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Stable Rank of C^*-Algebras of Continuous Fields

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Abstract. The C^*-algebras of continuous fields are enlarged and embedded into the associated direct products, and their stable rank and connected stable rank are estimated in terms of their base spaces and fibers. Using these estimates, we compute these ranks of C^*-algebras of continuous fields of elementary C^*-algebras, and those of group C^*-algebras of the discrete Heisenberg groups.

0. Introduction

The theory for the C^*-algebras of continuous fields over locally compact Hausdorff spaces were studied and developed by Fell [F], Dixmier [Dx], [TT], [Le1, 2], [APT], etc. in an early stage. In a recent stage, there are some remarkable researches about this topic by [AP], [Rf3], [Wl], [LP1], etc. The C^*-algebras of continuous fields (or the section algebras of C^*-bundles in the sense of [FD]) provide many kinds of examples in the theory of C^*-algebras, for example, such as the group C^*-algebras of the (generalized) discrete Heisenberg groups. On the other hand, the stable rank and connected stable rank of C^*-algebras were introduced by Rieffel [Rf1] for study of the non-stable K-theory, and the stable rank is regarded as a noncommutative counterpart to the covering dimension for spaces. So far some experts have tried to compute these ranks for some concrete examples (cf. References). In particular, these ranks for some group C^*-algebras were computed by [Sh], [ST] and [Sd1, 2] in the case of some connected Lie groups including important examples, and by [DHR] and [DH] in the case of some important discrete groups including the free groups.

Our first motivation is to determine the stable rank of the C^*-algebras of the discrete Heisenberg groups, which has been unsettled so far. Since the algebras can be regarded as the C^*-algebras of continuous fields (or sections), it is natural to seek the stable rank formula for these C^*-algebras in general, but there has been no such formula till now. Under this motivation, in this paper we first give an approach to enlarge and embed the C^*-algebras of continuous fields over locally compact Hausdorff spaces into more tractable subalgebras of direct products associated with their base spaces and fibers (Lemma 1.1). Using this lemma

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we give the stable rank and connected stable rank estimates for the C^*-algebras of continuous fields in terms of their base spaces and fibers (Theorem 1.3), which is the main result in this paper. We apply these estimates to some cases in the following sections. In Section 2, as an easy consequence we consider the ranks of C^*-algebras of continuous fields of elementary C^*-algebras. In Section 3, as a highly non-trivial consequence we compute the ranks of group C^*-algebras of the (generalized) discrete Heisenberg groups in terms of groups. Moreover, our method for the main result would be useful for computing the stable ranks of C^*-algebras in similar or other situations, in particular, certain twisted group C^*-algebras (cf. [LP1, 2], [PR]).

Notation and facts. Let $\mathfrak{A}$ be a C^*-algebra. We denote by $\mathrm{sr}(\mathfrak{A})$ and $\mathrm{csr}(\mathfrak{A})$ the stable rank and connected stable rank of $\mathfrak{A}$ respectively ([Rf1]). By definition, $\mathrm{sr}(\mathfrak{A}), \mathrm{csr}(\mathfrak{A}) \in \{1, 2, \cdots, \infty\}$, and $\mathrm{sr}(\mathfrak{A}) \leq n$ if and only if $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, and $\mathrm{csr}(\mathfrak{A}) \leq n$ if and only if $L_m(\mathfrak{A})$ is connected for any $m \geq n$, where $L_n(\mathfrak{A}) = \{(a_j) \in \mathfrak{A}^n \mid \sum_{j=1}^n a_j^* a_j \in \mathfrak{A}^{-1}\}$. If $\mathfrak{A}$ is nonunital, we define its ranks by those of its unitization $\mathfrak{A}^+$. If $\mathfrak{A}$ is unital, we let $\mathfrak{A}^+ = \mathfrak{A}$. We give some formulas of the ranks used later as follows.

(F1): $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ for any C^*-algebra $\mathfrak{A}$ by [Rf1, Corollary 4.10]. For an exact sequence of C^*-algebras: $0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0$, we have that

$$\text{(F2)}: \quad \begin{cases} \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{sr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{I}) \vee \mathrm{sr}(\mathfrak{A}/\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \\ \mathrm{csr}(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{I}) \vee \mathrm{csr}(\mathfrak{A}/\mathfrak{I}), \end{cases}$$

where $\vee$ means the maximum ([Rf1, Theorem 4.3, 4.4 and 4.11], [Sh, Theorem 3.9]). We denote by $C_0(X)$ the C^*-algebra of all continuous functions vanishing at infinity on a locally compact Hausdorff space X. If X is compact, we set $C_0(X) = C(X)$. By [Rf1, Proposition 1.7], [Ns1],

$$\text{(F3)}: \quad \mathrm{sr}(C(X)) = [\dim X/2] + 1 \equiv \dim_{\mathbf{C}} X, \quad \mathrm{csr}(C(X)) \leq [(\dim X + 1)/2] + 1,$$

where $\dim X$ is the covering dimension of X, and $[x]$ means the maximal integer $\leq x$. We note that $C_0(X)^+ \cong C(X^+)$ where X^+ means the one-point compactification of X. For the $n \times n$ matrix algebra $M_n(\mathfrak{A})$ over a C^*-algebra $\mathfrak{A}$, by [Rf1, Theorem 6.1], [Rf2],

$$\text{(F4)}: \quad \mathrm{sr}(M_n(\mathfrak{A})) = \{(\mathrm{sr}(\mathfrak{A}) - 1)/n\} + 1, \quad \mathrm{csr}(M_n(\mathfrak{A})) \leq \{(\mathrm{csr}(\mathfrak{A}) - 1)/n\} + 1,$$

where $\{x\}$ means the least integer $\geq x$. Let $\mathbf{K}$ be the C^*-algebra of all compact operators on a countably infinite dimensional Hilbert space. Then

$$\text{(F5)}: \quad \mathrm{sr}(\mathfrak{A} \otimes \mathbf{K}) = \mathrm{sr}(\mathfrak{A}) \wedge 2, \quad \mathrm{csr}(\mathfrak{A} \otimes \mathbf{K}) \leq \mathrm{csr}(\mathfrak{A}) \wedge 2,$$

where $\wedge$ means the minimum. See [Rf1, Theorem 3.6 and 6.4] and ([Sh, Theorem 3.10], [Ns1]) respectively.

1. C^*-algebras of continuous fields

Let X be a locally compact Hausdorff space and $\{\mathfrak{A}_t\}_{t\in X}$ a family of C^*-algebras $\mathfrak{A}_t$ indexed by $t \in X$. Then we denote by $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ the C^*-algebra of a continuous field on X, that is, the C^*-algebra of all continuous operator fields on X with respect to a family $\mathfrak{F}$ of continuous operator fields f vanishing at infinity on X such that

$$\begin{cases} f : t \mapsto f(t) \in \mathfrak{A}_t\,, \quad \{f(t) \mid f \in \mathfrak{F}\} \text{ is dense in } \mathfrak{A}_t\,, \\ X \ni t \mapsto \|f(t)\| \text{ is continuous}\,, \end{cases}$$

and $\mathfrak{F}$ (or the C^*-algebra) is closed under the local convergence at all $t \in X$, pointwise algebraic operations and involution (cf. [F], [Dx, Chapter 10], [Wl]), and note that the C^*-algebra associated to a maximal full algebra of operator fields is the section algebra of a C^*-bundle (cf. [FD]). If X is compact, we let $\Gamma(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F}) = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$. If $\mathfrak{A}_t = \mathfrak{A}$ for all $t \in X$, we put $\Gamma_0(X, \mathfrak{A}, \mathfrak{F}) = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$. We omit $\mathfrak{F}$ in some cases in what follows.

Recall that the direct product $\prod_{t\in X} \mathfrak{A}_t$ of $\{\mathfrak{A}_t\}_{t\in X}$ is the C^*-algebra of all elements $a = (a_t)_{t\in X}$ with the norm $\|a\| = \sup_{t\in X} \|a_t\|$ finite. For a C^*-algebra $\mathfrak{A}$, we denote by $C_0(X, \mathfrak{A})$ the C^*-algebra of all $\mathfrak{A}$-valued continuous functions on X vanishing at infinity. It is well known that $C_0(X, \mathfrak{A})$ is isomorphic to the C^*-tensor product $C_0(X) \otimes \mathfrak{A}$ (cf. [Mp, Theorem 6.4.17]).

First of all, we give the following key definition in this paper:

DEFINITION. Let X be a locally compact Hausdorff space and $\{\mathfrak{A}_t\}_{t\in X}$ a family of C^*-algebras $\mathfrak{A}_t$. For any $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$, we define a C^*-subalgebra $\mathfrak{F}'$ of $\prod_{t\in X} C_0(X, \mathfrak{A}_t)$ to be $\mathfrak{F}' = \Gamma_0(X, \{C_0(X, \mathfrak{A}_t)\}_{t\in X})$ such that the restriction of any $f = \{f(t, \cdot)\}_{t\in X} \in \mathfrak{F}'$ with $f(t, \cdot) = f(t) \in C_0(X, \mathfrak{A}_t)$ to the diagonal $(t, t) \in X \times X$ is in $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$, in particular, $t \mapsto f(t, t) \in \mathfrak{A}_t$ for every $t \in X$. Moreover, we assume that every element of $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ can be such a restriction from an element of $\mathfrak{F}'$ (However, it would be enough to assume continuity along the diagonal of the direct product).

Then it is clear that

LEMMA 1.1. *Let X be a locally compact Hausdorff space and $\{\mathfrak{A}_t\}_{t\in X}$ a family of C^*-algebras $\mathfrak{A}_t$. Then the algebra $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ is a quotient of the C^*-subalgebra $\mathfrak{F}'$ of $\prod_{t\in X} C_0(X, \mathfrak{A}_t)$.*

PROOF. Define the quotient map q from $\mathfrak{F}'$ to $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ by

$$q(f)(x) = q_x(\{f(t, \cdot)\}_{t\in X})(x) = f(x, x)\,, \quad q_x : \textstyle\prod_{t\in X} C_0(X, \mathfrak{A}_t) \to C_0(X, \mathfrak{A}_x)\,. \quad \square$$

On the other hand, we need the following lemma for the proof of Theorem 1.3:

LEMMA 1.2. *Let $\mathcal{A}$ be a C^*-algebra. Define the map Φ from $\mathcal{A}^n$ to the positive part $\mathcal{A}_+$ of $\mathcal{A}$ by $(a_j) \mapsto \sum_{j=1}^n a_j^* a_j$. Then Φ is continuous, and the quotient topology induced by Φ on $\mathcal{A}_+$ is stronger than the relative topology of $\mathcal{A}_+$ in $\mathcal{A}$.*

PROOF. It is clear that Φ is continuous. Also, the map Φ induces the quotient topology in $\mathcal{A}_+$. Then Φ is open with respect to this topology. Moreover, it is easy to show that the quotient topology is stronger than the relative topology of $\mathcal{A}_+$ in $\mathcal{A}$, which is proved by using a usual argument for inclusions of open neighborhoods in the theory of topological spaces. □

Recall that a topological space is paracompact if for any open covering of the space, there exists a locally finite refinement of the covering (cf. [Pd, 1.7.10]). See also [En, Chapter 5] in which paracompactness of spaces implies Hausdorffness. By technical reasons, a space will be assumed to be paracompact in what follows. Using the lemmas above, we have the following main result, which is used frequently later.

THEOREM 1.3. *Let X be a locally compact, paracompact Hausdorff space and $\{\mathfrak{A}_t\}_{t\in X}$ a family of C^*-algebras $\mathfrak{A}_t$. Then we have*

$$\begin{cases} \mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \le \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t)), \\ \mathrm{csr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \le \sup_{t\in X}(\mathrm{csr}(C_0(X, \mathfrak{A}_t)) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t))). \end{cases}$$

PROOF. By Lemma 1.1 and (F2), we have $\mathrm{sr}(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \le \mathrm{sr}(\mathfrak{F}')$. We note that $\prod_{t\in X} C_0(X, \mathfrak{A}_t)$ is a closed ideal of $\prod_{t\in X}(C_0(X, \mathfrak{A}_t)^+)$. We define $\mathfrak{B}$ to be the C^*-subalgebra of all elements $\{(f(t,\cdot), \lambda_t)\}_{t\in X}$ of $\prod_{t\in X}(C_0(X, \mathfrak{A}_t)^+)$ such that $f \in \mathfrak{F}'$, $\lambda_t \in \mathbf{C}$ as in the above definition. By (F2), we have that $\mathrm{sr}(\mathfrak{F}') \le \mathrm{sr}(\mathfrak{B})$. Moreover, when X is noncompact we may replace $\mathfrak{B}$ with $\mathfrak{B}' = \Gamma(X^+, \{C_0(X, \mathfrak{A}_t)^+\}_{t\in X} \cup \{\mathbf{C}\}) \supset \mathfrak{B}$ for X^+ the one-point compactification of X since $\mathfrak{B}$ is a closed ideal of $\mathfrak{B}'$.

Now suppose that $M = \sup_{t\in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t)^+) < \infty$. Then for any $\varepsilon > 0$ and any $(b_j)_{j=1}^M \in \mathfrak{B}^M$ with $b_j = (b_j(t))_{t\in X}$ and $b_j(t) \in C_0(X, \mathfrak{A}_t)^+$, we can find $c_j(t) \in C_0(X, \mathfrak{A}_t)^+$ for all $t \in X$ such that $\|c_j(t) - b_j(t)\| < \varepsilon_j(t) < \varepsilon$, and $e(t) \equiv \sum_{j=1}^M c_j(t)^* c_j(t)$ is invertible in $C_0(X, \mathfrak{A}_t)^+$. For a large constant $L > 0$, we may assume that $e(t) \ge \varepsilon/L > 0$ if necessary, by taking $\varepsilon_j(t)$ small enough, and replacing $\varepsilon_j(t)$ with $\varepsilon_j(t)' < \varepsilon$, and taking a suitable perturbation of $c_j(t)$, when $e(t) \ge \delta(t) > 0$ and $\delta(t) < \varepsilon/L$ for some $t \in X$.

In fact, for a unital C^*-algebra $\mathcal{A}$, by using Lemma 1.2 we have a continuous map Φ from $L_n(\mathcal{A})$ to the positive part $\mathcal{A}_+$ of $\mathcal{A}$ by $(a_j) \mapsto \sum_{j=1}^n a_j^* a_j$. We let $\mathcal{S} = \{b \in \mathcal{A}_+ \mid \|\sum_{j=1}^n a_j^* a_j - b\| < \eta$, and $b - (\sum_{j=1}^n a_j^* a_j + \eta' 1)$ is invertible$\}$ for some $\eta, \eta' > 0$. Then $\mathcal{S}$ is open in $\mathcal{A}_+$ since for $b' \in \mathcal{A}_+$ with $\|b - b'\|$ small, we can make the distance of their spectrums small. Taking η, η' suitably, we make the distance between $\sum_{j=1}^n a_j^* a_j$ and $\mathcal{S}$ small enough. Then we can find a small open neighborhood of (a_j) such that its image under Φ has the nonzero intersection with $\mathcal{S}$.

Moreover, we can assume that the function $t \mapsto c_j(t, t)$ belongs to $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ (or its unitization). Indeed, for given $t \in X$, there exists $\{c_j(t)\}_{j=1}^M$ satisfying the above required conditions. By definition of $\mathfrak{B}$, we can find $\{h_j\}_{j=1}^M \in \mathfrak{B}^M$ such that $\|h_j(t) - c_j(t)\|$ $(1 \le j \le M)$ are small enough so that $\sum_{j=1}^M h_j^*(t) h_j(t)$ is invertible, and $\|h_j(s) - b_j(s)\|$

$(1 \leq j \leq M)$ are small enough for s in an open neighborhood U_t of t (cf. [Dx, Lemma 10.1.11 and Proposition 10.2.2]). Note that if $\sum_{j=1}^{M} h_j^*(t)h_j(t)$ is invertible, then $\sum_{j=1}^{M} h_j^*(s)h_j(s)$ is also invertible in an open neighborhood V_t of t, which is deduced from a direct computation using continuity of the norm on fibers. Set $W_t = U_t \cap V_t$. Thus we continue this process inductively for the open covering $\{W_t\}$ of X for some t (or use compactness of X^+) and replace c_j with h_j. More precisely, since X is paracompact we may assume that the open covering $\{W_t\}$ is locally finite. Furthermore, let (p_t) be the partition of unity of continuous functions associated with $\{W_t\}$ such that $0 \leq p_t \leq 1$, $\sum_t p_t = 1$ on X (or X^+), and $p_t(s) = 0$ for $s \notin W_t$ (cf. [Pd, Proposition 1.7.12]). Then set $q_j^t = p_t h_j$ $(1 \leq j \leq M)$ for the elements h_j chosen above for some $t \in X$, where we further need to manipulate choosing h_j by using locally finiteness of X and openness of W_t since it is not sure whether the sum $\sum_{j=1}^{M}(\sum_t q_j^t(s))^*(\sum_t q_j^t(s))$ is invertible for s in finite intersections of $\{W_t\}$. However, if necessary, we may assume that $p_t = 1$ on a compact neighborhood K_t in W_t for those t such that K_t for t are disjoint (cf. [Pd, Proposition 1.7.5], and X is normal by the assumption on X [En, Theorem 5.1.5]), and enlarge K_t by adding inductively compact neighborhoods K_t' in W_t (or finite intersections of $\{W_t\}$ with W_t) such that $p_t = 1$ on $K_t \cup K_t'$, where in this process the relation $\sum_t p_t = 1$ may break, and the required elements are given by the limits of elements $\sum_t q_j^t$ associated with this adjustment (or by elements near the limits). Also, we may adopt another argument as follows. We take compact neighborhoods K_t in W_t chosen above. Since finite unions of $\{K_t\}$ are closed in X, there exist $\{l_j^p\}_{j=1}^{M} \in \mathfrak{B}^M$ for $p \geq 1$ such that $l_j^p = h_j$ on finite unions $\bigcup_{j=1}^{p} K_{t_j}$ of $\{K_t\}$ ([Dx, Proposition 10.1.12]). By repeating this process inductively and if necessary, by enlarging K_t as above, the required elements are given by the limits of $\{l_j^p\}_{p=1}^{\infty}$ $(1 \leq j \leq M)$ as $p \to \infty$ (or elements near the limits). Hence, $\mathrm{sr}(\mathfrak{B}) \leq \sup_{t \in X} \mathrm{sr}(C_0(X, \mathfrak{A}_t))$.

As for the connected stable rank estimate, we first suppose that X is compact and all $\{\mathfrak{A}_t\}_{t \in X}$ are unital. We let $\mathfrak{D} = \Gamma(X, \{\mathfrak{A}_t\}_{t \in X}, \mathfrak{F})$ and note that it is obtained from the elements $\{(f(t, \cdot)\}_{t \in X}$ of $\mathfrak{B}$ (defined as above) by evaluation to the diagonal components $\{f(t, t)\}_{t \in X}$. We now suppose that

$$N = \sup_{t \in X}(\mathrm{csr}(C(X, \mathfrak{A}_t)) \vee \mathrm{sr}(C(X, \mathfrak{A}_t))) < \infty .$$

By definition of the stable ranks (cf. [Rf2, Proposition 5.3]), any element $(c_j)_{j=1}^{N} \in \mathfrak{B}^N$ with $c_j = (c_j(t))_{t \in X}$ and $\sum_{j=1}^{N} c_j(t)^* c_j(t) = I_t$ in $C(X, \mathfrak{A}_t)$ can be mapped to $(I, 0', \cdots, 0')$ by an invertible matrix $(d_{ij})_{i,j=1}^{N}$ over $\mathfrak{B}$ with $d_{ij} = (d_{ij}(t))_{t \in X}$ and $(d_{ij}(t))_{i,j}^{N}$ invertible if necessary, perturbing the diagonal components $d_{ij}(t, t)$ of each d_{ij} as above, where $I = (I_t)_{t \in X}$, $0' = (0_t')_{t \in X}$, and I_t, $0_t'$ are the unit and the zero of $C(X, \mathfrak{A}_t)$ respectively. In fact, $(d_{ij})_{i,j=1}^{N}$ can be taken in $\mathfrak{B}$ as follows. For convenience, we may let $N = 2$. Then

$$\begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} I \\ 0' \end{pmatrix}$$

If we need to perturb d_{ij} by d'_{ij}, then we consider the following operation:

$$\begin{pmatrix} l & 0 \\ -(d'_{21}c_1 + d'_{22}c_2)l & 1 \end{pmatrix} \begin{pmatrix} d_{11} + d'_{11} & d_{12} + d'_{12} \\ d_{21} + d'_{21} & d_{22} + d'_{22} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} I \\ 0' \end{pmatrix}$$

where l is the inverse of $(d_{11} + d'_{11})c_1 + (d_{12} + d'_{12})c_2 \in \mathfrak{B}$ which is near I.

We obtain from the above argument that

$$\left(\sum_{j=1}^{N} d_{1j}(t)c_j(t)\right)_{t\in X} = (I_t)_{t\in X}\,, \quad \left(\sum_{j=1}^{N} d_{ij}(t)c_j(t)\right)_{t\in X} = (0'_t)_{t\in X}\,, \quad (2 \le i \le N)\,.$$

By taking evaluation,

$$\begin{cases} \left(\sum_{j=1}^{N} d_{1j}(t,t)c_j(t,t)\right)_{t\in X} = 1 = (1_t)_{t\in X}\,, \\ \left(\sum_{j=1}^{N} d_{ij}(t,t)c_j(t,t)\right)_{t\in X} = 0 = (0_t)_{t\in X}\,, \quad (2 \le i \le N) \end{cases}$$

where $1_t, 0_t$ are the unit and the zero of $\mathfrak{A}_t$ respectively. Hence $(\{c_j(t,t)\}_{t\in X})_{j=1}^N$ is mapped to $(1, 0, \cdots, 0)$ by an invertible matrix $(\{d_{ij}(t,t)\}_{t\in X})_{i,j=1}^N$ over $\mathfrak{D}$. And we may assume from the above argument for the stable rank estimate that any element $(d_j)_{j=1}^N$ of $\mathfrak{D}^N$ with $\sum_{j=1}^N d_j^* d_j$ invertible is extended to an element $(c_j)_{j=1}^N$ of $\mathfrak{B}^N$ with $\sum_{j=1}^N c_j(t)^* c_j(t)$ invertible. Therefore we obtain $\mathrm{csr}(\Gamma(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \le N$.

Next, we consider the cases where X is compact and $\mathfrak{A}_t$ for some $t \in X$ is nonunital, or X is noncompact. Then put $\mathcal{E} = \Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})^+$. Let $\mathfrak{D}$ be the C^*-subalgebra of $\prod_{t\in X} C_0(X, \mathfrak{A}_t)$ defined by replacing $C_0(X, \mathfrak{A}_t)^+$ with $C_0(X, \mathfrak{A}_t)$ in definition of $\mathfrak{B}$. Then $\mathcal{E}$ is obtained from $\mathfrak{D}^+$ by evaluation to the diagonal components. Thus $\mathcal{E}$ is a quotient of $\mathfrak{D}^+$ by Lemma 1.1. By [Eh, Theorem 1.1], we have

$$\mathrm{csr}(\mathcal{E}) \le \mathrm{csr}(\mathfrak{D}^+) \vee \mathrm{sr}(\mathfrak{D}^+)\,.$$

Note that $\mathfrak{D}$ is a closed ideal of $\mathfrak{B}$, and $\mathfrak{D}^+$ is identified with a C^*-subalgebra of $\mathfrak{B}$ by the inclusion: $(x, \lambda) \mapsto ((x(t), \lambda))_{t\in X}$ for $x = (x(t))_{t\in X} \in \mathfrak{D}$. Then any element $(x(t), \lambda_t)_{t\in X}$ of $\mathfrak{B}$ is deformed to an element of $\mathfrak{D}^+$ continuously with respect to λ_t. Hence we get $\mathrm{sr}(\mathfrak{D}^+) \le \mathrm{sr}(\mathfrak{B})$ and $\mathrm{csr}(\mathfrak{D}^+) = \mathrm{csr}(\mathfrak{B})$. Moreover, similarly as above we obtain

$$\mathrm{csr}(\mathfrak{B}) \le \sup_{t\in X}(\mathrm{csr}(C_0(X, \mathfrak{A}_t)^+) \vee \mathrm{sr}(C_0(X, \mathfrak{A}_t)^+)). \qquad \square$$

REMARK. The rank estimates of Theorem 1.3 tell us that the ranks of C^*-algebras of continuous fields are estimated in terms of their base spaces and fibers. The assumption on

X being paracompact can be replaced with either σ-compactness or second countability of X because if X is a locally compact, σ-compact Hausdorff space, it is paracompact and normal (cf. [Pd, Propositions 1.7.8 and 1.7.11]), and if X is a second countable, locally compact Hausdorff space, then it is metrizable (cf. [Pd, Corollary 1.7.9]). In fact, any locally compact, paracompact Hausdorff space is a disjoint union of σ-compact spaces ([Pd, 1.7.10]). Also, the rank estimates above would hold without the assumption on X being paracompact, but our inductive argument would be more complicated or collapse since X could have the dimension ∞. In the case $\dim X = \infty$, the right hand sides of the estimates can be infinite, but it is not always. For instance see (F5).

We now denote by $\mathrm{Prim}(\mathfrak{A})$ the space of all primitive ideals of a C^*-algebra $\mathfrak{A}$ with the hull-kernel topology. Then we have the following:

COROLLARY 1.4. *Let $\mathfrak{A}$ be a C^*-algebra. Suppose that there exists a continuous open map from* $\mathrm{Prim}(\mathfrak{A})$ *onto a locally compact*, *paracompact Hausdorff space X. Then*

$$\begin{cases} \mathrm{sr}(\mathfrak{A}) \leq \sup_{\mathfrak{I}\in\mathrm{Prim}(\mathfrak{A})} \mathrm{sr}(C_0(X, \mathfrak{A}/\mathfrak{I})), \\ \mathrm{csr}(\mathfrak{A}) \leq \sup_{\mathfrak{I}\in\mathrm{Prim}(\mathfrak{A})} (\mathrm{csr}(C_0(X, \mathfrak{A}/\mathfrak{I})) \vee \mathrm{sr}(C_0(X, \mathfrak{A}/\mathfrak{I}))). \end{cases}$$

PROOF. By [Le1, Theorem 4], we see that $\mathfrak{A}$ is regarded as the C^*-algebra of a continuous field on X with fibers $\mathfrak{A}/\mathfrak{I}$ for $\mathfrak{I} \in \mathrm{Prim}(\mathfrak{A})$. Then we use Theorem 1.3. □

For a locally compact Hausdorff space X, we now denote by $\Gamma^b(X, \{\mathfrak{B}_t\}_{t\in X}, \mathfrak{F})$ the C^*-algebra of a bounded continuous field on X with fibers $\{\mathfrak{B}_t\}_{t\in X}$, that is, the C^*-algebra of bounded continuous operator fields (with respect to) $\mathfrak{F}$. By following the similar procedure with Theorem 1.3, we get

THEOREM 1.5. *For $\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ the C^*-algebra of a bounded continuous field on a locally compact*, *paracompact Hausdorff space X*,

$$\begin{cases} \mathrm{sr}(\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \leq \sup_{t\in X} \mathrm{sr}(C^b(X) \otimes \mathfrak{A}_t)\,, \\ \mathrm{csr}(\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \leq \sup_{t\in X} (\mathrm{csr}(C^b(X) \otimes \mathfrak{A}_t) \vee \mathrm{sr}(C^b(X) \otimes \mathfrak{A}_t))\,, \end{cases}$$

where $C^b(X)$ is the C^*-algebra of all bounded continuous functions on X.

PROOF. We consider the quotient as in Lemma 1.1 as follows:

$$\prod_{t\in X}(C^b(X) \otimes \mathfrak{A}_t) \supset \mathfrak{F}' = \Gamma^b(X, \{C^b(X) \otimes \mathfrak{A}_t\}_{t\in X}) \to \Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F}) \to 0\,. \quad \square$$

REMARK. Note that it is not true in general that $C^b(X, \mathfrak{A}_t) = C^b(X) \otimes \mathfrak{A}_t$ even if $\mathfrak{A}_t$ is unital (and commutative) (cf. [APT, Theorem 3.8]). We have $C^b(X, \mathfrak{A}_t) \supset C^b(X) \otimes \mathfrak{A}_t$ in general. Also, we have $C^b(X) \cong C(\beta X)$ where βX means the Stone-Čech compactification (cf. [Ng, 6-2. Definition], [APT]).

REMARK. The algebras $\Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ as well as $\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$ are embeddable in $\prod_{t\in X}(C^b(X) \otimes \mathfrak{A}_t)$ by the identification: $f \leftrightarrow \{1 \otimes f(t)\}_{t\in X}$ for $f \in \Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$.

For a C^*-algebra $\mathfrak{A}$, we denote by $M(\mathfrak{A})$ its multiplier (cf. [Bl, Section 12], [Mp, 2.1]). We say that an operator field on a space is the unit (operator) field if it takes the unit in each fiber when all fibers are unital.

COROLLARY 1.6. *For $\Gamma_0(X, \{\mathfrak{A}_t\}, \mathfrak{F})$ with X a locally compact, paracompact Hausdorff space and the fibers $\{\mathfrak{A}_t\}_{t\in X}$ unital, we suppose that the unit field is continuous, that is, belongs to the C^*-algebra (in particular, $\mathfrak{F}$ contains the unit field). Then we have*

$$\begin{cases} \mathrm{sr}(M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F}))) \le \sup_{t\in X} \mathrm{sr}(C^b(X) \otimes \mathfrak{A}_t)\,, \\ \mathrm{csr}(M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F}))) \le \sup_{t\in X}(\mathrm{csr}(C^b(X) \otimes \mathfrak{A}_t) \vee \mathrm{sr}(C^b(X) \otimes \mathfrak{A}_t))\,. \end{cases}$$

PROOF. By assumption and [APT, Theorem 3.3 and a remark after Corollary 3.4], we obtain $M(\Gamma_0(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})) \cong \Gamma^b(X, \{\mathfrak{A}_t\}_{t\in X}, \mathfrak{F})$. □

REMARK. In the above isomorphism, if $\mathfrak{A}_t$ is nonunital, we must treat the strict topology on $M(\mathfrak{A}_t)$ (cf. [APT]). See [Rf1, Proposition 6.5], [Eh, Proposition 1.4].

2. Continuous fields of elementary C^*-algebras

We first define that an elementary C^*-algebra is a matrix algebra $M_n(\mathbf{C})$ over $\mathbf{C}$ $(n \ge 1)$ or $\mathbf{K}$ (cf. [Dx, 4.1.1]). For a C^*-algebra $\mathfrak{A}$, we denote by $\hat{\mathfrak{A}}$ its spectrum, that is, the space of all its irreducible representations up to unitary equivalence. For $\pi \in \hat{\mathfrak{A}}$, we denote by $\dim \pi$ the dimension of its representation space. We have that $\hat{\mathfrak{A}}$ is locally compact, but not necessarily a Hausdorff space (cf. [Dx, Chapter 3]). Recall that for n a natural number, an n-homogeneous C^*-algebra $\mathfrak{A}$ is a C^*-algebra with $\dim \pi = n$ for any $\pi \in \hat{\mathfrak{A}}$. Then $\mathfrak{A}$ has continuous trace, in particular, $\hat{\mathfrak{A}}$ is a Hausdorff space. Thus $\mathfrak{A}$ is isomorphic to $\Gamma_0(\hat{\mathfrak{A}}, \{M_n(\mathbf{C})\}_{t\in\hat{\mathfrak{A}}})$ with the local triviality (cf. [F], [TT], [Dx, Chapter 10]). Then

PROPOSITION 2.1. *Let $\mathfrak{A}$ be an n-homogeneous C^*-algebra. Then we have*

$$\begin{cases} \mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(C_0(\hat{\mathfrak{A}}) \otimes M_n(\mathbf{C})) = \{[\dim \hat{\mathfrak{A}}^+/2]/n\} + 1\,, \\ \mathrm{csr}(\mathfrak{A}) \le \mathrm{csr}(C_0(\hat{\mathfrak{A}}) \otimes M_n(\mathbf{C})) \le \{[(\dim \hat{\mathfrak{A}}^+ + 1)/2]/n\} + 1\,. \end{cases}$$

PROOF. By replacing $\mathrm{csr}(C_0(\hat{\mathfrak{A}}) \otimes M_n(\mathbf{C}))$ with $\mathrm{csr}(C_0(\hat{\mathfrak{A}}) \otimes M_n(\mathbf{C})) \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}) \otimes M_n(\mathbf{C}))$, for the first and second formulas we can use Theorem 1.3, (F3) and (F4) if $\hat{\mathfrak{A}}$ is paracompact. Moreover, for the both formulas, we can use the local triviality of $\mathfrak{A}$ inductively, and use (F3) and (F4). □

REMARK. The algebra $\mathfrak{A}$ is not always splitting into the tensor product $C_0(\hat{\mathfrak{A}})\otimes M_n(\mathbf{C})$ in general. When $\dim \hat{\mathfrak{A}}^+ = \infty$, the estimates hold trivially.

Next, we say that a C^*-algebra $\mathfrak{A}$ with its spectrum $\hat{\mathfrak{A}}$ a Hausdorff space is ∞-homogeneous if it is isomorphic to $\Gamma_0(\hat{\mathfrak{A}}, \{\mathbf{K}\}_{\pi\in\hat{\mathfrak{A}}})$ (cf. [Dx, Theorem 10.5.4]). Then

PROPOSITION 2.2. *Let $\mathfrak{A}$ be an ∞-homogeneous C^*-algebra. Then*

$$\mathrm{sr}(\mathfrak{A}) \le 2 \wedge \dim_{\mathbf{C}} \hat{\mathfrak{A}}^+ \le 2\,, \quad \mathrm{csr}(\mathfrak{A}) \le 2 \wedge ([(\dim \hat{\mathfrak{A}}^+ + 1)/2] + 1) \le 2\,.$$

PROOF. If $\hat{\mathfrak{A}}$ is paracompact, by using Theorem 1.3 and (F5) we have

$$\mathrm{sr}(\mathfrak{A}) \le \mathrm{sr}(C_0(\hat{\mathfrak{A}}) \otimes \mathbf{K}) = 2 \wedge \dim_{\mathbf{C}} \hat{\mathfrak{A}}^+\,,$$
$$\mathrm{csr}(\mathfrak{A}) \le \mathrm{csr}(C_0(\hat{\mathfrak{A}}) \otimes \mathbf{K}) \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}) \otimes \mathbf{K}) \le 2 \wedge \mathrm{csr}(C_0(\hat{\mathfrak{A}}))\,.$$

Even if $\hat{\mathfrak{A}}$ is not paracompact, note that $\Gamma_0(\hat{\mathfrak{A}}, \{\mathbf{K}\}_{\pi\in\hat{\mathfrak{A}}})$ has the local triviality when $\mathfrak{A}$ is of continuous trace ([Dx, Chapter 10]). Also, note that $\mathfrak{A}$ is liminal (or CCR), and that a C^*-algebra of type I has a composition series with its subquotients of continuous trace (cf. [Dx, Theorem 4.5.5]). Then use (F2) and (F5) inductively. □

REMARK. Note that $\mathrm{csr}(C_0(\mathbf{R}^n) \otimes \mathbf{K}) = 1$ if n even, and $= 2$ if n odd ([Sh, p. 386]). The above stable rank estimate was also obtained by [ST] using [Ns2, Lemma 2] inductively.

Recall that an n-subhomogeneous C^*-algebra $\mathfrak{A}$ is a C^*-algebra with $\dim \pi \le n$ for any $\pi \in \hat{\mathfrak{A}}$. Denote by $\hat{\mathfrak{A}}_k$ the subspace of all elements π of $\hat{\mathfrak{A}}$ with $\dim \pi = k$. Then

PROPOSITION 2.3. *Let $\mathfrak{A}$ be an n-subhomogeneous C^*-algebra. Then*

$$\begin{cases} \mathrm{sr}(\mathfrak{A}) \le (\{[\dim \hat{\mathfrak{A}}_n^+/2]/n\} + 1) \vee \max_{1\le k<n}(\{[(\dim \hat{\mathfrak{A}}_k^+ + 1)/2]/k\} + 1)\,, \\ \mathrm{csr}(\mathfrak{A}) \le \max_{1\le k\le n}(\{[(\dim \hat{\mathfrak{A}}_k^+ + 1)/2]/k\} + 1)\,. \end{cases}$$

PROOF. We may assume that the set of all dimensions of $\hat{\mathfrak{A}}$ is equal to a finite set $\{n_j\}_{j=1}^l$ with $n_j > n_{j+1}$ and $n_1 = n$. Then by [Dx, Proposition 3.6.3], $\mathfrak{A}$ has a finite composition series $\{\mathfrak{I}_j\}_{j=1}^l$ with $\mathfrak{I}_l = \mathfrak{A}$ such that $\mathfrak{I}_j/\mathfrak{I}_{j-1}$ is n_j-homogeneous. Applying (F2) and Proposition 2.1 to $\{\mathfrak{I}_j\}_{j=1}^l$ inductively, we obtain the formulas in the statement. □

REMARK. In general, subhomogeneous C^*-algebras are not of continuous trace. In particular, their spectrums are not Hausdorff spaces (cf. [Dx, Addenda 10.10.4]). As an example, we let $\mathfrak{A} = \Gamma([0, 1], \{\mathfrak{A}_t\}_{t\in[0,1]}, \mathfrak{F})$ with $\mathfrak{A}_0 = \mathbf{C}^n = \mathfrak{A}_1$ and $\mathfrak{A}_t = M_n(\mathbf{C})$ for $0 < t < 1$. Then using Theorem 1.3 we obtain $\mathrm{sr}(\mathfrak{A}) = \mathrm{csr}(\mathfrak{A}) = 1$.

For a C^*-algebra $\mathfrak{A}$, we respectively denote by $\hat{\mathfrak{A}}_f$ and $\hat{\mathfrak{A}}_\infty$ the subspaces of $\hat{\mathfrak{A}}$ of all elements π of $\hat{\mathfrak{A}}$ with $\dim \pi$ finite and infinite.

THEOREM 2.4. *Let $\mathfrak{A}$ be a C^*-algebra with $\hat{\mathfrak{A}}$ a paracompact Hausdorff space. Then*

$$\begin{cases} \mathrm{sr}(\mathfrak{A}) \le (2 \wedge \dim_{\mathbf{C}} \hat{\mathfrak{A}}^+) \vee \sup_{\pi \in \hat{\mathfrak{A}}_f} (\{[\dim \hat{\mathfrak{A}}^+/2]/\dim \pi\} + 1)\,, \\ \mathrm{csr}(\mathfrak{A}) \le (2 \wedge [\mathrm{csr}(C(\hat{\mathfrak{A}}^+)) \vee \mathrm{sr}(C(\hat{\mathfrak{A}}^+))]) \\ \qquad\qquad \vee \sup_{\pi \in \hat{\mathfrak{A}}_f} (\{[(\dim \hat{\mathfrak{A}}^+ + 1)/2]/\dim \pi\} + 1)\,. \end{cases}$$

Moreover, these estimates imply that

$$\mathrm{sr}(\mathfrak{A}) \le \dim_{\mathbf{C}} \hat{\mathfrak{A}}^+\,, \quad \mathrm{csr}(\mathfrak{A}) \le [(\dim \hat{\mathfrak{A}}^+ + 1)/2] + 1\,.$$

PROOF. Since $\hat{\mathfrak{A}}$ is a Hausdorff space, $\mathfrak{A}$ is liminal, that is, for any $\pi \in \hat{\mathfrak{A}}$, $\pi(\mathfrak{A})$ is isomorphic to $\mathbf{K}$ or $M_n(\mathbf{C})$ for $n \ge 1$. Then $\mathfrak{A}$ is isomorphic to $\Gamma_0(\hat{\mathfrak{A}}, \{\pi(\mathfrak{A})\}_{\pi \in \hat{\mathfrak{A}}})$ (cf. [Dx, Theorem 10.5.4]). Therefore, we obtain by Theorem 1.3 that

$$\begin{cases} \mathrm{sr}(\Gamma_0(\hat{\mathfrak{A}}, \{\pi(\mathfrak{A})\}_{\pi \in \hat{\mathfrak{A}}})) \le \sup_{\pi \in \hat{\mathfrak{A}}} \mathrm{sr}(C_0(\hat{\mathfrak{A}}, \pi(\mathfrak{A})))\,, \\ \mathrm{csr}(\Gamma_0(\hat{\mathfrak{A}}, \{\pi(\mathfrak{A})\}_{\pi \in \hat{\mathfrak{A}}})) \le \sup_{\pi \in \hat{\mathfrak{A}}} (\mathrm{csr}(C_0(\hat{\mathfrak{A}}, \pi(\mathfrak{A}))) \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}, \pi(\mathfrak{A}))))\,. \end{cases}$$

Then we apply Propositions 2.1 and 2.2 to imply the conclusion. In particular, by (F5)

$$\mathrm{csr}(C_0(\hat{\mathfrak{A}}, \mathbf{K})) \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}}, \mathbf{K})) \le 2 \wedge (\mathrm{csr}(C_0(\hat{\mathfrak{A}})) \vee \mathrm{sr}(C_0(\hat{\mathfrak{A}})))\,. \qquad \square$$

REMARK. The implied estimates are noncommutative versions of (F3) for commutative C^*-algebras, and they are the best possibles. The theorem above also suggests that the stable ranks of C^*-algebras (in more general) can be controlled by the dimension of their spectrums. Also, if $\dim \hat{\mathfrak{A}}^+ = \infty$ and $\hat{\mathfrak{A}}_f$ is non-empty, then the first two estimates hold trivially, and the implied estimates also do when $\dim \hat{\mathfrak{A}}^+ = \infty$.

3. Group C^*-algebras of the discrete Heisenberg groups

The discrete Heisenberg groups. Let $H_{2n+1}^{\mathbf{Z}}$ be the discrete Heisenberg group of rank $2n+1$ consisting of the following $(n+2) \times (n+2)$ matrices:

$$(c, b, a) = \begin{pmatrix} 1 & a & c \\ & 1_n & b^t \\ 0 & & 1 \end{pmatrix}, \quad a = (a_i)\,, \quad b = (b_i) \in \mathbf{Z}^n\,, \quad c \in \mathbf{Z}$$

where 1_n is the $n \times n$ identity matrix and b^t is the transpose of b. Then $H_{2n+1}^{\mathbf{Z}}$ is isomorphic to the semi-direct product $\mathbf{Z}^{n+1} \rtimes_\alpha \mathbf{Z}^n$ with the action α given by $\alpha_a(c, b) = (c + \sum_{i=1}^n a_i b_i, b)$. Note that $H_{2n+1}^{\mathbf{Z}}$ is a two-step nilpotent discrete group obtained by the following central extension: $1 \to \mathbf{Z} \to H_{2n+1} \to \mathbf{Z}^{2n} \to 1$. Then the group C^*-algebra $C^*(H_{2n+1}^{\mathbf{Z}})$ of $H_{2n+1}^{\mathbf{Z}}$ is isomorphic to the crossed product $C^*(\mathbf{Z}^{n+1}) \rtimes_\alpha \mathbf{Z}^n$ (cf. [Pd], [Tm] for crossed products).

Moreover, it is isomorphic to $C(\mathbf{T}^{n+1}) \rtimes_{\hat{\alpha}} \mathbf{Z}^n$, where $\hat{\alpha}$ is the action induced from α by the Fourier transform from $C^*(\mathbf{Z}^{n+1})$ to $C(\mathbf{T}^{n+1})$, and given by

$$\hat{\alpha}_a(w, z) = (w, w^{a_1} z_1, \cdots, w^{a_n} z_n)\,, \quad z = (z_i) \in \mathbf{T}^n\,, \quad w \in \mathbf{T}\,.$$

Then the crossed product $C(\mathbf{T} \times \mathbf{T}^n) \rtimes_{\hat{\alpha}} \mathbf{Z}^n$ is regarded as the C^*-algebra of a continuous field over $\mathbf{T}$ with fibers $\mathfrak{A}_w = C(\{w\} \times \mathbf{T}^n) \rtimes_{\hat{\alpha}} \mathbf{Z}^n$ for $w \in \mathbf{T}$, that is,

$$C^*(H_{2n+1}^{\mathbf{Z}}) \cong \Gamma(\mathbf{T}, \{\mathfrak{A}_w\}_{w\in\mathbf{T}}, \mathfrak{F})$$

for a family $\mathfrak{F}$ of continuous operator fields on $\mathbf{T}$ (cf. [AP]). Note that $\mathfrak{F}$ contains the unit field since $C^*(H_{2n+1}^{\mathbf{Z}})$ is unital. Since $\hat{\alpha}$ is trivial on $\{1\} \times \mathbf{T}^n$, and it is the multi-rotation on $\{w\} \times \mathbf{T}^n$, we have that

$$\mathfrak{A}_w \cong \begin{cases} \bigotimes^n \mathbf{T}_1^2 \cong C(\mathbf{T}^{2n}) & w = 1\,, \\ \bigotimes^n \mathbf{T}_w^2 & \text{otherwise} \end{cases}$$

where $\bigotimes^n \mathbf{T}_w^2$ means the n-times tensor product of the rotation algebra (= noncommutative 2-torus) $\mathbf{T}_w^2 = C(\{w\} \times \mathbf{T}) \rtimes_{\hat{\alpha}} \mathbf{Z}$ associated with the rotation on $\mathbf{T}$ by w.

By (F2), (F3), [Sh, p. 381] and [Eh, Theorem 2.2], we have

$$\mathrm{sr}(C^*(H_{2n+1}^{\mathbf{Z}})) \geq \mathrm{sr}(C(\mathbf{T}^{2n})) = n + 1\,, \quad \mathrm{csr}(C^*(H_{2n+1}^{\mathbf{Z}})) \geq 2\,.$$

By Theorem 1.3, it is obtained that

$$\begin{cases} \mathrm{sr}(C^*(H_{2n+1}^{\mathbf{Z}})) \leq \sup_{w\in\mathbf{T}} \mathrm{sr}(C(\mathbf{T}, \mathfrak{A}_w))\,, \\ \mathrm{csr}(C^*(H_{2n+1}^{\mathbf{Z}})) \leq \sup_{w\in\mathbf{T}} (\mathrm{csr}(C(\mathbf{T}, \mathfrak{A}_w)) \vee \mathrm{sr}(C(\mathbf{T}, \mathfrak{A}_w)))\,. \end{cases}$$

If $w = 1$, by (F3) and [Sh, p. 381] we have

$$\begin{cases} \mathrm{sr}(C(\mathbf{T}, C(\mathbf{T}^{2n}))) = \mathrm{sr}(C(\mathbf{T}^{2n+1})) = n + 1\,, \\ \mathrm{csr}(C(\mathbf{T}, C(\mathbf{T}^{2n}))) = \mathrm{csr}(C(\mathbf{T}^{2n+1})) = n + 2\,. \end{cases}$$

To make the above connected stable rank estimate sharper, we consider the following exact sequence:

$$0 \to \mathfrak{I} \to C^*(H_{2n+1}^{\mathbf{Z}}) \to C(\mathbf{T}^{2n}) \to 0\,, \quad \mathfrak{I} = C_0((\mathbf{T} \setminus \{1\}) \times \mathbf{T}^n) \rtimes \mathbf{Z}^n\,.$$

Then, by (F2) and Theorem 1.3,

$$\begin{aligned} \mathrm{csr}(C^*(H_{2n+1}^{\mathbf{Z}})) &\leq \mathrm{csr}(C(\mathbf{T}^{2n})) \vee \mathrm{csr}(\mathfrak{I}) \\ &\leq (n+1) \vee \sup_{w\in\mathbf{T}\setminus\{1\}} (\mathrm{csr}(C_0(\mathbf{T} \setminus \{1\}) \otimes \mathfrak{A}_w) \vee \mathrm{sr}(C_0(\mathbf{T} \setminus \{1\}) \otimes \mathfrak{A}_w))\,. \end{aligned}$$

Hence we estimate the stable rank of $C(\mathbf{T}) \otimes \mathfrak{A}_w$ and the connected stable rank of $C_0(\mathbf{T} \setminus \{1\}) \otimes \mathfrak{A}_w$ in the following.

Now we assume that $\mathbf{T}_w^2$ is irrational. It is known by [EE] that the irrational rotation algebras are inductive limits of 2-direct sums of matrix algebras over $C(\mathbf{T})$ with their matrix sizes going to infinity. It follows that $\mathfrak{A}_w = \bigotimes^n \mathbf{T}_w^2$, $C(\mathbf{T}) \otimes \mathfrak{A}_w$ and $C_0(\mathbf{T} \setminus \{1\}) \otimes \mathfrak{A}_w$ are inductive limits of 2^n-direct sums of matrix algebras over $C(\mathbf{T}^n)$, $C(\mathbf{T}^{n+1})$ and $C_0((\mathbf{T} \setminus \{1\}) \times \mathbf{T}^n)$ respectively, with their matrix sizes going to infinity. By [DNNP] and (F4), [Rf1, Theorem 5.1], we have that

$$\begin{cases} \mathrm{sr}(\mathfrak{A}_w) = 1\,, \quad \mathrm{sr}(C(\mathbf{T}) \otimes \mathfrak{A}_w) \le 2\,, \\ \mathrm{csr}(\mathfrak{A}_w) \le 2\,, \quad \mathrm{csr}(C_0(\mathbf{T} \setminus \{1\}) \otimes \mathfrak{A}_w) \le 2\,. \end{cases}$$

Note that $\mathfrak{A}_w \cong \mathfrak{B} \rtimes \mathbf{Z}$ where $\mathfrak{B} = (\bigotimes^{n-1} \mathbf{T}_w^2) \otimes C(\mathbf{T})$ with the unit.

If we assume $\mathrm{sr}(C(\mathbf{T}) \otimes \mathfrak{A}_w) = 1$, then it follows from taking its suitable quotient that $\mathrm{sr}(C([0,1]) \otimes \mathfrak{A}_w) = 1$. By [NOP, Proposition 5.2], we have the K_1-group of $\mathfrak{A}_w$ trivial. It is the contradiction since the K_0, K_1-groups of $\mathbf{T}_w^2$ are $\mathbf{Z}^2$ (cf. [Wo, 12.3]) so that the K_1-group of $\mathfrak{A}_w$ is also nontrivial by the Künneth formula (cf. [Wo, 9.3.3]).

Next we assume that the period of $w \in \mathbf{T}$ is $q \ge 2$. Then we have the following exact sequence (cf. [Bl, 10.3]):

$$0 \to C_0(\mathbf{R}) \otimes (C(\mathbf{T}) \rtimes_\beta \mathbf{Z}_q) \to \mathbf{T}_w^2 \to C(\mathbf{T}) \rtimes_\beta \mathbf{Z}_q \to 0$$

and $C(\mathbf{T}) \rtimes_\beta \mathbf{Z}_q \cong M_q(C(\mathbf{T}))$ (cf. [EL1, Lemma 3], [Dv, VIII.9]). Hence $\mathbf{T}_w^2$ is q-homogeneous. Thus $\mathfrak{A}_w = \bigotimes^n \mathbf{T}_w^2$ is q^n-homogeneous. Moreover, $\mathfrak{A}_w$ has a composition series $\{\mathfrak{I}_k\}_{k=1}^{2^n}$ such that $\mathfrak{I}_k/\mathfrak{I}_{k-1} \cong M_{q^n}(C_0(\mathbf{R}^{t_k} \times \mathbf{T}^n))$ for some $0 \le t_k \le t_{k-1} \le n$, and $\mathfrak{I}_1 = M_{q^n}(C_0(\mathbf{R}^n \times \mathbf{T}^n))$ and $\mathfrak{I}_{2^n}/\mathfrak{I}_{2^n-1} \cong M_{q^n}(C(\mathbf{T}^n))$. Then by tensoring this composition series with $C(\mathbf{T})$, we have a composition series $\{\mathcal{S}_k\}_{k=1}^{2^n}$ of $C(\mathbf{T}, \mathfrak{A}_w)$ such that $\mathcal{S}_k/\mathcal{S}_{k-1} \cong M_{q^n}(C_0(\mathbf{R}^{t_k} \times \mathbf{T}^{n+1}))$ with $\mathcal{S}_1 = M_{q^n}(C_0(\mathbf{R}^n \times \mathbf{T}^{n+1}))$ and $\mathcal{S}_{2^n}/\mathcal{S}_{2^n-1} \cong M_{q^n}(C(\mathbf{T}^{n+1}))$. Therefore, by (F2), (F3) and (F4), we obtain that

$$\mathrm{sr}(C(\mathbf{T}, \mathfrak{A}_w)) \ge \mathrm{sr}(M_{q^n}(C_0(\mathbf{R}^n \times \mathbf{T}^{n+1}))) = \{[(2n+1)/2]/q^n\} + 1 = 2\,,$$

$$\mathrm{sr}(C(\mathbf{T}, \mathfrak{A}_w)) \le \mathrm{sr}(M_{q^n}(C_0(\mathbf{R}^n \times \mathbf{T}^{n+1}))) \vee \mathrm{csr}(M_{q^n}(C_0(\mathbf{R}^{n-1} \times \mathbf{T}^{n+1}))) = 2\,.$$

Similarly, we get that

$$\mathrm{csr}(C_0(\mathbf{T} \setminus \{1\}, \mathfrak{A}_w)) \le \mathrm{csr}(M_{q^n}(C_0(\mathbf{R}^n \times \mathbf{T}^n \times (\mathbf{T} \setminus \{1\})))) \le 2\,.$$

Partly summing up, we obtain

PROPOSITION 3.1. *Let $\mathbf{T}_w^2$ be a noncommutative rotation algebra. Then*

$$\mathrm{sr}(C(\mathbf{T}, \bigotimes^n \mathbf{T}_w^2)) = 2\,, \quad \mathrm{csr}(C_0(\mathbf{T} \setminus \{1\}, \bigotimes^n \mathbf{T}_w^2)) \le 2\,,$$

for any $w \in \mathbf{T} \setminus \{1\}$ and $n \ge 1$.

REMARK. The same estimates for $C^b(\mathbf{T}\setminus\{1\})\otimes(\bigotimes^n \mathbf{T}_w^2)$ and $C_0(\mathbf{T}\setminus\{1\})\otimes(\bigotimes^n \mathbf{T}_w^2)$ hold on the same way as above.

From the above reasoning, we conclude the following:

THEOREM 3.2. *Let $H_{2n+1}^{\mathbf{Z}}$ be the discrete Heisenberg group of rank $2n+1$. Then*

$$\begin{cases} \mathrm{sr}(C^*(H_{2n+1}^{\mathbf{Z}})) = n+1 = \dim_{\mathbf{C}}(\hat{H}_{2n+1}^{\mathbf{Z}})_1 , \\ 2 \le \mathrm{csr}(C^*(H_{2n+1}^{\mathbf{Z}})) \le n+1 . \end{cases}$$

REMARK. Note that the quotient of $H_{2n+1}^{\mathbf{Z}}$ by its commutator is isomorphic to $\mathbf{Z}^{2n}$. Therefore, $(\hat{H}_{2n+1}^{\mathbf{Z}})_1$ is isomorphic to the dual group $\mathbf{T}^{2n}$ of $\mathbf{Z}^{2n}$. It has been known that the stable rank of $C^*(H_{2n+1}^{\mathbf{Z}})$ is estimated as follows:

$$n+1 \le \mathrm{sr}(C^*(H_{2n+1}^{\mathbf{Z}})) \le \mathrm{sr}(C^*(\mathbf{Z}^{n+1})) + n = [(n+1)/2] + n + 1 .$$

by [Rf1, Theorem 7.1] and some part of the above argument. In particular, $\mathrm{sr}(C^*(H_3^{\mathbf{Z}}))$ is 2 or 3. However, it has been unsettled to determine this alternative. As observed above, $C^*(H_{2n+1}^{\mathbf{Z}})$ as the algebras of continuous fields on $\mathbf{T}$ have no local triviality at all since their fibers vary continuously, which was a big obstruction to compute their ranks. Also, this theorem suggests that the stable ranks of $C^*(H_{2n+1}^{\mathbf{Z}})$ can be estimated by the dimension of the spaces of their 1-dimensional representations (which are subspaces of their spectrums) (cf. [Sd1, 2], [ST]).

The generalized discrete Heisenberg groups. Let $H_{2n+1}^{\mathbf{Z}}(d)$ be the generalized discrete Heisenberg group of rank $2n+1$ consisting of the $(n+2)\times(n+2)$ matrices (cf. [LP1, LP2]):

$$(c,b,a) = \begin{pmatrix} 1 & da & c \\ & 1_n & b^t \\ 0 & & 1 \end{pmatrix} , \quad a=(a_i) , \quad b=(b_i)\in\mathbf{Z}^n , \quad c\in\mathbf{Z}$$

with $da = (d_i a_i)$ for given positive integers $\{d_i\}$ with d_{i+1} divisible by d_i $(1 \le i \le n-1)$. Then it is known that any 2-step nilpotent group Γ obtained by a central extension of $\mathbf{Z}^k$ by $\mathbf{Z}$: $1 \to \mathbf{Z} \to \Gamma \to \mathbf{Z}^k \to 1$, is isomorphic to the direct product $H_{2n+1}^{\mathbf{Z}}(d)\times\mathbf{Z}^{k-2n}$ for some n and d [BPc, Corollary 3.4]. Note that if $k=1$, then $\Gamma \cong \mathbf{Z}^2$. Hence $C^*(\Gamma)\cong C(\mathbf{T}^2)$. Then by (F3) and [Sh, p. 381], $\mathrm{sr}(C^*(\Gamma)) = \mathrm{csr}(C^*(\Gamma)) = 2 = \dim\hat{\Gamma}$.

By the similar argument as given before Theorem 3.2, we obtain

THEOREM 3.3. *Let Γ be a 2-step nilpotent group obtained by a central extension of $\mathbf{Z}^k$ by $\mathbf{Z}$ $(k \ge 2)$. Then*

$$\begin{cases} \mathrm{sr}(C^*(\Gamma)) = [k/2] + 1 = \dim_{\mathbf{C}}(\hat{\Gamma})_1 , \\ 2 \le \mathrm{csr}(C^*(\Gamma)) \le [(k+1)/2] + 1 . \end{cases}$$

PROOF. Note that $C^*(\Gamma) \cong C^*(H^{\mathbf{Z}}_{2n+1}(d)) \otimes C(\mathbf{T}^{k-2n})$ for $k \geq 2$, $n \geq 1$ and some d, and $C^*(H^{\mathbf{Z}}_{2n+1}(d))$ is regarded as the C^*-algebra of a continuous field over $\mathbf{T}$ with fibers:

$$\mathfrak{A}_w = C(\{w\} \times \mathbf{T}^n) \rtimes \mathbf{Z}^n \cong \begin{cases} C(\mathbf{T}^{2n}) & w = 1\,, \\ \bigotimes_{i=1}^n \mathbf{T}^2_{w_i} & \text{otherwise} \end{cases}$$

where $\mathbf{T}^2_{w_i}$ is the rotation algebra associated with the rotation on $\mathbf{T}$ by w^{d_i}. And $\mathbf{T}^2_{w_i}$ is irrational or rational according to w, and nontrivial or trivial according to d_i in the rational cases. Next we apply some methods for the rank estimates in the case of the discrete Heisenberg groups to this generalized case similarly. □

As a remark, more generally, using a result of [LP1] about describing the twisted group C^*-algebras $C^*(H_{2n+1}(d), \sigma)$ with σ multipliers on $H_{2n+1}(d)$ $(n \geq 2)$ as the C^*-algebras of continuous fields over $\mathbf{T}$ with fibers isomorphic to matrix algebras over higher-dimensional rotation algebras, we would obtain their stable rank and connected stable rank estimates on the similar way. In fact, by [EL1, 2] some of simple noncommutative tori are isomorphic to inductive limits of finite direct sums of matrix algebras over $C(\mathbf{T})$. For rational noncommutative tori, it would be possible to follow the similar procedure as the rational rotation cases (before Proposition 3.1). See [Rf3], [PR] and [LP2] for more general cases of C^*-algebras of continuous fields.

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