $w_0(0) \in \mathfrak{p} \subset B$.

Hence we have $\bar{\varepsilon}_\eta > 0$. Set $\varepsilon_Z := \min_{\eta\in U_Z(TM)} \bar{\varepsilon}_\eta$ and

$$U' := \left(\bigcup_{Z\in UM} \{aZ \,|\, 0 \le a < \varepsilon_Z\}\right) \cap M^{\mathbf{c}}.$$

Assume $aZ \in U'$. Then we have $(w_1)_{\sqrt{-1}\mathfrak{p}}(a\sqrt{-1}) \neq 0$ and $w_0(a\sqrt{-1}) \in B$. These facts deduce $(\pi^{\mathbf{c}} \circ \exp)_*(w_1(a\sqrt{-1})) \neq 0$, where we note that $w_1(a\sqrt{-1}) \in T_{W(0)^{\mathbf{c}}(a\sqrt{-1})}\mathfrak{p}^{\mathbf{c}} = T_{w_0(a\sqrt{-1})}\mathfrak{p}^{\mathbf{c}}$. Hence, from (4.2), we obtain $f^{\mathbf{c}}_*(\eta_a) \neq 0$. From the arbitrarinesses of η, Z and a, this fact implies that the restriction of $f^{\mathbf{c}}$ to U' is an immersion. q.e.d.

Let $U'(\subset M^{\mathbf{c}})$ be a neighborhood of the 0-section of TM as in the proof of Proposition 1. Denote by $M^{\mathbf{c}}$ this neighborhood U' newly. Give $M^{\mathbf{c}}$ the Riemannian metric induced from that of $G^{\mathbf{c}}/K^{\mathbf{c}}$ by $f^{\mathbf{c}}$. Then $M^{\mathbf{c}}$ becomes an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$ immersed by $f^{\mathbf{c}}$. We call this anti-Kaehlerian submanifold $M^{\mathbf{c}}$ immersed by $f^{\mathbf{c}}$ the *extrinsic complexification* of the submanifold M. We consider the case where M is (extrinsically) homogeneous. Concretely we consider the case where $M = H(g_0K)$ and f is the inclusion map of M into G/K, where H is a closed subgroup of G. Let ι be a natural immersion of G/K into $G^{\mathbf{c}}/K^{\mathbf{c}}$, that is, $\iota(gK) = gK^{\mathbf{c}}$ $(g \in G)$. It is shown that ι is totally geodesic. Let $\mathfrak{g}^{\mathbf{c}}_H$ be the complexification of the Lie algebra of H and set $H^{\mathbf{c}} := \exp \mathfrak{g}^{\mathbf{c}}_H$.

PROPOSITION 2. *For a homogeneous submanifold $M = H(g_0K)$, the image $f^{\mathbf{c}}(M^{\mathbf{c}})$ is an open subset of the orbit $H^{\mathbf{c}}(g_0K^{\mathbf{c}})$.*

PROOF. Let $X \in M^{\mathbf{c}}$ and h_0g_0K be the base point of X (i.e., $X \in T_{h_0g_0K}M$). Let $W : \mathbf{R} \to \mathfrak{p}$ be the real analytic curve in $\mathfrak{p}$ with $(f \circ \gamma^M_X)(t) = \exp W(t)K$ $(t \in \mathbf{R})$ and $W^H : \mathbf{R} \to \mathfrak{g}_H$ be a real analytic curve in $\mathfrak{g}_H$ with $(f \circ \gamma^M_X)(t) = (\exp W^H(t))g_0K$. Denote by π (resp. $\pi^{\mathbf{c}}$) the natural projection of G onto G/K (resp. $G^{\mathbf{c}}$ onto $G^{\mathbf{c}}/K^{\mathbf{c}}$). Set $\alpha_1(z) := (\exp W^{\mathbf{c}}(z))K^{\mathbf{c}}$ and $\alpha_2(z) := (\exp(W^H)^{\mathbf{c}}(z))g_0K^{\mathbf{c}}$, where $W^{\mathbf{c}}$ (resp. $(W^H)^{\mathbf{c}}$) is the holomorphic extension of W (resp. W^H) to a neighborhood U of $\mathbf{R}$ in $\mathbf{C}$. We may assume $\sqrt{-1} \in U$ because of $X \in M^{\mathbf{c}}$. Clearly α_i $(i = 1, 2)$ are holomorphic and $\alpha_1(t) = \alpha_2(t)$ $(t \in \mathbf{R})$. Hence, it follows from the theorem of identity that $\alpha_1 = \alpha_2$ on U. In particular, we have $\alpha_1(\sqrt{-1}) = \alpha_2(\sqrt{-1})$. On the other hand, we have $f^{\mathbf{c}}(X) = \alpha_1(\sqrt{-1})$ and $\alpha_2(\sqrt{-1}) \in H^{\mathbf{c}}(g_0K^{\mathbf{c}})$. Hence we obtain $f^{\mathbf{c}}(X) \in H^{\mathbf{c}}(g_0K^{\mathbf{c}})$. Therefore, it follows from the arbitrariness of X that $f^{\mathbf{c}}(M^{\mathbf{c}}) \subset H^{\mathbf{c}}(g_0K^{\mathbf{c}})$. Further, it follows from $\dim M^{\mathbf{c}} = \dim H^{\mathbf{c}}(g_0K^{\mathbf{c}})$ $(= 2\dim M)$ that $f^{\mathbf{c}}(M^{\mathbf{c}})$ is an open subset of $H^{\mathbf{c}}(g_0K^{\mathbf{c}})$. q.e.d.

From this fact, we shall call $H^{\mathbf{c}}(g_0K^{\mathbf{c}})$ the *complete extrinsic complexification* of the homogeneous submanifold $M = H(g_0K)$ and denote it by $\widehat{M^{\mathbf{c}}}$. Let M be a complete real analytic submanifold in G/K and J be the complex structure on its extrinsic complexfication $M^{\mathbf{c}}$. Let R (resp. $\hat{R}$) be the curvature tensor of G/K (resp. $G^{\mathbf{c}}/K^{\mathbf{c}}$) and A and $R^\perp$ (resp. $\hat{A}$ and $\hat{R}^\perp$) be the shape tensor and the normal curvature tensor of M (resp. $M^{\mathbf{c}}$). Then we can show the following relations.

LEMMA 2. *Let* $\varepsilon_i = 1$ *or* 0 $(i = 1, 2, 3)$.
(i) *For any* $X, Y, Z \in T(G/K)$, *we have*

$$\widehat{R}(\widetilde{J}^{\varepsilon_1}\iota_* X, \widetilde{J}^{\varepsilon_2}\iota_* Y)\widetilde{J}^{\varepsilon_3}\iota_* Z = \widetilde{J}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\iota_* R(X, Y)Z\,.$$

(ii) *For any* $X \in TM$ *and any* $v \in T^{\perp}M$, *we have*

$$\widehat{A}_{\widetilde{J}^{\varepsilon_1}\iota_* v} J^{\varepsilon_2}\iota_* X = J^{\varepsilon_1+\varepsilon_2}\iota_* A_v X.$$

(iii) *For any* $X, Y \in TM$ *and any* $v \in T^{\perp}M$, *we have*

$$\widehat{R}^{\perp}(J^{\varepsilon_1}\iota_* X, J^{\varepsilon_2}\iota_* Y)\widetilde{J}^{\varepsilon_3}\iota_* v = \widetilde{J}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\iota_* R^{\perp}(X, Y)v\,.$$

PROOF. First we show the relation of (i). Let gK be the base point of X, Y, Z. Since ι is totally geodesic, we have $\widehat{R}(\iota_* X, \iota_* Y)\iota_* Z = \iota_* R(X, Y)Z$. This relation together with $(g_*^{-1} \circ \widetilde{J} \circ g_*)_{eK^{\mathbf{c}}} = \sqrt{-1}\mathrm{id}_{\mathfrak{p}^{\mathbf{c}}}$ ($\mathrm{id}_{\mathfrak{p}^{\mathbf{c}}}$: the identity transformation of $\mathfrak{p}^{\mathbf{c}} = T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$) deduces

$$\begin{aligned}
&\widehat{R}(\widetilde{J}^{\varepsilon_1}\iota_* X, \widetilde{J}^{\varepsilon_2}\iota_* Y)\widetilde{J}^{\varepsilon_3}\iota_* Z \\
&\quad = -g_*[[\sqrt{-1}^{\varepsilon_1} g_*^{-1}\iota_* X, \sqrt{-1}^{\varepsilon_2} g_*^{-1}\iota_* Y], \sqrt{-1}^{\varepsilon_3} g_*^{-1}\iota_* Z] \\
&\quad = -g_*\sqrt{-1}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}[[g_*^{-1}\iota_* X, g_*^{-1}\iota_* Y], g_*^{-1}\iota_* Z] \\
&\quad = \widetilde{J}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\widehat{R}(\iota_* X, \iota_* Y)\iota_* Z \\
&\quad = \widetilde{J}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\iota_* R(X, Y)Z\,,
\end{aligned}$$

where $[\ ,\ ]$ is the Lie bracket product of $\mathfrak{g}^{\mathbf{c}}$. Next we show the relation of (ii). Since ι is totally geodesic, we have $\hat{A}_{\iota_* v}\iota_* X = \iota_* A_v X$. This relation together with (3.4) deduces $\hat{A}_{\widetilde{J}^{\varepsilon_1}\iota_* v} J^{\varepsilon_2}\iota_* X = \widetilde{J}^{\varepsilon_1+\varepsilon_2}\iota_* A_v X$. Next we show the relation of (iii). Since ι is totally geodesic, we have

$$\widehat{R}^{\perp}(\iota_* X, \iota_* Y)\iota_* v = \iota_* R^{\perp}(X, Y)v\,. \tag{4.3}$$

From the Ricci equation, (3.4), (4.3) and the relation of (i), we have

$$\begin{aligned}
&\langle \widehat{R}^{\perp}(J^{\varepsilon_1}\iota_* X, J^{\varepsilon_2}\iota_* Y)\widetilde{J}^{\varepsilon_3}\iota_* v,\ \widetilde{J}^{\varepsilon_4}\iota_* w\rangle \\
&\quad = \langle \widehat{R}(J^{\varepsilon_1}\iota_* X, J^{\varepsilon_2}\iota_* Y)\widetilde{J}^{\varepsilon_3}\iota_* v,\ \widetilde{J}^{\varepsilon_4}\iota_* w\rangle + \langle [\hat{A}_{\widetilde{J}^{\varepsilon_3}\iota_* v},\ \hat{A}_{\widetilde{J}^{\varepsilon_4}\iota_* w}]J^{\varepsilon_1}\iota_* X,\ J^{\varepsilon_2}\iota_* Y\rangle \\
&\quad = \langle \widehat{R}(\iota_* X, \iota_* Y)\iota_* v,\ \widetilde{J}^{\varepsilon_1+\cdots+\varepsilon_4}\iota_* w\rangle + \langle [\hat{A}_{\iota_* v},\ \hat{A}_{\widetilde{J}^{\varepsilon_1+\cdots+\varepsilon_4}\iota_* w}]\iota_* X,\ \iota_* Y\rangle \\
&\quad = \langle \widehat{R}^{\perp}(\iota_* X, \iota_* Y)\iota_* v,\ \widetilde{J}^{\varepsilon_1+\cdots+\varepsilon_4}\iota_* w\rangle \\
&\quad = \langle \widetilde{J}^{\varepsilon_1+\varepsilon_2+\varepsilon_3}\iota_* R^{\perp}(X, Y)v,\ \widetilde{J}^{\varepsilon_4}\iota_* w\rangle\,,
\end{aligned}$$

where $\varepsilon_4 = 1$ or 0 and $w \in T^{\perp}M$. Thus we obtain the relation of (iii). q.e.d.

By using Lemmas 1 and 2, we prove the following result.

THEOREM 5. *Let M be a complete real analytic submanifold with globally flat and abelian normal bundle in a symmetric space G/K of non-compact type. The submanifold M is complex equifocal if and only if its extrinsic complexification $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal. In detail, for each unit normal vector v of M, the complex focal radii of M along γ_v coincide with those of $M^{\mathbf{c}}$ along $\gamma_{\iota_* v}$.*

PROOF. Let $p \in G/K$. Identify $(T_p(G/K))^{\mathbf{c}}$ with $T_{\iota(p)}(G^{\mathbf{c}}/K^{\mathbf{c}})$ by the one-to-one correspondence $X + \sqrt{-1}Y \leftrightarrow \iota_* X + \widetilde{J}\iota_* Y$ ($X, Y \in T_p(G/K)$). According to (i) of Lemma 2, the complexification $R_p^{\mathbf{c}}$ of R_p is identified with $\hat{R}_{\iota(p)}$ under this identification. Let $x \in M$. Identify $(T_x M)^{\mathbf{c}}$ (resp. $(T_x^\perp M)^{\mathbf{c}}$) with $T_{\iota(x)}M^{\mathbf{c}}$ (resp. $T_{\iota(x)}^\perp M^{\mathbf{c}}$) under this identification. According to (ii) (resp. (iii)) of Lemma 2, the complexification $A_x^{\mathbf{c}}$ of A_x (resp. the complexification $R_x^{\perp\mathbf{c}}$ of $R_x^\perp$) is identified with $\hat{A}_{\iota(x)}$ (resp. $\widehat{R}_{\iota(x)}^\perp$) under these identifications. Since M has abelian normal bundle and $\widehat{R}|_{\iota(G/K)}$ is identified with $R^{\mathbf{c}}$ as above, $M^{\mathbf{c}}$ has abelian normal bundle along $\iota(M)$. Further, according to the theorem of identity, it follows from the holomorphicities of $\widehat{R}$ and the normal bundle of $M^{\mathbf{c}}$ that $M^{\mathbf{c}}$ has abelian normal bundle (over the whole of $M^{\mathbf{c}}$), where we also use the fact that $\iota(M)$ is a half dimensional totally real submanifold of $M^{\mathbf{c}}$. Since M has globally flat normal bundle and $\widehat{R}^\perp|_{\iota(M)}$ is identified with $R^{\perp\mathbf{c}}$ as above, $\widehat{R}^\perp$ vanishes along $\iota(M)$. Further, according to the theorem of identity, it follows from the holomorphicities of $\widehat{R}^\perp$ and the normal bundle of $M^{\mathbf{c}}$ that $\widehat{R}^\perp$ vanishes (on the whole of $M^{\mathbf{c}}$). Hence $M^{\mathbf{c}}$ has flat normal bundle. Further, since $M^{\mathbf{c}}$ is a tubular neighborhood of the 0-section of the tangent bundle TM, we see that $M^{\mathbf{c}}$ has globally flat normal bundle. Let $z_0 = a+b\sqrt{-1}$ be a complex focal radius of $M^{\mathbf{c}}$ along $\gamma_{\iota_* v}$ ($v \in T_{gK}^\perp M$). Then there exists the Jacobi field Y along $\gamma_{a\iota_* v + b\widetilde{J}\iota_* v}$ such that $Y(0)(\neq 0) \in T_{gK^{\mathbf{c}}}M^{\mathbf{c}}$, $Y'(0) = -\hat{A}_{a\iota_* v + b\widetilde{J}\iota_* v}Y(0)$ and $Y(1) = 0$. The Jacobi field Y is described as

$$Y(s) = P_{\gamma_{a\iota_* v+b\widetilde{J}\iota_* v}|_{[0,s]}}((D^{co}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} - s(D^{si}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} \circ \hat{A}_{a\iota_* v+b\widetilde{J}\iota_* v}))Y(0)),$$

where

$$D^{co}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} = g_* \circ \cos(\sqrt{-1}\mathrm{ad}(s(ag_*^{-1}\iota_* v + bg_*^{-1}\widetilde{J}\iota_* v))) \circ g_*^{-1},$$

$$D^{si}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} = g_* \circ \frac{\sin(\sqrt{-1}\mathrm{ad}(s(ag_*^{-1}\iota_* v + bg_*^{-1}\widetilde{J}\iota_* v)))}{\sqrt{-1}\mathrm{ad}(s(ag_*^{-1}\iota_* v + bg_*^{-1}\widetilde{J}\iota_* v))} \circ g_*^{-1}.$$

By noticing $\widetilde{J}|_{T_{gK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})} = g_* \circ \sqrt{-1}id_{\mathfrak{p}^{\mathbf{c}}} \circ g_*^{-1}$ ($id_{\mathfrak{p}^{\mathbf{c}}}$: the identity transformation of $\mathfrak{p}^{\mathbf{c}}$), we have

$$D^{co}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} = g_* \circ \cos(\sqrt{-1}sz_0\mathrm{ad}(g_*^{-1}\iota_* v)) \circ g_*^{-1},$$

$$D^{si}_{s(a\iota_* v+b\widetilde{J}\iota_* v)} = g_* \circ \frac{\sin(\sqrt{-1}sz_0\mathrm{ad}(g_*^{-1}\iota_* v))}{\sqrt{-1}sz_0\mathrm{ad}(g_*^{-1}\iota_* v)} \circ g_*^{-1},$$

$$\hat{A}_{a\iota_* v+b\widetilde{J}\iota_* v} = g_* \circ \hat{A}_{z_0 g_*^{-1}\iota_* v} \circ g_*^{-1}.$$

Hence we have

$$
\begin{aligned}
Y(1) = P_{\gamma_{a\iota_* v + b\widetilde{J}\iota_* v}} \Bigg(& (g_* \circ \cos(\sqrt{-1} z_0 \mathrm{ad}(g_*^{-1}\iota_* v)) \circ g_*^{-1}) Y(0) \\
& - \left(g_* \circ \frac{\sin(\sqrt{-1} z_0 \mathrm{ad}(g_*^{-1}\iota_* v))}{\sqrt{-1} z_0 \mathrm{ad}(g_*^{-1}\iota_* v)} \circ \hat{A}_{z_0 g_*^{-1}\iota_* v} \circ g_*^{-1} \right) Y(0) \Bigg) \\
= 0 \, .
\end{aligned}
$$

Therefore, we can recognize that z is a complex focal radius of $M^{\mathbf{c}}$ along $\gamma_{\iota_* v}$ if and only if

$$
\begin{aligned}
\mathrm{Ker} \Bigg(& g_* \circ \cos(\sqrt{-1} z \mathrm{ad}^{\mathbf{c}}(g_*^{-1}\iota_* v)) \circ g_*^{-1} \\
& - g_* \circ \frac{\sin(\sqrt{-1} z \mathrm{ad}^{\mathbf{c}}(g_*^{-1}\iota_* v))}{\sqrt{-1} z \mathrm{ad}^{\mathbf{c}}(g_*^{-1}\iota_* v)} \circ \hat{A}_{z g_*^{-1}\iota_* v} \circ g_*^{-1} \Bigg) \neq \{0\} \, ,
\end{aligned}
$$

which is equivalent to $\mathrm{Ker}(D^{co}_{zv} - zD^{si}_{zv} \circ A^{\mathbf{c}}_v) \neq \{0\}$, that is, the fact that z is a complex focal radius of M along γ_v, where D^{co}_{zv} and D^{si}_{zv} are the operators stated in §2. Thus complex focal radii of $M^{\mathbf{c}}$ along $\gamma_{\iota_* v}$ coincide with those of M along γ_v. Since ι is totally geodesic, we see that a normal vector field $\widetilde{v}$ of M is parallel if and only if the normal vector field $\iota_*\widetilde{v}$ of $M^{\mathbf{c}}$ along $\iota(M)$ is parallel. Thus we see that if $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal, then M is complex equifocal. Now we show the converse. Assume that M is complex equifocal. Let w be a parallel normal vector field on $M^{\mathbf{c}}$. The normal vector field w is expressed as $w = \iota_*\widetilde{v}_1 + \widetilde{J}\iota_*\widetilde{v}_2$ ($\widetilde{v}_1$, $\widetilde{v}_2$: parallel normal vector fields of M) along $\iota(M)$ because of $\widetilde{\nabla}\widetilde{J} = 0$. Since M is complex equifocal, for an arbitrary parallel normal vector field $\widetilde{v}$ of M, the complex focal radii of $M^{\mathbf{c}}$ along $\gamma_{\iota_*\widetilde{v}_x}$ is independent of the choice of $x \in M$. Hence, from the definition of a complex focal radius, we see that so is also the complex focal radii of $M^{\mathbf{c}}$ along $\gamma_{(\iota_*\widetilde{v}_1 + \widetilde{J}\iota_*\widetilde{v}_2)_x}$. That is, the complex focal radius functions for w are constant along $\iota(M)$. Hence it follows from the holomorphicities of those complex focal radius functions (by Lemma 1) that those complex focal radius functions are constant over the whole of $M^{\mathbf{c}}$, where we also use the fact that $\iota(M)$ is a half-dimensional totally real submanifold of $M^{\mathbf{c}}$. Thus $M^{\mathbf{c}}$ is anti-Kaehlerian equifocal. This completes the proof. q.e.d.

Here we propose the following problem.

PROBLEM 1. Is the submanifold $M^{\mathbf{c}}$ in Theorem 5 extended to a complete anti-Kaehlerian equifocal submanifold?

For example, in the case where M is a homogeneous submanifold $H(g_0K)$, the submanifold $M^{\mathbf{c}}$ is extended to a complete anti-Kaehlerian equifocal submanifold $H^{\mathbf{c}}(g_0K^{\mathbf{c}})$.

At the end of this section, we explain the situation of the focal points of the complexification of a complete totally umbilical hypersurface in the hyperbolic space.

EXAMPLE 3. Let M be a complete totally umbilical hypersurface with principal curvature $\lambda\ (\geq 0)$ in the $(n+1)$-dimensional hyperbolic space $H^{n+1}(c)$ of constant curvature c and v be a unit normal vector field of M. The set F of all the complex focal radii of M (and hence $M^{\mathbf{c}}$) for v is given by

$$F = \begin{cases} \left\{ \dfrac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\dfrac{\sqrt{-c}}{\lambda} + j\pi\sqrt{-1}\right) \,\middle|\, j = 0, \pm 1, \cdots \right\} & (\lambda > \sqrt{-c}) \\ \emptyset & (\lambda = \sqrt{-c}) \\ \left\{ \dfrac{1}{\sqrt{-c}}\left(\operatorname{arctanh}\dfrac{\lambda}{\sqrt{-c}} + \left(j + \dfrac{1}{2}\right)\pi\sqrt{-1}\right) \,\middle|\, j = 0, \pm 1, \cdots \right\} & \\ & (0 \leq \lambda < \sqrt{-c})\,. \end{cases}$$

Note that, when $\lambda = \sqrt{-c}$, we should interpret as $F = \{\infty\}$ rather than $F = \emptyset$, where ∞ is the point at infinity of the complex sphere $S = \mathbf{C} \cup \{\infty\}$. Hence the situation of the focal points of $M^{\mathbf{c}}$ is as in Figures 2, 3 and 4.

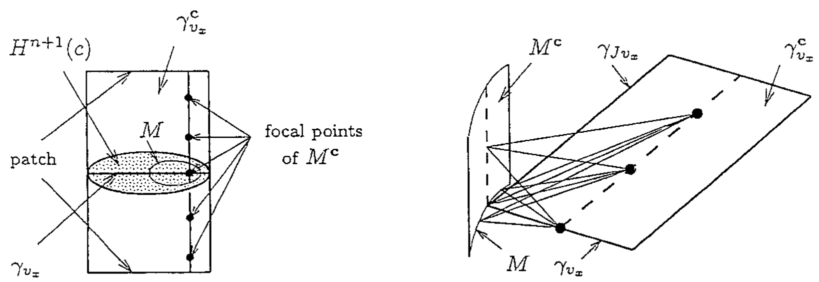

FIGURE 2.

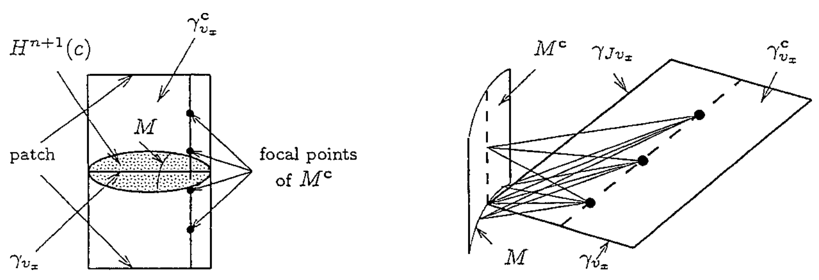

FIGURE 3.

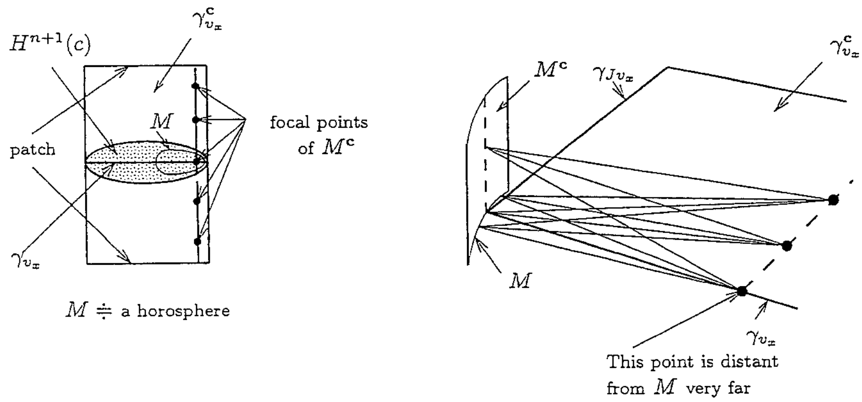

FIGURE 4.

5. Anti-Kaehlerian isoparametric submanifolds

In this section, we introduce some classes of submanifolds in the infinite dimensional anti-Kaehlerian space. We first define the notion of an infinite dimensional anti-Kaehlerian space. Let V be an infinite dimensional topological real vector space, $\widetilde{J}$ be a continuous linear operator of V such that $\widetilde{J}^2 = -\mathrm{id}$ and $\langle\ ,\ \rangle$ be a continuous non-degenerate symmetric bilinear form of V such that $\langle \widetilde{J}X, \widetilde{J}Y\rangle = -\langle X, Y\rangle$ holds for every $X, Y \in V$. If there exists an orthogonal time-space decomposition $V = V_- \oplus V_+$ (i.e., $\langle\ ,\ \rangle|_{V_-\times V_-}$: negative definite, $\langle\ ,\ \rangle|_{V_+\times V_+}$: positive definite) such that $\widetilde{J}V_\pm = V_\mp$, $(V, \langle\ ,\ \rangle_{V_\pm})$ is a Hilbert space and that the distance topology associated with $\langle\ ,\ \rangle_{V_\pm}$ coincides with the original topology of V, then we call $(V, \langle\ ,\ \rangle, \widetilde{J})$ the *infinite dimensional anti-Kaehlerian space*, where $\langle\ ,\ \rangle_{V_\pm} := -\pi^*_{V_-}\langle\ ,\ \rangle + \pi^*_{V_+}\langle\ ,\ \rangle$ ($\pi_{V_\pm}$: the projection of V onto $V_\pm$). Let $(M, \langle\ ,\ \rangle, J)$ be an $2n$-dimensional anti-Kaehlerian submanifold in an infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$ and A (resp. h) the shape tensor (resp. the second fundamental form) of M. Let H be the mean curvature vector of $(M, \langle\ ,\ \rangle, J)$, that is, $H = \frac{1}{2n}\sum_{i=1}^{2n}\langle e_i, e_i\rangle h(e_i, e_i)$ ($e_1, \cdots, e_{2n}$: an orthonormal base of M). If $h(X, Y) = \langle X, Y\rangle H - \langle JX, Y\rangle \widetilde{J}H$ for every $X, Y \in TM$, then we call $(M, \langle\ ,\ \rangle, J)$ a *totally anti-Kaehlerian umbilical submanifold*. Here we note that the above relation is rewritten as $h^{\mathbf{c}}(X^{(1,0)}, Y^{(1,0)}) = \langle X^{(1,0)}, Y^{(1,0)}\rangle H^{(1,0)}$, where $X^{(1,0)} := X - \sqrt{-1}JX$, $Y^{(1,0)} := Y - \sqrt{-1}JY$ and $H^{(1,0)} := H - \sqrt{-1}\widetilde{J}H$. Totally anti-Kaehlerian umbilical submanifolds will be characterized in §8.

Let M be a Hilbert manifold modelled on a separable Hilbert space $(V, \langle\ ,\ \rangle_V)$. Let $\langle\ ,\ \rangle$ be a section of the $(0, 2)$-tensor bundle $T^*M \otimes T^*M$ such that $\langle\ ,\ \rangle_x$ is a continuous non-degenerate symmetric bilinear form on T_xM for each $x \in M$ and J be a section of the $(1, 1)$-tensor bundle $T^*M \otimes TM$ such that $J^2 = -\mathrm{id}$, $\nabla J = 0$ (∇ : the Levi-Civita connection of $\langle\ ,\ \rangle$), J_x is a continuous linear operator of T_xM for each $x \in M$ and $\langle JX, JY\rangle = -\langle X, Y\rangle$ for every $X, Y \in TM$. If, for each $x \in M$, there exist distributions

$W_\pm$ on some neighborhood U of x satisfying the following condition (AKH), then we call $(M, \langle\ ,\ \rangle, J)$ an *anti-Kaehlerian Hilbert manifold.*

(AKH) For each $y \in U$, $W_{\pm y}$ gives an orthogonal time-space decomposition of $(T_yM, \langle\ ,\ \rangle_y)$, $(T_yM, \langle\ ,\ \rangle_{y,W_{\pm y}})$ is isometric to $(V, \langle\ ,\ \rangle_V)$ and $J_y W_{\pm y} = W_{\mp y}$.

Let f be an isometric immersion of an anti-Kaehlerian Hilbert manifold $(M, \langle\ ,\ \rangle, J)$ into an anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. If $\widetilde{J} \circ f_* = f_* \circ J$ holds, then we call M an *anti-Kaehlerian Hilbert submanifold in* $(V, \langle\ ,\ \rangle, \widetilde{J})$ *immersed by* f. If M is of finite codimension and, for each $v \in T^\perp M$, the shape operator A_v is a compact operator with respect to $f^*\langle\ ,\ \rangle_{V_\pm}$, then we call M an *anti-Kaehlerian Fredholm submanifold* (rather than *anti-Kaehlerian Fredholm Hilbert submanifold*). Let $(M, \langle\ ,\ \rangle, J)$ be an anti-Kaehlerian Fredholm submanifold in an anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$ and A be the shape tensor of $(M, \langle\ ,\ \rangle, J)$. Fix a unit normal vector v of $(M, \langle\ ,\ \rangle, J)$. If there exists $X(\neq 0) \in TM$ with $A_vX = aX + bJX$, then we call the complex number $a + b\sqrt{-1}$ a *J-eigenvalue of A_v* (or a *complex principal curvature of direction v*) and call X a *J-eigenvector for $a + b\sqrt{-1}$*. Here we note that this relation is rewritten as $A_v^{\mathbf{c}} X^{(1,0)} = (a + b\sqrt{-1})X^{(1,0)}$, where $X^{(1,0)}$ is as above. Also, we call the space of all J-eigenvectors for $a + b\sqrt{-1}$ a *J-eigenspace for $a + b\sqrt{-1}$*. For J-eigenspaces of A_v, we have the following fact.

LEMMA 3. (i) *J-eigenvectors for distinct J-eigenvalues are orthogonal to each other.*

(ii) *Each J-eigenspace of A_v is J-invariant and non-degenerate* (*i.e.*, *the restriction of* $\langle\ ,\ \rangle$ *to the J-eigenspace is non-degenerate*).

PROOF. From $A_v \circ J = J \circ A_v$, it follows that each J-eigenspace is J-invariant. Let $a_i + b_i\sqrt{-1}$ $(i = 1, 2)$ be distinct J-eigenvalues of A_v and X_i $(i = 1, 2)$ be J-eigenvector for $a_i + b_i\sqrt{-1}$. From $\langle A_vX_1, X_2\rangle = \langle X_1, A_vX_2\rangle$ and the symmetricness of J, we have

$$(a_1 - a_2)\langle X_1, X_2\rangle + (b_1 - b_2)\langle JX_1, X_2\rangle = 0\,. \tag{5.1}$$

From $A_v \circ J = J \circ A_v$ and the symmetricness of J, we have

$$\langle A_vJX_1, X_2\rangle = a_1\langle JX_1, X_2\rangle - b_1\langle X_1, X_2\rangle$$

and

$$\langle A_vJX_1, X_2\rangle = \langle JX_1, A_vX_2\rangle = a_2\langle JX_1, X_2\rangle - b_2\langle X_1, X_2\rangle\,.$$

Hence we have $(b_1 - b_2)\langle X_1, X_2\rangle - (a_1 - a_2)\langle JX_1, X_2\rangle = 0$. This together with (5.1) deduces

$$\det\begin{pmatrix} \langle X_1, X_2\rangle & \langle JX_1, X_2\rangle \\ -\langle JX_1, X_2\rangle & \langle X_1, X_2\rangle \end{pmatrix} = \langle X_1, X_2\rangle^2 + \langle JX_1, X_2\rangle^2 = 0$$

because of $(a_1 - a_2,\ b_1 - b_2) \neq (0, 0)$. That is, we obtain $\langle X_1, X_2\rangle = 0$. Thus the statement (i) is shown. According to the statement (i), each J-eigenspace is non-degenerate because of the non-degeneracy of $\langle\ ,\ \rangle$. q.e.d.

We call the set of all J-eigenvalues of A_v the *J-spectrum of A_v* and denote it by $\mathrm{Spec}_J A_v$. For $\mathrm{Spec}_J A_v$, we have the following fact.

PROPOSITION 3. *The set* $\mathrm{Spec}_J A_v \setminus \{0\}$ *is described as follows*:

$$\mathrm{Spec}_J A_v \setminus \{0\} = \{\lambda_i \mid i = 1, 2, \cdots\}$$

$$\left(\begin{array}{c} |\lambda_i| > |\lambda_{i+1}| \ \text{ or } \ \text{“}|\lambda_i| = |\lambda_{i+1}| \ \& \ \mathrm{Re}\,\lambda_i > \mathrm{Re}\,\lambda_{i+1}\text{”} \\ \text{or } \ \text{“}|\lambda_i| = |\lambda_{i+1}| \ \& \ \mathrm{Re}\,\lambda_i = \mathrm{Re}\,\lambda_{i+1} \ \& \ \mathrm{Im}\,\lambda_i = -\mathrm{Im}\,\lambda_{i+1} > 0\text{”} \end{array}\right).$$

Also, the J-eigenspace for each J-eigenvalue of A_v other than 0 *is of finite dimension.*

PROOF. Let $a + b\sqrt{-1} \in \mathrm{Spec}_J A_v$ and X be a J-eigenvector for $a + b\sqrt{-1}$. Let $X = X_- + X_+$ $(X_\pm \in V_\pm)$. Then we have

$$\begin{aligned} \langle A_v X, A_v X\rangle_{V_\pm} &= \langle aX_+ + bJX_-, aX_+ + bJX_-\rangle - \langle aX_- + bJX_+, aX_- + bJX_+\rangle \\ &= a^2\langle X_+, X_+\rangle - b^2\langle X_-, X_-\rangle - a^2\langle X_-, X_-\rangle + b^2\langle X_+, X_+\rangle \\ &= (a^2 + b^2)\langle X, X\rangle_{V_\pm}\,. \end{aligned}$$

Since A_v is a compact operator with respect to $\langle\ ,\ \rangle_{V_\pm}$, this relation deduces that $\mathrm{Spec}_J A_v \setminus\{0\}$ is described as in the statement and the J-eigenspace for each J-eigenvalue of A_v other than 0 is of finite dimension. q.e.d.

We call the J-eigenvalue λ_i as in the statement of Proposition 3 the *i-th complex principal curvature of direction v*. Assume that $(M, \langle\ ,\ \rangle, J)$ has globally flat normal bundle. Fix a parallel normal vector field $\widetilde{v}$ of M. Assume that the number (which may be ∞) of distinct complex principal curvatures of direction $\widetilde{v}_x$ is independent of the choice of $x \in M$. Then we can define functions $\widetilde{\lambda}_i$ $(i = 1, 2, \cdots)$ on M by assigning the i-th complex principal curvature of direction $\widetilde{v}_x$ to each $x \in M$. We call this function $\widetilde{\lambda}_i$ the *i-th complex principal curvature function of direction $\widetilde{v}$*. We consider the following condition:

(AKI) For each parallel normal vector field $\widetilde{v}$, the number of distinct complex principal curvatures of direction $\widetilde{v}_x$ is independent of the choice of $x \in M$, each complex principal curvature function of direction $\widetilde{v}$ is constant on M and it has constant multiplicity.

If $(M, \langle\ ,\ \rangle, J)$ satisfies this condition (AKI), then we call $(M, \langle\ ,\ \rangle, J)$ an *anti-Kaehlerian isoparametric submanifold*. Let $\{e_i\}_{i=1}^\infty$ be an orthonormal system of $(T_x M, \langle\ ,\ \rangle_x)$. If $\{e_i\}_{i=1}^\infty \cup \{Je_i\}_{i=1}^\infty$ is an orthonormal base of $T_x M$, then we call $\{e_i\}_{i=1}^\infty$ a *J-orthonormal base*. If there exists a J-orthonormal base consisting of J-eigenvectors of A_v, then A_v is said to *be diagonalized with respect to the J-orthonormal base*. If $(M, \langle\ ,\ \rangle, J)$ is anti-Kaehlerian isoparametric and, for each $v \in T^\perp M$, the shape operator A_v is diagonalized with respect to an J-orthonormal base, then we call $(M, \langle\ ,\ \rangle, J)$ a *proper anti-Kaehlerian isoparametric submanifold*.

LEMMA 4. *Let $(M, \langle\ ,\ \rangle, J)$ be a proper anti-Kaehlerian isoparametric submanifold. Then A_v's $(v \in T_x^\perp M)$ are simultaneously diagonalized with respect to a J-orthonormal base of $T_x M$.*

PROOF. Let $\{v_1, \cdots, v_r\}$ be an orthonormal base of $T_x^\perp M$. Let $\{E_{ij} \mid j \in I_i\}$ be the set of J-eigenspaces of A_{v_i} $(i = 1, \cdots, r)$. Let $A_{v_i}|_{E_{ij}} = a_{ij}\mathrm{id} + b_{ij}J$ $(i = 1, \cdots, r,\ j \in I_i)$. Fix $j_0 \in I_1$. Take $X \in E_{1j_0}$. Let $X = \sum_{j\in I_2} X_j$ $(X_j \in E_{2j})$. Then we have

$$A_{v_1}A_{v_2}X = \sum_{j\in I_2} A_{v_1}(a_{2j}X_j + b_{2j}JX_j)$$

and

$$A_{v_2}A_{v_1}X = (a_{1j_0}\mathrm{id} + b_{1j_0}J)\sum_{j\in I_2}(a_{2j}X_j + b_{2j}JX_j)\,.$$

Since $[A_{v_1}, A_{v_2}] = 0$, we have

$$\sum_{j\in I_2}(A_{v_1} - a_{1j_0}\mathrm{id} - b_{1j_0}J)(a_{2j}X_j + b_{2j}JX_j) = 0\,.$$

Also, by using $[A_{v_1}, A_{v_2}] = 0$ again, we can show

$$(A_{v_1} - a_{1j_0}\mathrm{id} - b_{1j_0}J)(a_{2j}X_j + b_{2j}JX_j) \in E_{2j}\,.$$

Hence we have $a_{2j}X_j + b_{2j}JX_j \in E_{1j_0}$, which implies $X_j \in E_{1j_0}$ because E_{1j_0} is J-invariant. After all we have $X \in \overline{\bigoplus_{j\in I_2}(E_{1j_0}\cap E_{2j})}$. Thus we have $E_{1j_0} = \overline{\bigoplus_{j\in I_2}(E_{1j_0}\cap E_{2j})}$ and hence

$$T_xM = \overline{\bigoplus_{(j_1,j_2)\in I_1\times I_2}(E_{1j_1}\cap E_{2j_2})}\,.$$

By repeating the same process, we have

$$T_xM = \overline{\bigoplus_{(j_1,\cdots,j_r)\in I_1\times\cdots\times I_r}(E_{1j_1}\cap\cdots\cap E_{rj_r})}\,.$$

This relation implies that A_v's $(v \in T_x^\perp M)$ are simultaneously diagonalized with respect to a J-orthonormal base. q.e.d.

Let $(M, \langle\ ,\ \rangle, J)$ be a proper anti-Kaehlerian isoparametric submanifold in an infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. Let $\{E_i \mid i \in I\}$ be the family of distributions on M such that, for each $x \in M$, $\{E_i(x) \mid i \in I\}$ is the set of all common J-eigenspaces of A_v's $(v \in T_x^\perp M)$. Let λ_i $(i \in I)$ be the section of $(T^\perp M)^* \otimes \mathbf{C}$ such that $A_v = \mathrm{Re}\lambda_i(v)\mathrm{id} + \mathrm{Im}\lambda_i(v)J$ on $E_i(\pi(v))$ for each $v \in T^\perp M$, where π is the bundle projection of $T^\perp M$. We call λ_i $(i \in I)$ *complex principal curvatures* of $(M, \langle\ ,\ \rangle, J)$ and call distributions E_i $(i \in I)$ *complex curvature distributions* of $(M, \langle\ ,\ \rangle, J)$.

LEMMA 5. *There uniquely exists a normal vector field* v_i *of* $(M, \langle\ ,\ \rangle, J)$ *with* $\lambda_i(\cdot) = \langle v_i, \cdot\rangle - \sqrt{-1}\langle \widetilde{J} v_i, \cdot\rangle$.

PROOF. We can express as $\lambda_i(\cdot) = \langle v_i, \cdot\rangle + \sqrt{-1}\langle w_i, \cdot\rangle$, where v_i and w_i are normal vector fields of $(M, \langle\ ,\ \rangle, J)$. We shall show $w_i = -\widetilde{J} v_i$. Let $v \in T^{\perp}M$. From $\lambda_i(\widetilde{J}v) = \sqrt{-1}\lambda_i(v)$, we have

$$\lambda_i(\widetilde{J}v) = -\langle w_i, v\rangle + \sqrt{-1}\langle v_i, v\rangle\,.$$

On the other hand, we have

$$\lambda_i(\widetilde{J}v) = \langle v_i, \widetilde{J}v\rangle + \sqrt{-1}\langle w_i, \widetilde{J}v\rangle = \langle \widetilde{J}v_i, v\rangle + \sqrt{-1}\langle \widetilde{J}w_i, v\rangle\,.$$

Hence we obtain $\langle w_i + \widetilde{J}v_i, v\rangle = 0$. It follows from the arbitrariness of v that $w_i = -\widetilde{J}v_i$. Thus the existenceness is shown. The uniqueness is trivial. q.e.d.

We call v_i $(i \in I)$ the *complex curvature normals* of $(M, \langle\ ,\ \rangle, J)$. Note that v_i is parallel with respect to the normal connection $\nabla^{\perp}$.

6. The parallel transport map for the complexification of a semi-simple Lie group

In this section, we define the parallel transport map for the complexification $G^{\mathbf{c}}$ of a connected semi-simple Lie group G. Take an $\mathrm{Ad}(G)$-invariant non-degenerate symmetric bilinear form $\langle\ ,\ \rangle$ of the Lie algebra $\mathfrak{g}$ of G. Fix an orthogonal time-space decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$. Let $\langle\ ,\ \rangle'$ be the real part of the complexification $\langle\ ,\ \rangle^{\mathbf{c}}$ (: $\mathfrak{g}^{\mathbf{c}} \times \mathfrak{g}^{\mathbf{c}} \to \mathbf{C}$) of $\langle\ ,\ \rangle$. Denote by the same symbol $\langle\ ,\ \rangle'$ the bi-invariant pseudo-Riemannian metric of $G^{\mathbf{c}}$ induced from $\langle\ ,\ \rangle'$. It is clear that $\mathfrak{g}^{\mathbf{c}} = (\mathfrak{g}_- \oplus \sqrt{-1}\mathfrak{g}_+) \oplus (\mathfrak{g}_+ \oplus \sqrt{-1}\mathfrak{g}_-)$ is an orthogonal time-space decomposition of $(\mathfrak{g}^{\mathbf{c}}, \langle\ ,\ \rangle')$. Set $\mathfrak{g}^{\mathbf{c}}_- := \mathfrak{g}_- \oplus \sqrt{-1}\mathfrak{g}_+$, $\mathfrak{g}^{\mathbf{c}}_+ := \mathfrak{g}_+ \oplus \sqrt{-1}\mathfrak{g}_-$ and $\langle\ ,\ \rangle'_{\mathfrak{g}^{\mathbf{c}}_{\pm}} := -\pi^*_{\mathfrak{g}^{\mathbf{c}}_-}\langle\ ,\ \rangle' + \pi^*_{\mathfrak{g}^{\mathbf{c}}_+}\langle\ ,\ \rangle'$, where $\pi_{\mathfrak{g}^{\mathbf{c}}_-}$ (resp. $\pi_{\mathfrak{g}^{\mathbf{c}}_+}$) is the projection of $\mathfrak{g}^{\mathbf{c}}$ onto $\mathfrak{g}^{\mathbf{c}}_-$ (resp. $\mathfrak{g}^{\mathbf{c}}_+$). Let $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}^{\mathbf{c}}$ (with respect to $\langle\ ,\ \rangle'_{\mathfrak{g}^{\mathbf{c}}_{\pm}}$). Note that $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ is independent of the choice of the orthogonal time-space decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_+$. Let $H^0([0,1], \mathfrak{g}^{\mathbf{c}}_-)$ (resp. $H^0([0,1], \mathfrak{g}^{\mathbf{c}}_+)$) be the space of all L^2-integrable paths $u : [0,1] \to \mathfrak{g}^{\mathbf{c}}_-$ (resp. $u : [0,1] \to \mathfrak{g}^{\mathbf{c}}_+$) with respect to $-\langle\ ,\ \rangle'|_{\mathfrak{g}^{\mathbf{c}}_- \times \mathfrak{g}^{\mathbf{c}}_-}$ (resp. $\langle\ ,\ \rangle'|_{\mathfrak{g}^{\mathbf{c}}_+ \times \mathfrak{g}^{\mathbf{c}}_+}$). It is clear that $H^0([0,1], \mathfrak{g}^{\mathbf{c}}) = H^0([0,1], \mathfrak{g}^{\mathbf{c}}_-) \oplus H^0([0,1], \mathfrak{g}^{\mathbf{c}}_+)$. Define a non-degenerate symmetric bilinear form $\langle\ ,\ \rangle'_0$ of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ by $\langle u, v\rangle'_0 := \int_0^1 \langle u(t), v(t)\rangle' dt$. It is easy to show that the decomposition $H^0([0,1], \mathfrak{g}^{\mathbf{c}}) = H^0([0,1], \mathfrak{g}^{\mathbf{c}}_-) \oplus H^0([0,1], \mathfrak{g}^{\mathbf{c}}_+)$ is an orthogonal time-space decomposition with respect to $\langle\ ,\ \rangle'_0$. For simplicity, set $H^{0,\mathbf{c}}_{\pm} := H^0([0,1], \mathfrak{g}^{\mathbf{c}}_{\pm})$ and $\langle\ ,\ \rangle_{0,H^{0,\mathbf{c}}_{\pm}} := -\pi^*_{H^{0,\mathbf{c}}_-}\langle\ ,\ \rangle_0 + \pi^*_{H^{0,\mathbf{c}}_+}\langle\ ,\ \rangle_0$, where $\pi_{H^{0,\mathbf{c}}_-}$ (resp. $\pi_{H^{0,\mathbf{c}}_+}$) is the projection of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ onto $H^{0,\mathbf{c}}_-$ (resp. $H^{0,\mathbf{c}}_+$). It is clear that $\langle u, v\rangle_{0,H^{0,\mathbf{c}}_{\pm}} = \int_0^1 \langle u(t), v(t)\rangle'_{\mathfrak{g}^{\mathbf{c}}_{\pm}} dt$ $(u, v \in H^0([0,1], \mathfrak{g}^{\mathbf{c}}))$. Hence $(H^0([0,1], \mathfrak{g}^{\mathbf{c}}), \langle\ ,\ \rangle'_{0,H^{0,\mathbf{c}}_{\pm}})$ is a Hilbert space, that is, $(H^0([0,1],\mathfrak{g}^{\mathbf{c}}),\ \langle\ ,\ \rangle_0')$ is a pseudo-Hilbert space. Let J be the endomorphism of $\mathfrak{g}^{\mathbf{c}}$ defined by $JX=\sqrt{-1}X$ $(X\in\mathfrak{g}^{\mathbf{c}})$. Denote by the same symbol J the bi-invariant almost complex structure of $G^{\mathbf{c}}$ induced from J. Define the endomorphism $\widetilde{J}$ of $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ by $\widetilde{J}u=\sqrt{-1}u$ $(u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}}))$. Since $\widetilde{J}H^{0,\mathbf{c}}_{\pm}=H^{0,\mathbf{c}}_{\mp}$ and $\langle\widetilde{J}u,\widetilde{J}v\rangle_0'=-\langle u,v\rangle_0'$ $(u,v\in H^0([0,1],\mathfrak{g}^{\mathbf{c}}))$, the space $(H^0([0,1],\mathfrak{g}^{\mathbf{c}}),\langle\ ,\ \rangle_0',\widetilde{J})$ is an anti-Kaehlerian space. Let $H^1([0,1],G^{\mathbf{c}})$ be the Hilbert Lie group of all absolutely continuous paths $g:[0,1]\to G^{\mathbf{c}}$ such that the weak derivative g' of g exists and that g' is squared integrable (with respect to $\langle\ ,\ \rangle'_{\mathfrak{g}^{\mathbf{c}}_{\pm}}$), that is, $g_*^{-1}g'\in H^0([0,1],\mathfrak{g}^{\mathbf{c}})$. Define a map $\phi^{\mathbf{c}}:H^0([0,1],\mathfrak{g}^{\mathbf{c}})\to G^{\mathbf{c}}$ by $\phi^{\mathbf{c}}(u)=g_u(1)$ $(u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}}))$, where g_u is the element of $H^1([0,1],G^{\mathbf{c}})$ satisfying $g_u(0)=e$ and $g_{u*}^{-1}g_u'=u$. We call this map the *parallel transport map* (*from* 0 *to* 1) *for* $G^{\mathbf{c}}$. It is shown that this map $\phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion of $(H^0([0,1],\mathfrak{g}^{\mathbf{c}}),\ \langle\ ,\ \rangle_0',\widetilde{J})$ onto $(G^{\mathbf{c}},\ \langle\ ,\ \rangle',J)$ (i.e., it is a pseudo-Riemannian submersion, its vertical distribution is $\widetilde{J}$-invariant and $\phi^{\mathbf{c}}_*(\widetilde{J}X)=J(\phi^{\mathbf{c}}_*X)$ for every horizontal vector X).

Imitating the proof of Lemma 2.1 in [24], we can show the following relations.

LEMMA 6. *For* $v\in T_{\hat{0}}H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ $(\hat{0}$: *the constant path at the zero element* 0 *of* $\mathfrak{g}^{\mathbf{c}})$, $\phi^{\mathbf{c}}_{*\hat{0}}(v)=\int_0^1 v(t)dt$ *holds, where we identify* $T_{\hat{0}}H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ *with* $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$.

Imitating the proof of Lemma 2.3 in [24], we can show the following relation.

LEMMA 7. *For the horizontal lift* $\widetilde{v}^L$ *of a vector field* $\widetilde{v}$ *on* $G^{\mathbf{c}}$ *to* $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ *and* $\xi\in T_{\hat{0}}H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ $(=H^0([0,1],\mathfrak{g}^{\mathbf{c}}))$, *we have*

$$\widetilde{\widetilde{\nabla}}_\xi\widetilde{v}^L=(\nabla^*_{\phi^{\mathbf{c}}_*\xi}\widetilde{v})^L_{\hat{0}}-\left[\int_0^t\xi\,dt,\ \widetilde{v}_e\right]+\frac{1}{2}[\phi^{\mathbf{c}}_*\xi,\ \widetilde{v}_e]^L_{\hat{0}}\,,$$

where $\widetilde{\widetilde{\nabla}}$ (*resp.* ∇^*) *is the Levi-Civita connection of* $\langle\ ,\ \rangle_0'$ (*resp.* $\langle\ ,\ \rangle'$).

Set $P(G^{\mathbf{c}},e\times G^{\mathbf{c}}):=\{g\in H^1([0,1],G^{\mathbf{c}})\,|\,g(0)=e\}$ and $\Omega_e(G^{\mathbf{c}}):=\{g\in H^1([0,1],G^{\mathbf{c}})\,|\,g(0)=g(1)=e\}$. The group $H^1([0,1],G^{\mathbf{c}})$ acts on $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ by gauge transformations, that is,

$$g*u:=\mathrm{Ad}(g)u-g'g_*^{-1}\quad(g\in H^1([0,1],G^{\mathbf{c}}),\ u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}}))\,.$$

It is shown that the following facts hold:

(i) The above action of $H^1([0,1],G^{\mathbf{c}})$ on $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ is isometric,

(ii) The above action of $P(G^{\mathbf{c}},e\times G^{\mathbf{c}})$ on $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ is transitive and free,

(iii) $\phi^{\mathbf{c}}(g*u)=g(0)\phi^{\mathbf{c}}(u)g(1)^{-1}$ for $g\in H^1([0,1],G^{\mathbf{c}})$ and $u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}})$,

(iv) $\phi^{\mathbf{c}}:H^0([0,1],\mathfrak{g}^{\mathbf{c}})\to G^{\mathbf{c}}$ is regarded as a $\Omega_e(G^{\mathbf{c}})$-bundle.

(v) If $\phi^{\mathbf{c}}(u)=x_0\phi^{\mathbf{c}}(v)x_1^{-1}$ $(u,v\in H^0([0,1],\mathfrak{g}^{\mathbf{c}}),\ x_0,x_1\in G^{\mathbf{c}})$, then there exists $g\in H^1([0,1],G^{\mathbf{c}})$ such that $g(0)=x_0,\ g(1)=x_1$ and $u=g*v$. In particular, it follows that each $u\in H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ is described as $u=g*\hat{0}$ in terms of some $g\in P(G^{\mathbf{c}},G^{\mathbf{c}}\times e)$.

Let $\widetilde{\iota}$ be the inclusion map of $H^0([0,1],\mathfrak{g})$ into $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ and $\bar{\iota}$ be the inclusion map of G into $G^{\mathbf{c}}$. Also, let ϕ be the parallel transport map for G. Then we have

$$\phi^{\mathbf{c}}\circ\widetilde{\iota}=\bar{\iota}\circ\phi\,. \tag{6.1}$$

7. Proof of Theorem 1

In this section, we prove Theorem 1. First we prepare the following proposition.

PROPOSITION 4. *Let M be an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$ with globally flat and abelian normal bundle, where $G^{\mathbf{c}}/K^{\mathbf{c}}$ is the complexification of a symmetric space G/K of non-compact type. Then M is anti-Kaehlerian equifocal in $G^{\mathbf{c}}/K^{\mathbf{c}}$ if and only if each component of $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$ is anti-Kaehlerian isoparametric in $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$. In detail, for each unit normal vector v of M, complex focal radii along the geodesic γ_v coincide with the inverse numbers of complex principal curvatures of the horizontal lift v^L-direction, where γ_v is the maximal geodesic in $G^{\mathbf{c}}/K^{\mathbf{c}}$ with $\dot{\gamma}_v(0)=v$.*

PROOF. Let M be an anti-Kaehlerian submanifold with globally flat and abelian normal bundle in $G^{\mathbf{c}}/K^{\mathbf{c}}$. Denote by the same symbol $\widetilde{J}$ the almost complex structures of $G^{\mathbf{c}}/K^{\mathbf{c}}$ and $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ and by the same symbol J those of M and $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$. Fix a normal vector v of M at x and $\tilde{x}\in(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(x)$. Since $\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}}$ is a pseudo-Riemannian submersion, it is shown that $\exp^{\perp}(av+b\widetilde{J}v)$ is a focal point with multiplicity m along the geodesic $\gamma_{av+b\widetilde{J}v}$ if and only if $\widehat{\exp}^{\perp}((av+b\widetilde{J}v)^L_{\tilde{x}})$ is a focal point with multiplicity m along the geodesic $\gamma_{(av+b\widetilde{J}v)^L_{\tilde{x}}}$ by imitating the proof of Lemma 5.12 in [49], where $\exp^{\perp}$ (resp. $\widehat{\exp}^{\perp}$) is the normal exponential map of M (resp. $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$). Further, since $\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}}$ is an anti-Kaehlerian submersion, we have $(av+b\widetilde{J}v)^L_{\tilde{x}}=av^L_{\tilde{x}}+b\widetilde{J}v^L_{\tilde{x}}$. Hence we see that $a+b\sqrt{-1}$ is a complex focal radius of direction v if and only if $a+b\sqrt{-1}$ is a complex focal radius of direction $v^L_{\tilde{x}}$. Complex focal radii of direction $v^L_{\tilde{x}}$ coincide with the inverse numbers of complex principal curvatures of direction $v^L_{\tilde{x}}$. Thus we see that complex focal radii of direction v coincide with the inverse numbers of complex principal curvatures of direction $v^L_{\tilde{x}}$. On the other hand, since M has abelian normal bundle, we can show that for each normal vector field $\tilde{v}$ of M, $\nabla^{\perp}\tilde{v}=0$ is equivalent to $\widehat{\nabla}^{\perp}\tilde{v}^L=0$ (see the proof of Lemma 5.6 in [49]), where $\nabla^{\perp}$ (resp. $\widehat{\nabla}^{\perp}$) is the normal connection of M (resp. $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$). Hence since M has globally flat normal bundle, $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$ also has globally flat normal bundle. These facts deduce that M is anti-Kaehlerian equifocal if and only if each component of $(\pi^{\mathbf{c}}\circ\phi^{\mathbf{c}})^{-1}(M)$ is anti-Kaehlerian isoparametric. q.e.d.

The statement (i) of Theorem 1 directly follows from this proposition and Theorem 5. Next we prepare some facts to prove the statement (ii) of Theorem 1. Let M^* be one of components of $\pi^{\mathbf{c}-1}(M)$ and $\widetilde{M}$ be one of components of $\phi^{\mathbf{c}-1}(M^*)$. Then we have the following fact.

LEMMA 8. *Take $u \in \widetilde{M}$. Set $g_0 := \phi^{\mathbf{c}}(u)$ and $x := (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(u)$. Then there exists $g \in H^1([0,1], G^{\mathbf{c}})$ satisfying $g * u = \hat{0}$, $g(0) = g_0^{-1}$ (and hence $g(0)x = eK^{\mathbf{c}}$) and $\phi^{\mathbf{c}}(g * \widetilde{M}) = g(0)M^*$ (and hence $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})(g * \widetilde{M}) = g(0)M$).*

PROOF. According to the facts (iii) and (iv) stated in Section 6, we can find $g \in H^1([0,1], G^{\mathbf{c}})$ satisfying $g(0) = g_0^{-1}$, $g(1) = e$ and $g * u = \hat{0}$. Then it follows from the fact (iii) that $\phi^{\mathbf{c}}(g * \widetilde{M}) = g_0^{-1} M^*$. q.e.d.

Denote by X^* the horizontal lift of a vector (or a vector field) X of $G^{\mathbf{c}}/K^{\mathbf{c}}$ to $G^{\mathbf{c}}$ and by Y^L the horizontal lift of a vector (or a vector field) Y of $G^{\mathbf{c}}$ to $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. In particular, the horizontal lift X_e^* of $X \in T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$ to e is identified with X. Fix $v \in T^{\perp}_{g_0K^{\mathbf{c}}}M$ with $||v|| = 1$ and $u \in (\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(g_0K^{\mathbf{c}})$, where $g_0 \in M^*$. According to Lemma 8, we may assume $g_0K^{\mathbf{c}} = eK^{\mathbf{c}}$ and $u = \hat{0}$. Also, we may assume $e \in M^*$. Let $\mathfrak{g}^{\mathbf{c}} = \mathfrak{k}^{\mathbf{c}} + \mathfrak{p}^{\mathbf{c}}$ be the canonical decomposition of $\mathfrak{g}^{\mathbf{c}}$. Take a maximal abelian subspace $\mathfrak{h}^{\mathbf{c}}$ (in $\mathfrak{p}^{\mathbf{c}} = T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$) containing v. Let $\mathfrak{p}^{\mathbf{c}} = \mathfrak{h}^{\mathbf{c}} + \sum_{\alpha \in \triangle_+} \mathfrak{p}^{\mathbf{c}}_\alpha$ be the root space decomposition. Take a maximal abelian subalgebra $\tilde{\mathfrak{h}}^{\mathbf{c}}$ in $\mathfrak{g}^{\mathbf{c}}$ containing $\mathfrak{h}^{\mathbf{c}}$ and let $\mathfrak{h}^{\mathbf{c}}_{\mathfrak{k}} := \tilde{\mathfrak{h}}^{\mathbf{c}} \cap \mathfrak{k}^{\mathbf{c}}$. Denote by $\hat{X}$ the constant path at $X \in \mathfrak{g}^{\mathbf{c}}$ and by $X^L_{\hat{0}}$ the horizontal lift of $X \in \mathfrak{g}^{\mathbf{c}}$ to $\hat{0}$. Note that $\hat{X}$ coincides with $X^L_{\hat{0}}$ under the identification of $T_{\hat{0}}H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ and $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$.

For $X \in \mathfrak{p}^{\mathbf{c}}_\alpha$, let $X_{\mathfrak{k}^{\mathbf{c}}}$ be the element of $\mathfrak{k}^{\mathbf{c}}$ such that $\mathrm{ad}(a)X = \alpha^{\mathbf{c}}(a)X_{\mathfrak{k}^{\mathbf{c}}}$ for all $a \in \mathfrak{h}^{\mathbf{c}}$, where ad is the adjoint representation of $\mathfrak{g}^{\mathbf{c}}$. Note that $\mathrm{ad}(a)X_{\mathfrak{k}^{\mathbf{c}}} = \alpha^{\mathbf{c}}(a)X$ for all $a \in \mathfrak{h}^{\mathbf{c}}$. For $X \in \mathfrak{p}^{\mathbf{c}}_\alpha$ (resp. $\tilde{\mathfrak{h}}^{\mathbf{c}}$), we define loop vectors $l^i_{X,j} \in H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ ($i = 1, 2$, $j \in \mathbf{Z}$) by

$$l^1_{X,j}(t) := X\cos(2j\pi t) + \sqrt{-1}X_{\mathfrak{k}^{\mathbf{c}}}\sin(2j\pi t)$$
$$(\text{resp. } l^1_{X,j}(t) := X\cos(2j\pi t))$$

and

$$l^2_{X,j}(t) := \sqrt{-1}X\sin(2j\pi t) + X_{\mathfrak{k}^{\mathbf{c}}}\cos(2j\pi t)$$
$$(\text{resp. } l^2_{X,j}(t) := \sqrt{-1}X\sin(2j\pi t))\,.$$

According to Lemma 6, these loop vectors are vertical with respect to $\phi^{\mathbf{c}}$. Under the identification of $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ and $T_{\hat{0}}H^0([0,1], \mathfrak{g}^{\mathbf{c}})$, the constant path $\hat{X}$ and loop vectors $l^i_{X,j}$ ($i = 1, 2$) are regarded as elements of $T_{\hat{0}}H^0([0,1], \mathfrak{g}^{\mathbf{c}})$. Now we prepare the following lemma.

LEMMA 9. *Let $\{X_1, \cdots, X_n\}$ be an orthonormal base of $T_{eK^{\mathbf{c}}}M$, $\{Y_1, \cdots, Y_{m_{\mathfrak{k}^{\mathbf{c}}}}\}$ be that of $\mathfrak{k}^{\mathbf{c}}$, $\{e^0_1, \cdots, e^0_{m_0}\}$ be that of $\tilde{\mathfrak{h}}^{\mathbf{c}}$ and $\{e^\alpha_1, \cdots, e^\alpha_{m_\alpha}\}$ ($\alpha \in \triangle_+$) be that of $\mathfrak{p}^{\mathbf{c}}_\alpha$. Then the system*

$$\{\hat{X}_1, \cdots, \hat{X}_n\} \cup \{\hat{Y}_1, \cdots, \hat{Y}_{m_{\mathfrak{k}^{\mathbf{c}}}}\}$$
$$\cup \{\sqrt{2}l^i_{e^0_j,k} \,|\, i = 1, 2,\ j = 1, \cdots, m_0,\ k \in \mathbf{N}\}$$

$$\cup\left(\bigcup_{\alpha\in\triangle_+}\{l^i_{e^\alpha_j,k}\,|\,i=1,2,\ \ j=1,\cdots,m_\alpha,\ \ k\in\mathbf{Z}\setminus\{0\}\}\right)$$

is a J-orthonormal base of $(T_{\hat{0}}\widetilde{M},\ \langle\ ,\ \rangle_0)$.

PROOF. This fact is shown by imitating the proof of Lemma 5.3 in [25]. q.e.d.

From Lemma 7, the following relations are directly deduced.

LEMMA 10. *Let $\tilde{v}$ be a vector field on a neighborhood of e in $G^{\mathbf{c}}$ with $\tilde{v}_e = v$ $(\in T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}) = (T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}}))^L_e \subset T_eG^{\mathbf{c}})$ and ∇^* (resp. $\widetilde{\nabla}$) be the Levi-Civita connection of $G^{\mathbf{c}}$ (resp. $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$). Then the following statements* (i)–(iii) *hold.*

(i) *For $X\in\tilde{\mathfrak{h}}^{\mathbf{c}}$, the relation $\widetilde{\nabla}_{\hat{X}}\tilde{v}^L = (\nabla^*_X\tilde{v})^L_{\hat{0}}$ holds.*

(ii) *For $X\in\mathfrak{p}^{\mathbf{c}}_\alpha$, the relations $\widetilde{\nabla}_{\hat{X}}\tilde{v}^L = (\nabla^*_X\tilde{v})^L_{\hat{0}} + t\alpha^{\mathbf{c}}(v)X_{\mathfrak{f}^{\mathbf{c}}} - \frac{1}{2}\alpha^{\mathbf{c}}(v)\hat{X}_{\mathfrak{f}^{\mathbf{c}}}$ and $\widetilde{\nabla}_{\hat{X}_{\mathfrak{f}^{\mathbf{c}}}}\tilde{v}^L = (\nabla^*_{X_{\mathfrak{f}^{\mathbf{c}}}}\tilde{v})^L_{\hat{0}} + t\alpha^{\mathbf{c}}(v)X - \frac{1}{2}\alpha^{\mathbf{c}}(v)\hat{X}$ hold.*

(iii) *For $X\in\mathfrak{g}^{\mathbf{c}}$, the relations $\widetilde{\nabla}_{l^i_{X,j}}\tilde{v}^L = -[\int_0^t l^i_{X,j}(t)dt,\ v]$ $(i=1,2,\ \ j\in\mathbf{Z})$ hold.*

Also, since $\pi^{\mathbf{c}}$ is a pseudo-Riemannian submersion, we have the following relations.

LEMMA 11. (i) *For $X\in T_{eK^{\mathbf{c}}}M$ $(= (T_{eK^{\mathbf{c}}}M)^L_e \subset T_eM^*)$, the relation $A^*_vX = A_vX + \frac{1}{2}[v,X]$ holds.*

(ii) *For $X\in\mathfrak{f}^{\mathbf{c}}$ $(\subset T_eG^{\mathbf{c}})$, the relation $\nabla^*_Xv^* = \frac{1}{2}[v,X]$ holds.*

PROOF. Since π is a pseudo-Riemannian submersion, the statement (i) is shown by imitating the proof of Proposition 7.3 in [49]. Also, since v^* is a right-invariant vector field along the fibre $K^{\mathbf{c}}$ of $\pi^{\mathbf{c}}$, the statement (ii) is directly deduced. q.e.d.

From Lemmas 10 and 11, the following facts are directly deduced.

LEMMA 12. *Let $X\in\tilde{\mathfrak{h}}^{\mathbf{c}}$. Then the following statements* (i)–(iii) *hold.*

(i) *If $X\in\mathfrak{h}^{\mathbf{c}}\cap T_{eK^{\mathbf{c}}}M$ and $A_vX = \lambda X (= (\mathrm{Re}\lambda)X + (\mathrm{Im}\lambda)JX)$, then we have $\widetilde{A}_{\hat{v}}\hat{X} = \lambda\hat{X}$.*

(ii) *If $X\in\mathfrak{h}^{\mathbf{c}}_{\mathfrak{f}}$, then we have $\widetilde{A}_{\hat{v}}\hat{X} = 0$.*

(iii) *For the vertical loop vectors $l^i_{X,j}$ $(i=1,2,\ \ j\in\mathbf{N})$, we have $\widetilde{A}_{\hat{v}}l^i_{X,j} = 0$.*

In the sequel, for a family $\{a_k\}_{k\in\mathbf{Z}\setminus\{0\}}$ of complex numbers or vectors, the notation $\sum_{k\in\mathbf{Z}\setminus\{0\}}a_k$ implies $\sum_{k=1}^\infty(a_k+a_{-k})$, where $\mathbf{Z}$ is the set of all integers. Also, we obtain the following relations in terms of Lemmas 10 and 11.

LEMMA 13. *Let $X\in\mathfrak{p}^{\mathbf{c}}_\alpha$. The following statements* (i)–(v) *hold.*

(i) *If $X\in T_{eK^{\mathbf{c}}}M$, $A_vX = \lambda X (= (\mathrm{Re}\lambda)X + (\mathrm{Im}\lambda)JX)$ and $\alpha^{\mathbf{c}}(v) = 0$, then we have $\widetilde{A}_{\hat{v}}\hat{X} = \lambda\hat{X}$, $\widetilde{A}_{\hat{v}}\hat{X}_{\mathfrak{f}^{\mathbf{c}}} = \widetilde{A}_{\hat{v}}l^i_{X,j} = 0$ $(i=1,2,\ \ j\in\mathbf{Z})$.*

(ii) *If* $X \in T_{eK^{\mathbf{c}}}M$, $A_v X = \lambda X (= (\mathrm{Re}\lambda)X + (\mathrm{Im}\lambda)JX)$ $(\lambda \neq \pm\alpha^{\mathbf{c}}(v))$ *and* $\alpha^{\mathbf{c}}(v) \neq 0$, *then*

$$\hat{X} + a_{\lambda,\alpha}\hat{X}_{\mathfrak{k}^{\mathbf{c}}} + \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{b_{\lambda,\alpha}}{b_{\lambda,\alpha} - k\pi}(l^1_{X,k} + a_{\lambda,\alpha}l^2_{X,k})$$

is a J*-eigenvector for a* J*-eigenvalue* $\frac{\alpha^{\mathbf{c}}(v)\sqrt{-1}}{2b_{\lambda,\alpha}}$ *of* $\widetilde{A}_{\hat{v}}$, *where* $a_{\lambda,\alpha} = \frac{\alpha^{\mathbf{c}}(v)}{\lambda+\sqrt{\lambda^2-\alpha^{\mathbf{c}}(v)^2}}$ (2-*valued*), $b_{\lambda,\alpha} = \arctan(a_{\lambda,\alpha}\sqrt{-1})$ (∞-*valued*).

(iii) *If* $X \in T_{eK^{\mathbf{c}}}M$, $A_v X = \pm\alpha^{\mathbf{c}}(v)X$ *and* $\alpha^{\mathbf{c}}(v) \neq 0$, *then there exists no* J*-eigenvector of* $\widetilde{A}_{\hat{v}}$ *belonging to* $\mathrm{Span}\big((\bigcup_{k\in\mathbf{Z}\setminus\{0\}}\{l^1_{X,k},\ l^2_{X,k}\}) \cup \{\hat{X}, \hat{X}_{\mathfrak{k}^{\mathbf{c}}}\}\big)$.

(iv) *If* $X \in T^{\perp}_{eK^{\mathbf{c}}}M$ *and* $\alpha^{\mathbf{c}}(v) = 0$, *then we have* $\widetilde{A}_{\hat{v}}\hat{X}_{\mathfrak{k}^{\mathbf{c}}} = \widetilde{A}_{\hat{v}}l^i_{X,k} = 0$ $(\,i = 1, 2,\ k \in \mathbf{Z}\setminus\{0\})$.

(v) *If* $X \in T^{\perp}_{eK^{\mathbf{c}}}M$ *and* $\alpha^{\mathbf{c}}(v) \neq 0$, *then we have* $\widetilde{A}_{\hat{v}}l^1_{X,j} = \frac{\alpha^{\mathbf{c}}(v)\sqrt{-1}}{2j\pi}l^1_{X,j}$ $(j \in \mathbf{Z}\setminus\{0\})$ *and*

$$\hat{X}_{\mathfrak{k}^{\mathbf{c}}} + \sum_{k\in\mathbf{Z}\setminus\{0\}} \frac{2j+1}{2j-2k+1}l^2_{X,k} \quad (j \in \mathbf{Z})$$

is a J*-eigenvector for a* J*-eigenvalue* $\frac{\alpha^{\mathbf{c}}(v)\sqrt{-1}}{(2j+1)\pi}$ *of* $\widetilde{A}_{\hat{v}}$.

PROOF. These facts are shown by imitating the proof of Proposition 5.7 in [25]. q.e.d.

Now we obtain the following fact in terms of these lemmas.

PROPOSITION 5. *Let* M *be an anti-Kaehlerian submanifold with globally flat and abelian normal bundle in* $G^{\mathbf{c}}/K^{\mathbf{c}}$, *where* $G^{\mathbf{c}}/K^{\mathbf{c}}$ *is the complexification of a symmetric space* G/K *of non-compact type. Assume that* M *is curvature adapted. Then* M *is anti-Kaehlerian equifocal in* $G^{\mathbf{c}}/K^{\mathbf{c}}$ *and for each* $v \in T^{\perp}M$ *and each* $\alpha^{\mathbf{c}} \in \triangle^{\mathbf{c}}_+$ *with* $\alpha^{\mathbf{c}}(g_*^{-1}v) \neq 0$, $\pm\alpha^{\mathbf{c}}(g_*^{-1}v)$ *is not a* J*-eigenvalue of* $A^{\mathbf{c}}_v|_{g_*\mathfrak{p}^{\mathbf{c}}_\alpha}$ *if and only if* $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M)$ *is proper anti-Kaehlerian isoparametric in* $H^0([0, 1], \mathfrak{g}^{\mathbf{c}})$, *where* g *is a representative element of the base point of* v *and* $\triangle^{\mathbf{c}}_+$ *is the positive root system with respect to a maximal abelian subspace (equipped with some lexicographical ordering) of* $\mathfrak{p}^{\mathbf{c}} = T_{eK^{\mathbf{c}}}(G^{\mathbf{c}}/K^{\mathbf{c}})$ *containing* $g_*^{-1}v$.

PROOF. This proposition directly follows from Proposition 4 and Lemmas 9, 12 and 13. q.e.d.

Now we prove the statement (ii) of Theorem 1.

PROOF OF (ii) OF THEOREM 1. The first half of the statement (ii) of Theorem 1 directly follows from this proposition and Theorem 5. It remains to show that each component of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ extends to a complete proper anti-Kaehlerian isoparametric submanifold. Let $\widetilde{M}^{\mathbf{c}}$ be one component of $(\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}})^{-1}(M^{\mathbf{c}})$ and $\{E_i \mid i \in I\}$ be the set of all complex curvature distributions on $\widetilde{M}^{\mathbf{c}}$. According to Theorem 2 (which will be proved in the next section), the distribution E_i is totally geodesic in $\widetilde{M}^{\mathbf{c}}$ (and hence integrable) and each leaf L of E_i is an open potion of a complex sphere or complex affine subspace. Denote by $S_L^{\mathbf{c}}$ this complex sphere or this complex affine subspace. Set $(\widetilde{M}^{\mathbf{c}})_1 := \bigcup_{i\in I}\bigcup_{L\in\widetilde{M}^{\mathbf{c}}/E_i} S_L^{\mathbf{c}}$, where $\widetilde{M}^{\mathbf{c}}/E_i$ is the leaf space of E_i. It is shown that $(\widetilde{M}^{\mathbf{c}})_1$ is a real analytic extension of $\widetilde{M}^{\mathbf{c}}$. From this fact, it follows that $(\widetilde{M}^{\mathbf{c}})_1$ is also proper anti-Kaehlerian isoparametric. Let $\{E_i^1 \mid i \in I\}$ be the set of all complex curvature distributions on $(\widetilde{M}^{\mathbf{c}})_1$. For each leaf L_1 of E_i^1, denote by $S_{L_1}^{\mathbf{c}}$ the complex sphere or the complex affine subspace containing L_1 as an open subset. Set $(\widetilde{M}^{\mathbf{c}})_2 := \bigcup_{i\in I}\bigcup_{L_1\in(\widetilde{M}^{\mathbf{c}})_1/E_i^1} S_{L_1}^{\mathbf{c}}$, which is a real analytic extension of $(\widetilde{M}^{\mathbf{c}})_1$. From this fact, it follows that $(\widetilde{M}^{\mathbf{c}})_2$ is also proper anti-Kaehlerian isoparametric. In the sequel, we define $(\widetilde{M}^{\mathbf{c}})_j$ $(j = 2, 3, 4, \cdots)$ inductively. Set $(\widetilde{M}^{\mathbf{c}})\widehat{\ } := \bigcup_{j=1}^{\infty}(\widetilde{M}^{\mathbf{c}})_j$. It is clear that $(\widetilde{M}^{\mathbf{c}})\widehat{\ }$ is a desired extension of $\widetilde{M}^{\mathbf{c}}$. q.e.d.

8. Proof of Theorem 2

In this section, we prove Theorem 2. First we characterize totally anti-Kaehlerian umbilical submanifold in an infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. Let W be a $(2m+2)$-dimensional $\widetilde{J}$-invariant vector space of $(V, \langle\ ,\ \rangle, \widetilde{J})$. Identify W with $\mathbf{C}^{m+1}$. Hence we have $\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\rangle^{\mathbf{c}}|_{W^{\mathbf{c}}\times W^{\mathbf{c}}} = \frac{1}{2}\delta_{ij}$, $\langle \frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\rangle^{\mathbf{c}}|_{W^{\mathbf{c}}\times W^{\mathbf{c}}} = 0$ $(1 \le i, j \le m+1)$ and $\widetilde{J}|_W = \sqrt{-1}\,\mathrm{id}_{\mathbf{C}^{m+1}}$, where $(z_1, \cdots, z_{m+1})$ is the natural coordinate of $\mathbf{C}^{m+1}$. Define the complex hypersurface $S_{\mathbf{c}}^m(\kappa)$ in $W = \mathbf{C}^{m+1}$ by the equation $z_1^2 + \cdots + z_{m+1}^2 = \kappa^2$ $(\kappa \in \mathbf{C})$. Let $\kappa = a + b\sqrt{-1}$ $(a, b \in \mathbf{R})$. It is clear that $S_{\mathbf{c}}^m(\kappa)$ is an anti-Kaehlerian submanifold in W (and hence $(V, \langle\ ,\ \rangle, \widetilde{J})$). We call $S_{\mathbf{c}}^m(\kappa)$ a *complex sphere of radius* κ. The position vector field $\widetilde{v}$ (: $(z_1, \cdots, z_{m+1})\,(\in S_{\mathbf{c}}^m(\kappa)) \to (z_1, \cdots, z_{m+1})$) is a parallel normal vector field on $S_{\mathbf{c}}^m(\kappa)$ and the normal space $T_z^{\perp}S_{\mathbf{c}}^m(\kappa)$ $(z = (z_1, \cdots, z_{m+1}))$ in W is spanned by $\widetilde{v}_z$ and $\widetilde{J}\widetilde{v}_z$. Denote by J the almost complex structure of $S_{\mathbf{c}}^m(\kappa)$. For the shape tensor A of $S_{\mathbf{c}}^m(\kappa)(\subset V)$, we have $A_{\widetilde{v}} = -\mathrm{id}$ and $A_{\widetilde{J}\widetilde{v}} = -J$, that is, $\mathrm{Spec}_J A_{\widetilde{v}} = \{-1\}$ and $\mathrm{Spec}_J A_{\widetilde{J}\widetilde{v}} = \{-\sqrt{-1}\}$. Also, we have $A_w = 0$ for every $w \in \mathrm{Span}\{\widetilde{v}, \widetilde{J}\widetilde{v}\}^{\perp}$. On the other hand, we have $\langle \widetilde{v}, \widetilde{v}\rangle = a^2 - b^2$, $\langle \widetilde{J}\widetilde{v}, \widetilde{J}\widetilde{v}\rangle = -(a^2 - b^2)$ and $\langle \widetilde{v}, \widetilde{J}\widetilde{v}\rangle = 2ab$. From these relations, we see that $\widetilde{v}_1 := -\frac{1}{|\kappa|^2}(a\widetilde{v} + b\widetilde{J}\widetilde{v})$ and $\widetilde{J}\widetilde{v}_1$ become an orthonormal frame field of $T^{\perp}S_{\mathbf{c}}^m(\kappa)$. Note that $\langle \widetilde{v}_1, \widetilde{v}_1\rangle = 1$. Denote by h the second fundamental form of $S_{\mathbf{c}}^m(\kappa)(\subset V)$. Then we have

$$
\begin{aligned}
h(X, Y) &= \langle h(X, Y), \widetilde{v}_1\rangle \widetilde{v}_1 - \langle h(X, Y), \widetilde{J}\widetilde{v}_1\rangle \widetilde{J}\widetilde{v}_1 \\
&= \langle A_{\widetilde{v}_1}X, Y\rangle \widetilde{v}_1 - \langle A_{\widetilde{J}\widetilde{v}_1}X, Y\rangle \widetilde{J}\widetilde{v}_1 \\
&= \frac{1}{|\kappa|^2}\langle aX + bJX, Y\rangle \widetilde{v}_1 - \frac{1}{|\kappa|^2}\langle aJX - bX, Y\rangle \widetilde{J}\widetilde{v}_1
\end{aligned}
$$

$$= \langle X, Y\rangle \frac{1}{|\kappa|^2}(a\widetilde{v}_1 + b\widetilde{J}\widetilde{v}_1) - \langle JX, Y\rangle \frac{1}{|\kappa|^2}\widetilde{J}(a\widetilde{v}_1 + b\widetilde{J}\widetilde{v}_1)$$

$$= \langle X, Y\rangle \left(\frac{\mathrm{Re}(\kappa^2)}{|\kappa|^4}\widetilde{v} + \frac{\mathrm{Im}(\kappa^2)}{|\kappa|^4}\widetilde{J}\widetilde{v}\right) - \langle JX, Y\rangle \widetilde{J}\left(\frac{\mathrm{Re}(\kappa^2)}{|\kappa|^4}\widetilde{v} + \frac{\mathrm{Im}(\kappa^2)}{|\kappa|^4}\widetilde{J}\widetilde{v}\right).$$

Thus the complex sphere $S^m_{\mathbf{C}}(\kappa)$ becomes a totally anti-Kaehlerian umbilical submanifold with the mean curvature vector $\frac{\mathrm{Re}(\kappa^2)}{|\kappa|^4}\widetilde{v} + \frac{\mathrm{Im}(\kappa^2)}{|\kappa|^4}\widetilde{J}\widetilde{v}$. Set $H := \frac{\mathrm{Re}(\kappa^2)}{|\kappa|^4}\widetilde{v} + \frac{\mathrm{Im}(\kappa^2)}{|\kappa|^4}\widetilde{J}\widetilde{v}$. Then we can show

$$\kappa^2 = \frac{\langle H, H\rangle - \sqrt{-1}\langle \widetilde{J}H, H\rangle}{\langle H, H\rangle^2 + \langle H, \widetilde{J}H\rangle^2}\,. \tag{8.1}$$

Now we prove the following characterization theorem for totally anti-Kaehlerian umbilical submanifolds.

THEOREM 6. *A totally anti-Kaehlerian umbilical submanifold with parallel mean curvature vector in the infinite dimensional anti-Kaehlerian space is an open potion of a complex sphere or a complex affine subspace.*

PROOF. Let $(M, \langle\ ,\ \rangle, J)$ be a totally anti-Kaehlerian umbilical submanifold in the infinite dimensional anti-Kaehlerian space $(V, \langle\ ,\ \rangle, \widetilde{J})$. Denote by ∇, A, h, $\nabla^\perp$ and H the induced connection, the shape tensor, the second fundamental form, the normal connection and the mean curvature vector of $(M, \langle\ ,\ \rangle, J)$, respectively. When $H = 0$, it is clear that M is an open potion of a complex affine subspace. In the sequel, we consider the case of $H \neq 0$. The first normal space N_1 of $(M, \langle\ ,\ \rangle, J)$ is spanned by H and $\widetilde{J}H$. From the assumption, we obtain $\nabla^\perp_X H = 0$ and $\nabla^\perp_X(\widetilde{J}H) = 0$. Thus the first normal space N_1 is parallel with respect to $\nabla^\perp$. Therefore, according to the reduction theorem, there exists an $(n+2)$-dimensional $\widetilde{J}$-invariant subspace W of $(V, \langle\ ,\ \rangle, \widetilde{J})$ containing $(M, \langle\ ,\ \rangle, J)$. Set

$$v_x := x + \frac{1}{\langle H_x, H_x\rangle^2 + \langle H_x, \widetilde{J}H_x\rangle^2}(\langle H_x, H_x\rangle H_x + \langle H_x, \widetilde{J}H_x\rangle \widetilde{J}H_x)\,,$$

where $x \in M$. Then we can show

$$\widetilde{\nabla}_X v = X - \frac{\langle H, H\rangle}{\langle H, H\rangle^2 + \langle H, \widetilde{J}H\rangle^2}A_H X - \frac{\langle H, \widetilde{J}H\rangle}{\langle H, H\rangle^2 + \langle H, \widetilde{J}H\rangle^2}A_{\widetilde{J}H}X \tag{8.2}$$

$(X \in TM)$ in terms of $\nabla^\perp H = \nabla^\perp \widetilde{J}H = 0$. On the other hand, we have $A_H X = \langle H, H\rangle X - \langle H, \widetilde{J}H\rangle JX\ (X \in TM)$ and $A_{\widetilde{J}H}X = \langle H, \widetilde{J}H\rangle X + \langle H, H\rangle JX\ (X \in TM)$. By substituting these relations into (8.2), we obtain $\widetilde{\nabla}_X v = 0$. That is, v is a constant vector. Identify W with $\mathbf{C}^{\frac{n}{2}+1}$. For each $z = (z_1, \cdots, z_{\frac{n}{2}+1}) \in \mathbf{C}^{\frac{n}{2}+1}$, we denote $z_1^2 + \cdots + z_{\frac{n}{2}+1}^2$ by z^2. Then we have

$$(x - v_x)^2 = \frac{(\langle H_x, H_x\rangle^2 - 2\sqrt{-1}\langle H_x, H_x\rangle\langle H_x, \widetilde{J}H_x\rangle - \langle H_x, \widetilde{J}H_x\rangle^2)H_x^2}{(\langle H_x, H_x\rangle^2 + \langle H_x, \widetilde{J}H_x\rangle^2)^2}\,.$$

It follows from $\nabla^{\perp} H = \nabla^{\perp}(\widetilde{J}H) = 0$ that the right-hand side is independent of $x \in M$. This means that $(M, \langle\ ,\ \rangle, J)$ is contained in a complex sphere. q.e.d.

REMARK 4. A real $n(\geq 4)$-dimensional totally anti-Kaehlerian umbilical submanifold in the infinite dimensional anti-Kaehlerian space has automatically parallel mean curvature vector.

Now we prove Theorem 2.

PROOF OF THEOREM 2. First we prove the statement (i). Let $v \in T^{\perp}_x M$. The point $x + v$ is a focal point of (M, x) if and only if $\mathrm{Ker}(\mathrm{id} - A_v) \neq \{0\}$, that is, $1 \in \mathrm{Spec}_J A_v$. The J-spectrum $\mathrm{Spec}_J A_v$ coincides with $\{\lambda_i(x)(v) \mid i \in I\}$. Hence, $x + v$ is a focal point of (M, x) if and only if $v \in \bigcup_{i\in I} \lambda_i(x)^{-1}(1)$. That is, the focal set of (M, x) coincides with $\bigcup_{i\in I} \lambda_i(x)^{-1}(1)$ (under the identification of v with $x + v$). Next we prove the statement (ii). Denote by ∇ the Levi-Civita connection of M and by h_i $(i \in I)$ the second fundamental form of the distribution E_i on M. Let $X, Y \in \Gamma(E_i)$, $Z \in \Gamma(E_j)$ $(i \neq j)$ and v be a parallel normal vector field of $(M, \langle\ ,\ \rangle, J)$ satisfying $\lambda_i(v) \neq \lambda_j(v)$. Then we have

$$(\nabla_Z A)_v X = \lambda_i(v)\nabla_Z X - A_v(\nabla_Z X)$$

and

$$(\nabla_X A)_v Z = \lambda_j(v)\nabla_X Z - A_v(\nabla_X Z)\,,$$

where $\sqrt{-1}*$ implies $J*$ and we use the fact that $\lambda_i(v)$ and $\lambda_j(v)$ are constant because of $\nabla^{\perp} v = 0$. Since $(\nabla_Z A)_v X = (\nabla_X A)_v Z$ by the Codazzi equation, we have $(A_v - \lambda_i(v)\mathrm{id})\nabla_Z X = (A_v - \lambda_j(v)\mathrm{id})\nabla_X Z$. By taking the inner product with Y, we have $(\lambda_i(v) - \lambda_j(v))\langle\nabla_X Z, Y\rangle = 0$, that is, $\langle\nabla_X Z, Y\rangle = 0$. On the other hand, we have $\langle\nabla_X Z, Y\rangle = -\langle Z, \nabla_X Y\rangle = -\langle h_i(X, Y), Z\rangle$. Hence we have $\langle h_i(X, Y), Z\rangle = 0$. From the arbitrariness of $Z \in E_j$ and $j(\neq i)$, we obtain $h_i(X, Y) = 0$. Thus E_i is totally geodesic in M. Denote by h the second fundamental form of M and by $\widetilde{h}_i$ the second fundamental form of leaves of E_i in V. Let $X, Y \in (E_i)_x$ and $v \in T^{\perp}_x M$. From $h_i = 0$ and $A_v X = \mathrm{Re}(\lambda_i(v))X + \mathrm{Im}(\lambda_i(v))JX = \langle v_i, v\rangle - \langle\widetilde{J}v_i, v\rangle JX$ (by Lemma 5),

$$\begin{aligned}\langle\widetilde{h}_i(X, Y), v\rangle &= \langle h(X, Y), v\rangle = \langle A_v X, Y\rangle\\ &= \langle\langle v_i, v\rangle X - \langle\widetilde{J}v_i, v\rangle JX,\ Y\rangle = \langle\langle X, Y\rangle v_i - \langle JX, Y\rangle\widetilde{J}v_i,\ v\rangle\,.\end{aligned}$$

From the arbitrariness of v, we have $\widetilde{h}_i(X, Y) = \langle X, Y\rangle v_i - \langle JX, Y\rangle\widetilde{J}v_i$. This means that leaves of E_i are a totally anti-Kaehlerian umbilical submanifold with the mean curvature vector v_i in $(V, \langle\ ,\ \rangle, \widetilde{J})$. Further, it is easy to show that v_i is parallel with respect to the normal connection of each leaf of E_i in $(V, \langle\ ,\ \rangle, \widetilde{J})$. Hence it follows from Theorem 6 and (8.1) that the leaves of E_i are open potions of complex spheres of radius $\sqrt{\frac{\langle v_i, v_i\rangle - \sqrt{-1}\langle\widetilde{J}v_i, v_i\rangle}{\langle v_i, v_i\rangle^2 + \langle v_i, \widetilde{J}v_i\rangle^2}} = \frac{\sqrt{\lambda_i(v_i)}}{|\lambda_i(v_i)|}$ (when $\lambda_i \neq 0$) or complex affine subspaces (when $\lambda_i = 0$). q.e.d.

9. Proof of Theorem 3

In this section, we prove Theorem 3. For its purpose, we prepare some propositions. First we prove the following fact for isometric actions on a semi-simple Lie group G equipped with a bi-invariant pseudo-Riemannian metric $\langle\ ,\ \rangle$.

PROPOSITION 6. *Let H be a closed subgroup of $G\times G$, which is an isometric action on G by the adjoint action. Also, let ν be the cohomogeneity of the H-action. If the normal space $T_e^{\perp}(H\cdot e)$ of the orbit $H\cdot e$ at e contains a ν-dimensional non-degenerate abelian subspace $\mathfrak{T}$, then principal orbits through $\Sigma := \exp_G(\mathfrak{T})$ of the H-action are complex equifocal, where $\exp_G$ is the exponential map of G.*

PROOF. By imitating the proof of Lemma 2.2 of [18], we can show that all H-orbits through Σ meet Σ orthogonally. Now we shall show that the principal orbit $H\cdot g_1$ through $g_1\in\Sigma$ is complex equifocal. Since the dimension of $\mathfrak{T}$ is equal to the cohomogeneity of the H-action, we have $T_{g_1}^{\perp}(H\cdot g_1)=T_{g_1}\Sigma$. Hence, since $\mathfrak{T}$ is non-degenerate and abelian, the orbit $H\cdot g_1$ is a pseudo-Riemannian submanifold and it has abelian normal bundle. Also, since $H\cdot g_1$ is a principal orbit, there exists a normal frame field $(v_1,\cdots,v_\nu)$ of $H\cdot g_1$ consisting of H-equivariant normal vector fields. Let U be an open neighborhood of $H\cdot g_1$ consisting of principal orbits of the H-action and $\psi : U\to U/H$ be the natural submersion. It is clear that there exist the pseudo-Riemannian metric on U/H such that ψ is a pseudo-Riemannian submersion, where we use the non-degeneracy of $\mathfrak{T}$. Since the fibres of the pseudo-Riemannian submersion ψ are orthogonal to Σ, we see that the horizontal distribution of ψ is integrable. Hence H-equivariant normal vector fields v_i $(i=1,\cdots,\nu)$ of $H\cdot g_1$ are parallel with respect to the normal connection of $H\cdot g_1$. Also, since v_i is H-equivariant, we see that the complex focal radii along $(v_i)_x$ is independent of the choice of $x\in H\cdot g_1$. These facts imply that $H\cdot g_1$ is complex equifocal. q.e.d.

Next we prove the following fact for isometric actions on a symmetric space G/K of non-compact type.

PROPOSITION 7. *Let H be a closed subgroup of G and ν be the cohomogeneity of the H-action. If the normal space $T_{eK}^{\perp}(H(eK))$ of the orbit $H(eK)$ at eK contains a ν-dimensional abelian subspace $\mathfrak{T}$, then principal orbits through $\Sigma := \exp(\mathfrak{T})$ of the H-action are complex equifocal, where $\exp$ is the exponential map of G/K at eK.*

PROOF. Let $\widetilde{\mathfrak{T}}$ be the horizontal lift of $\mathfrak{T}$ to e and set $\widetilde{\Sigma} := \exp_G\widetilde{\mathfrak{T}}$. The group $H\times K$ acts on G by the adjoint action. Since $\pi\circ(h,k)=h\circ\pi$ $((h,k)\in H\times K)$, the $H\times K$-orbits are the inverse images of the H-orbits by π. Hence the cohomogeneity of the $H\times K$-action is equal to ν and $\widetilde{\mathfrak{T}}\subset T_e^{\perp}((H\times K)\cdot e)$ holds. Therefore, it follows that all $H\times K$-orbits through $\widetilde{\Sigma}$ meet it orthogonally. This implies that all H-orbits through $\pi(\widetilde{\Sigma})$ meet $\pi(\widetilde{\Sigma})$ orthogonally. Since $\pi\circ\exp_G|_{\mathfrak{p}}=\exp$, we have $\pi(\widetilde{\Sigma})=\Sigma$. Hence, by imitating the proof of Proposition 6, we can show that the principal orbits through Σ are complex equifocal. q.e.d.

Next we prove the following fact for isometric actions on the pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$, where $\mathfrak{g}$ is the Lie algebra of a semi-simple Lie group G (see [25] about the definition of the pseudo-Hilbert space $H^0([0,1],\mathfrak{g})$).

PROPOSITION 8. *Let H be a closed subgroup of $G\times G$ and ν be the cohomogeneity of the H-action. Also, set $P(G,H):=\{g\in H^1([0,1],G)\,|\,(g(0),g(1))\in H\}$, which acts on $H^0([0,1],\mathfrak{g})$ by the gauge action. If the normal space $T^{\perp}_{\hat{0}}(P(G,H)*\hat{0})$ of the orbit $P(G,H)*\hat{0}$ at $\hat{0}$ contains a ν-dimensional non-degenerate subspace $\mathfrak{T}$ such that $\phi_*\mathfrak{T}$ is abelian, then principal orbits through $\mathfrak{T}$ (which is regarded as the subspace of $H^0([0,1],\mathfrak{g})$) of the $P(G,H)$-action are complex isoparametric, where ϕ is the parallel transport map for G (see* [25] *about this definition).*

PROOF. Let $\Sigma:=\exp_G\phi_*\mathfrak{T}$. Since $\phi\circ(g*\cdot)=(g(0),g(1))\circ\phi$ (see §4 of [25]), the $P(G,H)$-orbits are the inverse images of the H-orbits by ϕ. Hence the cohomogeneity of the $P(G,H)$-action is equal to ν, $\phi_*\mathfrak{T}\subset T^{\perp}_e(H\cdot e)$ holds and the $P(G,H)$-orbits are Fredholm. Therefore, since $\phi_*\mathfrak{T}$ is abelian, all H-orbits through Σ meet it orthogonally (see the first half of the proof of Proposition 6). It is shown that there exists the horizontal lift Σ^L of Σ through $\hat{0}$ (see the proof of (i)⇔(ii) in Theorem 4). Clearly we have $\Sigma^L=\mathfrak{T}$. This fact implies that $P(G,H)$-orbits through $\mathfrak{T}$ meet it orthogonally. Hence, by imitating the second half of the proof of Proposition 6, we can show that principal orbits through $\mathfrak{T}$ have a global parallel normal frame field consisting of $P(G,H)$-equivariant normal fields. Therefore, we see that principal orbits through $\mathfrak{T}$ are complex isoparametric. q.e.d.

Next we prove the following fact for isometric actions on the complexification $G^{\mathbf{c}}$ of a semi-simple Lie group G.

PROPOSITION 9. *Let H be a closed subgroup of $G^{\mathbf{c}}\times G^{\mathbf{c}}$ and ν be the cohomogeneity of the H-action. If the normal space $T^{\perp}_e(H\cdot e)$ of the orbit $H\cdot e$ at e contains a ν-dimensional J-invariant abelian subspace $\mathfrak{T}$, then principal orbits through $\Sigma:=\exp_{G^{\mathbf{c}}}(\mathfrak{T})$ of the H-action are anti-Kaehlerian equifocal, where J is the complex structure of $G^{\mathbf{c}}$ and $\exp_{G^{\mathbf{c}}}$ is the exponential map of $G^{\mathbf{c}}$.*

PROOF. By imitating the proof of Lemma 2.2 of [18], we can show that all H-orbits through Σ meet it orthogonally. Take the principal orbit $H\cdot g_1$ of the H-action through $g_1\in\Sigma$. Since the dimension of $\mathfrak{T}$ is equal to the cohomogeneity of the H-action, we have $T^{\perp}_{g_1}(H\cdot g_1)=T_{g_1}\Sigma$. Hence since $\mathfrak{T}$ is J-invariant and abelian, it is shown that the orbit $H\cdot g_1$ is an anti-Kaehlerian submanifold with abelian normal bundle. By imitating the proof of Proposition 6, we can show the existence of a global parallel normal frame field of $H\cdot g_1$ consisting of H-equivariant normal vector fields. This implies that $H\cdot g_1$ is anti-Kaehlerian equifocal. q.e.d.

By imitating the proof of Proposition 7, we obtain the following fact for isometric actions on the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$ associated with a symmetric space G/K of non-compact type.

PROPOSITION 10. *Let H be a closed subgroup of $G^{\mathbf{c}}$ and ν be the cohomogeneity of the H-action. If the normal space $T^{\perp}_{eK}(H(eK^{\mathbf{c}}))$ of the orbit $H(eK^{\mathbf{c}})$ at $eK^{\mathbf{c}}$ contains a ν-dimensional J-invariant abelian subspace $\mathfrak{T}$, then principal orbits through $\Sigma := \exp(\mathfrak{T})$ of the H-action are anti-Kaehlerian equifocal, where J is the complex structure of $G^{\mathbf{c}}/K^{\mathbf{c}}$ and* exp *is the exponential map of $G^{\mathbf{c}}/K^{\mathbf{c}}$ at $eK^{\mathbf{c}}$.*

By imitating the proof of Proposition 8, we obtain the following fact for isometric actions on the anti-Kaehlerian space $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$, where $\mathfrak{g}^{\mathbf{c}}$ implies the Lie algebra of the complexification $G^{\mathbf{c}}$ of a semi-simple Lie group G.

PROPOSITION 11. *Let H be a closed subgroup of $G^{\mathbf{c}}\times G^{\mathbf{c}}$ and ν be the cohomogeneity of the H-action. Also, set $P(G^{\mathbf{c}},H) := \{g \in H^1([0,1],G^{\mathbf{c}}) \mid (g(0),g(1)) \in H\}$, which acts on $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ by the gauge action. If the normal space $T^{\perp}_{\hat{0}}(P(G^{\mathbf{c}},H)*\hat{0})$ of the orbit $P(G^{\mathbf{c}},H)*\hat{0}$ at $\hat{0}$ contains a ν-dimensional J-invariant subspace $\mathfrak{T}$ such that $\phi^{\mathbf{c}}_*\mathfrak{T}$ is abelian, then principal orbits through $\mathfrak{T}$ (which is regarded as the subspace of $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$) are anti-Kaehlerian isoparametric, where J is the complex structure of $H^0([0,1],\mathfrak{g}^{\mathbf{c}})$ and $\phi^{\mathbf{c}}$ is the parallel transport map for $G^{\mathbf{c}}$.*

Now we prove Theorem 3 in terms of these propositions.

PROOF OF THEOREM 3. In the case where the H-action is of cohomogeneity one, the statements (i) $\sim$ (vi) are directly shown by imitating the proof of Proposition 6. So we shall consider the case where H is the group of all fixed points of an involution $\sigma(\neq \mathrm{id}_G)$ of G. Let $\rho : G \to G$ be the Cartan involution associated with G/K. Also, let $\mathfrak{f}$ (resp. $\mathfrak{g}_H$) be the Lie algebra of K (resp. H) and $\mathfrak{p}$ (resp. $\mathfrak{p}_H$) be the eigenspace of ρ_{*e} (resp. σ_{*e}) for the eigenvalue -1. Note that $\mathfrak{p}$ (resp. $\mathfrak{p}_H$) is the orthogonal complement of $\mathfrak{f}$ (resp. $\mathfrak{g}_H$). It is clear that the isotropy group $(H\times K)_e$ of $H\times K$ at e is equal to the diagonal group $\triangle(H\cap K)$. Also, we have $T_e((H\times K)\cdot e) = \{X - Y \mid X \in \mathfrak{g}_H,\ Y \in \mathfrak{f}\} = \mathfrak{g}_H + \mathfrak{f}$, we have $T^{\perp}_e((H\times K)\cdot e) = \mathfrak{p}_H \cap \mathfrak{p}$. Let L be the group of all fixed points of $\rho\circ\sigma$ and $\mathfrak{l}$ be the Lie algebra of L. It is easy to show that $\rho = \sigma$ on L and that $H\cap K$ is the group of all fixed points of $\rho|_L(=\sigma|_L)$. The eigenspace decomposition of $\rho_{*e}|_{\mathfrak{l}}$ is given by $\mathfrak{l} = (\mathfrak{g}_H\cap\mathfrak{f}) + (\mathfrak{p}_H\cap\mathfrak{p})$. It is clear that this decomposition is a Cartan decomposition, that is, $L/H\cap K$ is a symmetric space of non-compact type. From these facts, we can show that the slice representation of $(H\times K)_e(=\triangle(H\cap K))$ on $T^{\perp}_e((H\times K)\cdot e)(=\mathfrak{p}_H\cap\mathfrak{p})$ is equivalent to the linear isotropy representation of the symmetric space $L/H\cap K$. This fact deduces that the section $\mathfrak{T}$ of the slice representation is an abelian subspace of $\mathfrak{g}$ and $\langle\ ,\ \rangle|_{\mathfrak{T}\times\mathfrak{T}}$ is positive definite. On the other hand, the cohomogeneity of the slice representation coincides with that of the $H\times K$-action. That is, the dimension of $\mathfrak{T}$ is equal to the cohomogeneity of the $H\times K$-action. Therefore, it follows from Propositions 6 $\sim$ 11 that principal orbits of the H-action through

$(\pi \circ \exp_G)(\mathfrak{T})$ and those of the $H \times K$-action through $\exp_G(\mathfrak{T})$ are complex equifocal, those of the $P(G, H \times K)$-action through $\mathfrak{T}^L_{\hat{0}}$ are complex isoparametric, those of $H^{\mathbf{c}}$-action through $(\pi^{\mathbf{c}} \circ \exp_{G^{\mathbf{c}}})(\mathfrak{T}^{\mathbf{c}})$ and those of the $H^{\mathbf{c}} \times K^{\mathbf{c}}$-action through $\exp_{G^{\mathbf{c}}}(\mathfrak{T}^{\mathbf{c}})$ are anti-Kaehlerian equifocal and that those of the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action through $(\mathfrak{T}^{\mathbf{c}})^L_{\hat{0}}$ are anti-Kaehlerian isoparametric. By the same argument for $g^{-1}Hg\,(g \in G)$ instead of H, we can show that other principal orbits of those actions also are complex equifocal, complex isoparametric, anti-Kaehlerian equifocal or anti-Kaehlerian isoparametric, where we note that $g^{-1}Hg$ is the group of all fixed points of the involution $\mathrm{Ad}(g^{-1}) \circ \sigma \circ \mathrm{Ad}(g)$. q.e.d.

Now we classify homogeneous complex equifocal submanifolds in the m-dimensional hyperbolic space $SO^0(m,1)/SO(m)$. For its purpose, we recall the classification of cohomogeneity one actions on $SO^0(m,1)/SO(m)$.

THEOREM 7 ([6]). *A cohomogeneity one action on $SO^0(m,1)/SO(m)$ is orbit equivalent to one of the $SO^0(m-k,1) \times SO(k)$-action $(1 \le k \le m)$ or the N-action, where N is the nilpotent part in the Iwasawa decomposition of $SO^0(m,1)$.*

From this classification, we can obtain the following fact.

THEOREM 8. *A homogeneous complex equifocal submanifold in $SO^0(m,1)/SO(m)$ is congruent to one of the following* (I)–(IV)*:*

(I) *A complete totally umbilical hypersphere* (= *a geodesic sphere*).

(II) *A tube over a complete totally geodesic submanifold of codimension bigger than one.*

(III) *A complete totally umbilical hyperbolic space of codimension one*

(IV) *A horosphere.*

PROOF. Since $SO^0(m,1)/SO(m)$ is of rank one, complex equifocal submanifolds in the space are hypersurfaces. Hence homogeneous ones are principal orbits of cohomogeneity one actions on the space. Therefore, we obtain the statement from Theorem 7. q.e.d.

REMARK 5. (i) The hypersurfaces (I), (II), (III) and (IV) are a principal orbit of the actions $SO(m)$, $SO^0(m-k,1) \times SO(k)\ (2 \le k \le m-1)$, $SO^0(m-1,1)$ and N, respectively.

(ii) The anti-Kaehlerian isoparametric hypersurface in $H^0([0,1], \mathfrak{so}(m,1)^{\mathbf{c}})$ arising from the hypersurfaces (I), (II) and (III) are proper anti-Kaehlerain isoparametric but so is not the hypersurface arising from the hypersurface (IV). Here we note that a horosphere has imaginary focal points on the ideal boundary of $SO^0(m,1)/SO(m)$.

(iii) The hypersurfaces (I) and (II) have the only real focal radius and the hypersurfaces (III) and (IV) have no real focal radius.

For cohomogeneity one actions on irreducible symmetric spaces of non-compact type, the following facts have recently been shown by J. Berndt and H. Tamaru.

THEOREM 9 ([4]). *Let G/K be an irreducible symmetric space of non-compact type and $G = KAN$ be the Iwasawa decomposition of G. A cohomogeneity one action on G/K having no singular orbit is orbit equivalent to one of the following actions*:

(I) *The S_l-action, where l is a linear line of $\mathfrak{h}$ ($\mathfrak{h}$: the Lie algebra of A) and S_l is a connected Lie subgroup of AN associated with the orthogonal complement $(\mathfrak{h}+\mathfrak{n}) \ominus l$ of l in $\mathfrak{h}+\mathfrak{n}$ ($\mathfrak{n}$: the Lie algebra of N).*

(II) *The S_ξ-action, where ξ is an element of the root space $\mathfrak{g}_\alpha := \{X \in \mathfrak{g} \mid \mathrm{ad}(H)(X) = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$ for a simple root α and S_ξ is a connected Lie subgroup of AN associated with the Lie algebra $\mathfrak{h} + (\mathfrak{n} \ominus \mathbf{R}\xi)$.*

THEOREM 10 ([5]). *Let F be a totally geodesic singular orbit of a cohomogeneity one action on an irreducible symmetric space G/K of non-compact type. If* $\dim F \geq 2$, *then F is one of totally geodesic submanifolds in Theorems* 3.3 *and* 4.2 *of* [5].

From these classifications, we can obtain the following fact.

THEOREM 11. *Let M be a homogeneous hypersurface* (*hence a complex equifocal one*) *in an irreducible symmetric space of non-compact type.*

(i) *If M has no real focal radius, then M is congruent to an orbit of the above S_l-action or S_ξ-action.*

(ii) *If M has a real focal radius and the focal submanifold is totally geodesic, then M is congruent to a tube over one of totally geodesic submanifolds in Theorems* 3.3 *and* 4.2 *of* [5] *or a tube over geodesics.*

REMARK 6. (i) Orbits of the S_l-action are curvature adapted but so are not those of the S_ξ-action. Also, the anti-Kaehlerian isoparametric hypersurfaces arising from orbits of both the S_l-action and the S_ξ-action are not proper anti-Kaehlerian isoparametric.

(ii) Tubes over totally geodesic submanifolds in Theorems 3.3 and 4.2 of [5] are curvature adapted and the anti-Kaehlerian isoparametric hypersurfaces arising from the tubes are proper anti-Kaehlerian isoparametric.

(iii) Homogeneous complex equifocal hypersurface in a irreducible symmetric space of non-compact type has at most one real focal radius (by Proposition 1 of [3]). Also, if the hypersurface has a real focal radius with multiplicity bigger than $\left[\frac{n}{2}\right]$, then the focal submanifold is automatically totally geodesic (by Proposition 2 of [3]).

10. Complex and anti-Kaehlerian hyperpolar actions

In this section, we introduce the new notions of complex hyperpolar actions on a symmetric space G/K of non-compact type, a connected semi-simple Lie group G equipped with a bi-invariant pseudo-Riemannian metric and a pseudo-Hilbert space V. It is shown that the principal orbits of those actions become complex equifocal or complex isoparametric submanifolds. Also, we introduce the new notions of anti-Kaehlerian hyperpolar actions on an anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$, the complexification $G^{\mathbf{c}}$ of G and an infinite dimensional anti-Kaehlerian space $V^{\mathbf{c}}$. It is shown that the principal orbits of those actions become anti-Kaehlerian equifocal or anti-Kaehlerian isoparametric submanifolds. A complex hyperpolar action is interpreted as an action having common focal points of the complexifications of its orbits as its poles and imaginary poles. Also, an anti-Kaehlerian hyperpolar action is interpreted as an action having common focal points of its orbits as its poles. We investigate the relations among those actions. First we define the notion of a complex hyperpolar action on the symmetric space G/K.

DEFINITION 1. Let H be a closed subgroup of G. If there exists an embedded (i.e., properly and injectively immersed) submanifold Σ of G/K which meets all H-orbits orthogonally, then we call the H-action the *complex polar action* and call Σ its *section*, where we note that Σ is automatically totally geodesic. Further, if the section Σ is flat (with respect to the induced metric), then we call the action a *complex hyperpolar action* and Σ its *flat section*.

Here we explain the situation of the poles and the imaginary poles of complex hyperpolar actions on the hyperbolic space $SO^0(m,1)/SO(m)$.

EXAMPLE 4. Since $SO^0(m,1)/SO(m)$ is of rank one, complex hyperpolar actions on the space are automatically cohomogeneity one actions. According to Theorem 7, a cohomogeneity one action on the space is one of the $SO^0(m-k,1)\times SO(k)$-action $(1\leq k\leq m)$ or the N-action, where N is as in Theorem 7. It is clear that these actions admit a flat section, that is, they are complex hyperpolar. For example, the orbits, the poles and the imaginary poles of the $SO(m)$-action are as in Figure 5, those of the $SO^0(m-1,1)$-action are as in Figure 6 and those of the N-action are as in Figure 7.

Now we define a complex hyperpolar action on a simply connected semi-simple Lie group G equipped with a bi-invariant pseudo-Riemannian metric.

DEFINITION 2. Let H be a closed subgroup of $G\times G$, which acts on G by the adjoint action. We call the action of H a *complex polar action on* G if the following conditions (i) and (ii) hold:

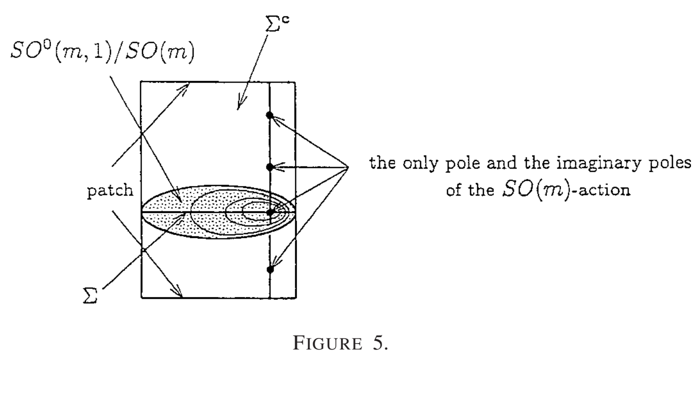

FIGURE 5.

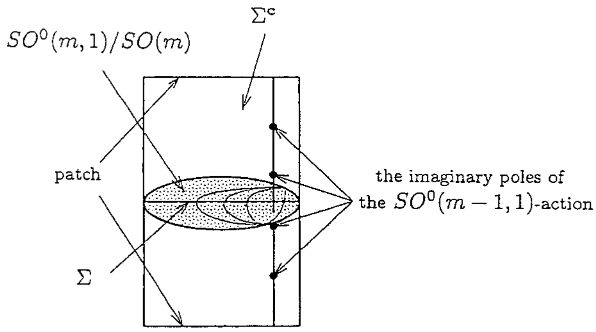

FIGURE 6.

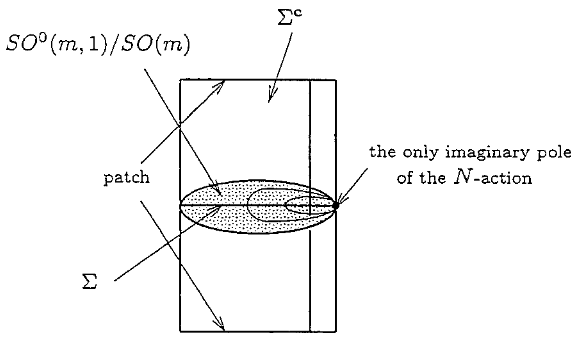

FIGURE 7.

(i) each H-orbit is a pseudo-Riemannian submanifold in G,

(ii) there exists an embedded submanifold Σ of G which meets all H-orbits orthogonally.

Also, we call Σ its *section*, where we note that Σ is automatically totally geodesic. Further, if the section Σ is flat, then we call the action a *complex hyperpolar action* and Σ its *flat section*.

Next we define the notion of a complex hyperpolar action on a pseudo-Hilbert space V.

DEFINITION 3. Let H be a Hilbert Lie group consisting of isometries of V. We call the H-action a *complex hyperpolar* (or simply *complex polar*) *action on* V if the following conditions (i)–(iii) hold:

(i) the H-action is Fredholm, that is, each orbit map $h(\in H) \to hu(\in V)$ is a Fredholm map for each $u \in V$,

(ii) each H-orbit is a pseudo-Riemannian submanifold,

(iii) there exists a subspace Σ of V which meets all H-orbits orthogonally.

Also, we call Σ its *flat section*.

REMARK 7. Each orbit of an isometric action H on V satisfying the above conditions (i) and (ii) becomes a Fredholm pseudo-Riemannian submanifold.

For these complex hyperpolar actions, we have the following fact.

THEOREM 12. (i) *Let G/K be a symmetric space of non-compact type. Principal orbits of a complex hyperpolar action on G/K are complex equifocal.*

(ii) *Let V be a pseudo-Hilbert space. Principal orbits of a complex hyperpolar action on V are complex isoparametric.*

PROOF. The statement (i) is shown by using the argument in the second half of the proof of Proposition 6. Assume that the H-action on V is complex hyperpolar. Let M be a principal orbit of the H-action. Since H is a Fredholm action, the normal exponential map of M is a Fredholm map, that is, M is a Fredholm submanifold. Further, by imitating the argument in the second half of the proof of Proposition 6, it is shown that M is complex isoparametric. Thus the statement (ii) is also shown. q.e.d.

For complex hyperpolar actions on a symmetric space G/K of non-compact type and the connected semi-simple Lie group G, we can show the following fact by imitating the proof of Proposition 2.11 in [18], where we give G a bi-invariant pseudo-Riemannian metric inducing the Riemannian metric of G/K.

THEOREM 13. *Let H be a closed subgroup of G. Then the H-action on G/K is complex hyperpolar if and only if so is the $H \times K$-action on G.*

PROOF. Let π be the natural projection of G onto G/K. Since each $H \times K$-orbit is the inverse image of an H-orbit by π, each $H \times K$-orbit is a pseudo-Riemannian submanifold. Assume that the $H \times K$-action is complex hyperpolar. Let Σ be a flat section of the $H \times K$-action. Then it is shown that $\pi(\Sigma)$ meets all orbits of the H-action orthogonally and that it is embedded. Denote by D_V (resp. D_H) the vertical (resp. horizontal) distribution of the pseudo-Riemannian submersion π. Define $B \in \Gamma(D_H^* \otimes D_H^* \otimes D_V)$ by $B(X, Y) := (\nabla^*_X Y)^V$ for $X, Y \in D_H$, where ∇^* is the Levi-Civita connection of G and $(\cdot)^V$ is the vertical component of $\cdot$. For an arbitrary tangent 2-plane σ of $\pi(\Sigma)$, we have

$$\bar{K}(\sigma) = K(\sigma^L) + 3\langle e_1, e_1\rangle\langle e_2, e_2\rangle\langle B(e_1, e_2), B(e_1, e_2)\rangle \tag{10.1}$$

(see [32,33]), where σ^L is the horizontal lift of σ to Σ, $\bar{K}(\sigma)$ (resp. $K(\sigma^L)$) is the sectional curvature of σ (resp. σ^L) and $\{e_1, e_2\}$ is an orthonormal base of σ^L. Since $B(e_1, e_2) = 0$ by the existence of Σ and $K(\sigma^L) = 0$ by the flatness of Σ, we have $\bar{K}(\sigma) = 0$. After all $\pi(\Sigma)$ is a flat section of the H-action, that is, the H-action is complex hyperpolar. Conversely assume that the H-action is complex hyperpolar. Let $\bar{\Sigma}$ be a flat section of the H-action. For an arbitrary tangent 2-plane σ of $\bar{\Sigma}$, we have (10.1). From the flatness of $\bar{\Sigma}$, we have $\bar{K}(\sigma) = 0$. Since G is of non-positive curvature, we have $K(\sigma^L) \leq 0$. Also, since $\langle\ ,\ \rangle|_{D_H \times D_H}$ (resp. $\langle\ ,\ \rangle|_{D_V \times D_V}$) is positive (resp. negative) definite, we have $\langle e_1, e_1\rangle\langle e_2, e_2\rangle\langle B(e_1, e_2), B(e_1, e_2)\rangle \leq 0$. It follows from these relations that $K(\sigma^L) = 0$

and $B(e_1, e_2) = 0$. These facts imply that the horizontal lift $\bar{\Sigma}^L$ of $\bar{\Sigma}$ exists and $\bar{\Sigma}^L$ is flat. Further, it is shown that $\bar{\Sigma}^L$ meets all $H \times K$-orbits orthogonally and that it is embedded. After all $\bar{\Sigma}^L$ is a flat section of the $H \times K$-action, that is, the $H \times K$-action is complex hyperpolar. q.e.d.

Now we prove the equivalenceness of (i) and (ii) in Theorem 4.

PROOF OF (i) $\Leftrightarrow$ (ii) IN THEOREM 4. Let $\phi : H^0([0,1], \mathfrak{g}) \to G$ be the parallel transport map and $\pi : G \to G/K$ be the natural projection. Take $g \in P(G, H \times K)$. Then we have $\phi \circ (g * \cdot) = (g(0), g(1)) \circ \phi$. Hence $P(G, H \times K)$-orbits are the inverse images of $H \times K$-orbits by ϕ. This fact implies that each $H \times K$-orbit is a pseudo-Riemannian submanifold if and only if so is each $P(G, H \times K)$-orbit. Since $P(G, H \times K)$ is closed and of finite codimension in $H^1([0,1], G)$, the $P(G, H \times K)$-action is Fredholm. First we show (ii) $\Rightarrow$ (i). Assume that the $P(G, H \times K)$-action is complex hyperpolar. Let $\widetilde{\Sigma}$ be a flat section of the $P(G, H \times K)$-action. Then it is clear that $\phi(\widetilde{\Sigma})$ meets all orbits of the $H \times K$-action orthogonally. It follows from this fact that $\phi(\widetilde{\Sigma})$ is totally geodesic. Denote by $\widetilde{D}_V$ (resp. $\widetilde{D}_H$) the vertical (resp. horizontal) distribution of the pseudo-Riemannian submersion ϕ. Define $\widetilde{B} \in \Gamma(\widetilde{D}_H^* \otimes \widetilde{D}_H^* \otimes \widetilde{D}_V)$ by $\widetilde{B}(X, Y) := (\widetilde{\nabla}_X Y)^V$ for $X, Y \in \widetilde{D}_H$, where $\widetilde{\nabla}$ is the Levi-Civita connection of $H^0([0,1], \mathfrak{g})$ and $(\cdot)^V$ is the vertical component of $\cdot$. For an arbitrary non-degenerate 2-plane σ of $\phi(\widetilde{\Sigma})$, we have

$$K(\sigma) = \widetilde{K}(\sigma^L) + 3\langle e_1, e_1\rangle\langle e_2, e_2\rangle\langle \widetilde{B}(e_1, e_2), \widetilde{B}(e_1, e_2)\rangle,$$

where σ^L is the horizontal lift of σ to $\widetilde{\Sigma}$, $K(\sigma)$ (resp. $\widetilde{K}(\sigma^L)$) is the sectional curvature of σ (resp. σ^L) and $\{e_1, e_2\}$ is an orthonormal base of σ^L. From the existence of $\widetilde{\Sigma}$, we have $\widetilde{B}(e_1, e_2) = 0$. Also, from the flatness of $\widetilde{\Sigma}$, we have $\widetilde{K}(\sigma^L) = 0$. Hence we have $K(\sigma) = 0$. Thus $\phi(\widetilde{\Sigma})$ is flat. Hence, by imitating the proof of Theorem 13, we can show that $\pi(\phi(\widetilde{\Sigma}))$ is flat section of the H-action. That is, the H-action is complex hyperpolar. Next we show (i) $\Rightarrow$ (ii). Assume that the H-action is complex hyperpolar. According to Theorem 13, the $H \times K$-action is also complex hyperpolar. Let Σ be a flat section of the $H \times K$-action. Take an arbitrary non-degenerate 2-plane σ of Σ and an orthonormal base $\{e_1, e_2\}$ of the horizontal lift σ^L of σ. Let u be the base point of σ^L. According to the fact (v) stated in §2, we can express $u = g * \hat{0}$ for some $g \in P(G, G \times e)$. By using Lemma 7, we have

$$\begin{aligned}(g * \cdot)_*^{-1}\widetilde{B}(e_1, e_2) &= \widetilde{B}((g * \cdot)_*^{-1}e_1, (g * \cdot)_*^{-1}e_2) \\ &= (-t[\phi_*(g * \cdot)_*^{-1}e_1,\ \phi_*(g * \cdot)_*^{-1}e_2])^V \\ &= (-t[g(0)_*^{-1}\phi_* e_1,\ g(0)_*^{-1}\phi_* e_2])^V .\end{aligned}$$

On the other hand, since $\phi_* e_i$ $(i = 1, 2)$ are normal to the $H \times K$-orbit through $\phi(u)$ and the orbit has abelian normal bundle, we have $[g(0)_*^{-1}\phi_* e_1,\ g(0)_*^{-1}\phi_* e_2] = 0$. Therefore, we have $\widetilde{B}(e_1, e_2) = 0$. Hence the existence of the horizonal lift Σ^L of Σ is assured. Further, since Σ meets all $H \times K$-orbits orthogonally and each $P(G, H \times K)$-orbit is the inverse image of a $H \times K$-orbit by ϕ, Σ^L meets all $P(G, H \times K)$-orbits orthogonally. This fact implies that Σ^L is a complete totally geodesic submanifold (i.e., a subspace) in $H^0([0, 1], \mathfrak{g})$. Thus Σ^L is a flat section of the $P(G, H \times K)$-action, that is, the $P(G, H \times K)$-action is complex hyperpolar. q.e.d.

Now we define a notion of an anti-Kaehlerian hyperpolar action on the anti-Kaehlerian symmetric space $G^{\mathbf{c}}/K^{\mathbf{c}}$ associated with a symmetric space G/K of non-compact type.

DEFINITION 4. Let H be a closed subgroup of $G^{\mathbf{c}}$. We call the action of H on $G^{\mathbf{c}}/K^{\mathbf{c}}$ an *anti-Kaehlerian polar action* on $G^{\mathbf{c}}/K^{\mathbf{c}}$ if the following conditions (i) and (ii) hold:

(i) each H-orbit is an anti-Kaehlerian submanifold in $G^{\mathbf{c}}/K^{\mathbf{c}}$,

(ii) there exists an embedded submanifold Σ of $G^{\mathbf{c}}/K^{\mathbf{c}}$ which meets all H-orbits orthogonally.

Also, we call Σ its *section*, which is automatically totally geodesic. Further, if the section Σ is flat, then we call the action an *anti-Kaehlerian hyperpolar action* and Σ its *flat section*.

The complexificaton $G^{\mathbf{c}}$ of a connected semi-simple Lie group G equipped with a bi-invariant pseudo-Riemannian metric is regarded as the anti-Kaehlerian symmetric space associated with $G = G \times G/\triangle(G)$, where $\triangle$ is the diagonal map. Hence an anti-Kaehlerian hyperpolar action on $G^{\mathbf{c}}$ is defined as above. Next we define a notion of an anti-Kaehlerian hyperpolar action on an infinite dimensional anti-Kaehlerian space $(V, \widetilde{J})$.

DEFINITION 5. Let H be a Hilbert Lie group consisting of isometries of $(V, \widetilde{J})$ preserving $\widetilde{J}$. We call the H-action an *anti-Kaehlerian hyperpolar* (or simply *anti-Kaehlerian polar*) *action on* $(V, \widetilde{J})$ if the following conditions (i)–(iii) hold:

(i) the H-action is Fredholm,

(ii) each H-orbit is an anti-Kaehlerian submanifold in V,

(iii) there exists a subspace Σ of V which meets all H-orbits orthogonally.

Also, we call Σ its *flat section*.

For these anti-Kaehlerian hyperpolar actions, we have the following fact.

THEOREM 14. (i) *Let $G^{\mathbf{c}}/K^{\mathbf{c}}$ be the anti-Kaehlerian symmetric space associated with a symmetric space G/K of non-compact type. Principal orbits of an anti-Kaehlerian hyperpolar action on $G^{\mathbf{c}}/K^{\mathbf{c}}$ are anti-Kaehlerian equifocal.*

(ii) *Let V be an infinite dimensional anti-Kaehlerian space. Principal orbits of an anti-Kaehlerian hyperpolar action on V are anti-Kaehlerian isoparametric.*

PROOF. These facts are proved by imitating the proof of Theorem 12. q.e.d.

Next we prove the equivalenceness of (ii) and (iii) in Theorem 4.

PROOF OF (ii) $\Leftrightarrow$ (iii) IN THEOREM 4. It is clear that the $P(G, H \times K)$-action and the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action are Fredholm, each $P(G, H \times K)$-orbit is a pseudo-Riemannian submanifold and each $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-orbit is an anti-Kaehlerian submanifold. Assume that the $P(G, H \times K)$-action is complex hyperpolar. Let Σ be a flat section of the $P(G, H \times K)$-action. Note that Σ is a finite dimensional subspace of V. It is clear that the complexification $\Sigma^{\mathbf{c}}$ of the subspace Σ is a flat section of the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action. That is, $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action is anti-Kaehlerian hyperpolar. Conversely assume that the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action is anti-Kaehlerian hyperpolar. Let $\widetilde{\Sigma}$ be a flat section through $u_0 \in V\ (\subset V^{\mathbf{c}})$ of the $P(G^{\mathbf{c}}, H^{\mathbf{c}} \times K^{\mathbf{c}})$-action. Note that $\widetilde{\Sigma}$ is a finite dimensional anti-Kaehlerian subspace of $V^{\mathbf{c}}$. Let $\widetilde{\Sigma}' := \widetilde{\Sigma} \cap V$. It is clear that $\widetilde{\Sigma}'$ is a flat section of the $P(G, H \times K)$-action. That is, the $P(G, H \times K)$-action is complex hyperpolar. q.e.d.

At the end of this section, we propose the following problem.

PROBLEM 2. (i) Can any homogeneous complex equifocal submanifold in a symmetric space G/K of non-compact type be caught as a principal orbit of a complex hyperpolar action?

(ii) Can any homogeneous complex isoparametric submanifold in $H^0([0, 1], \mathfrak{g})$ which is the sum of fibres of $\pi \circ \phi$ be caught as a principal orbit of a complex hyperpolar action of $P(G, \cdot \times K)$-type?

(iii) Can any homogeneous anti-Kaehlerian isoparametric submanifold in $H^0([0,1], \mathfrak{g}^{\mathbf{c}})$ which is the sum of fibres of $\pi^{\mathbf{c}} \circ \phi^{\mathbf{c}}$ be caught as a principal orbit of an anti-Kaehlerian hyperpolar action of $P(G^{\mathbf{c}}, \cdot \times K^{\mathbf{c}})$-type?

11. Isoparametric submanifolds in the sense of Heintze-Liu-Olmos

In this section, we first investigate the equivalence of the complex equifocality and the isoparametricness with flat section in the sense of [16] of a submanifold in a symmetric space of non-compact type. Heintze-Liu-Olmos [16] defined the notion of an isoparametric submanifold in an arbitrary (finite dimensional) Riemannian manifold as a submanifold M satisfying the following conditions:

(i) M has flat normal bundle and locally parallel submanifolds of M have constant mean curvature in radial directions,

(ii) M has a section at each point, where a section means a totally geodesic submanifold meeting M orthogonally whose dimension is equal to the codimension of M.

In the above definition, Heintze-Liu-Olmos assumed only the (not necessarily globally) flatness of the normal bundle of M. However, we may assume that M has globally flat normal bundle by letting the universal covering of M be M newly if necessary, where we note that the universal covering is immersed by the composition of the universal covering map and the original immersion of M. In particular, if the sections are flat with respect to the induced metric, then M is called an *isoparametric submanifold with flat section*. According to Theorem 6.5 of [16], we see that a submanifold in a symmetric space G/K of compact type is an isoparametric submanifold with flat section in [16]-sense if and only if it is equifocal. So the following question is naturally proposed.

QUESTION. Is a submanifold in a symmetric space of non-compact type an isoparametric submanifold with flat section if and only if it is complex equifocal?

For this question, we can answer partially as follows.

THEOREM 15. *Let M be a submanifold in a symmetric space G/K of non-compact type. Then the following statements* (i) *and* (ii) *hold*:

(i) *If M is an isoparametric submanifold with flat section, then it is complex equifocal.*

(ii) *If M is complex equifocal and curvature adapted, then it is an isoparametric submanifold with flat section.*

(iii) *If M is complex equifocal submanifold of codimension $r(:=\mathrm{rank} G/K)$ and if there exists a parallel normal frame field $(v_1,\cdots,v_r)$ of M such that, for each $x(=gK)\in M$, $((g_*^{-1}v_{1x})_\sharp,\cdots,(g_*^{-1}v_{rx})_\sharp)$ is a simple root system for a maximal abelian subspace $g_*^{-1}T_x^\perp M$ of $\mathfrak{p}=T_{eK}G/K$, then it is an isoparametric submanifold with flat section, where $(g_*^{-1}v_{ix})_\sharp(\cdot)=\langle g_*^{-1}v_{ix},\cdot\rangle$ ($\langle\ ,\ \rangle$: the inner product of $g_*^{-1}T_x^\perp M$).*

PROOF. We first note that M has abelian normal bundle if and only if it has flat sections. Let v be a parallel unit normal vector field on M. Let $\eta_{sv}:=\exp^\perp\circ sv\,(: M\to G/K)$ and $M_{sv}:=\eta_{sv}(M)$, where s is sufficiently close to zero. Define a function F_{sv} on M by $\eta_{sv}^*\omega_{sv}=F_{sv}\omega$, where ω (resp. ω_{sv}) is the volume element of M (resp. M_{sv}). Set $\hat F_{v_x}(s):=F_{sv}(x)$ ($x\in M$). It is shown that $\hat F_{v_x}$ ($x\in M$) have the holomorphic extensions. Denote by $\hat F^h_{v_x}$ (: $\mathbf{C}\to\mathbf{C}$) its holomorphic extension. According to Corollary 2.6 of [16], M is isoparametric if and only if the projection from M to any (sufficiently close) parallel submanifold along the sections is volume preserving up to a constant factor. That is, M is isoparametric if and only if $\hat F^h_{v_{x_1}}=\hat F^h_{v_{x_2}}$ holds for every parallel normal vector field v of M and every $x_1,x_2\in M$. On the other hand, the complex focal radii along the geodesic γ_{v_x} are catched as zero points of $\hat F^h_{v_x}$. Hence we see that M is complex equifocal if and only if $(\hat F^h_{v_{x_1}})^{-1}(0)=(\hat F^h_{v_{x_2}})^{-1}(0)$ holds for every parallel normal vector field v of M and every $x_1,x_2\in M$. From these facts, the statement (i) is shown. We shall show the statements (ii) and (iii). Take a continuous orthonormal tangent frame field $(e_1,\cdots,e_n)$ of M defined on a connected open set U such that $R(e_i,v)v=-\beta_i^2e_i$ ($i=1,\cdots,n$), where β_i ($i=1,\cdots,n$) are continuous functions on U. Let $A_ve_i=\sum_{j=1}^n a_{ij}e_j$ ($i=1,\cdots,n$), where a_{ij} ($i,j=1,\cdots,n$) are continuous functions on U. The Jacobi field $J_{i,x}$ along γ_{v_x} ($x\in U$) with $J_{i,x}(0)=e_{ix}$ and $J'_{i,x}(0)=-A_{v_x}e_{ix}$ is described as

$$J_{i,x}(s)=\sum_{j=1}^n(\cosh(s\beta_i(x))\delta_{ij}-\frac{a_{ij}(x)\sinh(s\beta_j(x))}{\beta_j(x)})P_{\gamma_{v_x}|_{[0,s]}}e_{jx}\,,$$

where $\frac{\sinh(s\beta_j(x))}{\beta_j(x)}$ implies s when $\beta_j(x)=0$. From this description, we have

$$\hat{F}_{v_x}(s)=\det\left(\cosh(s\beta_i(x))\delta_{ij}-\frac{a_{ij}(x)\sinh(s\beta_j(x))}{\beta_j(x)}\right),$$

where $\left(\cosh(s\beta_i(x))\delta_{ij}-\frac{a_{ij}(x)\sinh(s\beta_j(x))}{\beta_j(x)}\right)$ is the matrix of (n,n)-type whose (i,j)-component is $\cosh(s\beta_i(x))\delta_{ij}-\frac{a_{ij}(x)\sinh(s\beta_j(x))}{\beta_j(x)}$. Hence we have

$$\hat{F}^h_{v_x}(z)=\det\left(\cos(\sqrt{-1}z\beta_i(x))\delta_{ij}-\frac{a_{ij}(x)\sin(\sqrt{-1}z\beta_j(x))}{\sqrt{-1}\beta_j(x)}\right). \tag{11.1}$$

Assume that M is complex equifocal and curvature adapted. Then since $R(\cdot,v)v$ and A_v are commutative, we may assume that $a_{ij}=0$ $(i\neq j)$. For simplicity, set $\lambda_i:=a_{ii}$ $(i=1,\cdots,n)$. From (11.1), the function $\hat{F}^h_{v_x}$ $(x\in U)$ is described as

$$\hat{F}^h_{v_x}(z)=\prod_{i=1}^n\left(\cos(\sqrt{-1}z\beta_i(x))-\frac{\lambda_i(x)\sin(\sqrt{-1}z\beta_i(x))}{\sqrt{-1}\beta_i(x)}\right). \tag{11.2}$$

Hence we have

$$(\hat{F}^h_{v_x})^{-1}(0)=\bigcup_{i=1}^n\left\{z\mid\cos(\sqrt{-1}z\beta_i(x))=\frac{\lambda_i(x)\sin(\sqrt{-1}z\beta_i(x))}{\sqrt{-1}\beta_i(x)}\right\}. \tag{11.3}$$

Take arbitrary two points x_1 and x_2 of U. Since M is complex equifocal, we have $(\hat{F}^h_{v_{x_1}})^{-1}(0)=(\hat{F}^h_{v_{x_2}})^{-1}(0)$. These facts together with (11.2) and (11.3) deduces $\hat{F}^h_{v_{x_1}}=\hat{F}^h_{v_{x_2}}$. From the arbitrarinesses of x_1, x_2 and U, we see that $\hat{F}^h_{v_x}$ is independent of the choice of $x\in M$. Thus M is an isoparametric submanifold with flat section. Next we assume that M is complex equifocal, $\operatorname{codim}M=\operatorname{rank}G/K$ and it admits a parallel normal frame field as in the statement (iii). Then we see that each β_i is constant over U because $\{\beta_i(x)\mid i=1,\cdots,n\}=\{\alpha(g_*^{-1}v_x)\mid\alpha\in\triangle_+\}$ for each $x(=gK)\in U$, where $\triangle_+$ is a positive root system for the maximal abelian subspace $g_*^{-1}T^\perp_xM$. Let W be the set of all parallel unit normal vector field v such that $\beta_1:\cdots:\beta_n$ is an integer ratio and that $\beta_i\neq 0$ $(i=1,\cdots,n)$. Also, let $W_x:=\{v_x\mid v\in W\}$ $(x\in M)$. It is easy to show that W_x is dense in the unit sphere of $T^\perp_xM$. Assume that $v\in W$. Since $\beta_1:\cdots:\beta_n$ is an integer ratio, they are expressed as $\beta_i=m_ib$ $(i=1,\cdots,n)$ in terms of some real constant b and integers $m_1,\cdots,m_n$ which are mutually primal. Hence the function $\hat{F}^h_{v_x}$ $(x\in U)$ is described as $\hat{F}^h_{v_x}(z)=e^{-bz\sum_{i=1}^n|m_i|}G_x(e^{2bz})$ in terms of some polynomial G_x of degree $|m_1|+\cdots+|m_n|$. Take arbitrary two points x_1 and x_2 of U. Let $c:[0,1]\to U$ be a continuous curve with $c(0)=x_1$ and $c(1)=x_2$. Since M is complex equifocal, $(\hat{F}^h_{v_{c(t)}})^{-1}(0)$ is independent of the choice of $t\in[0,1]$. Hence so is also $G^{-1}_{c(t)}(0)$. This implies that $G_{x_2}=aG_{x_1}$ (a : a non-zero complex constant). Hence we have $\hat{F}^h_{x_2} = a\hat{F}^h_{x_1}$. Further, since $\hat{F}^h_{v_{x_1}}(0) = \hat{F}^h_{v_{x_2}}(0) = 1$, we have $\hat{F}^h_{v_{x_1}} = \hat{F}^h_{v_{x_2}}$. From the arbitrarinesses of x_1, x_2 and U, we see that $\hat{F}^h_{v_x}$ is independent of the choice of $x \in M$. Further, since W_x is dense in the unit sphere of $T^{\perp}_x M$, we see that $\hat{F}^h_{v'_x}$ is independent of the choice of $x \in M$ for each parallel unit normal vector field v' with $v' \notin W$. Thus M is an isoparametric submanifold with flat section. q.e.d.

REMARK 8. Complex equifocal hypersurfaces in rank one symmetric spaces of non-compact type admit a parallel normal frame field as in the statement (iii). Also, if the H-action (H : the group of all fixed points of an involution of G) on a symmetric space G/K of non-compact type is of cohomogeneity rankG/K, then principal orbits of the action admit a parallel normal frame field as in the statement (iii) (see the proof of Theorem 3). Similarly, principal orbits of complex hyperpolar actions of cohomogeneity rankG/K on G/K also admit such a parallel normal frame field.

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