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Spectral Geometry of the Jacobi Operator of Totally Real Submanifolds of QP^n

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Abstract. We calculate some invariants determined by the spectrum of the Jacobi operator J of n -dimensional totally real submanifolds of the quaternionic projective space QP^n and we use such invariants to characterize parallel submanifolds of QP^n .

1. Introduction

The Jacobi operator J is a second order elliptic operator associated to an isometric immersion of a compact Riemannian manifold M into a Riemannian manifold \bar{M} . J is defined on the space of smooth sections of the normal bundle TM^\perp by the formula

$$J = D + \tilde{R} - \tilde{A},$$

where D is the rough Laplacian of the normal connection ∇^\perp on TM^\perp , \tilde{R} and \tilde{A} are linear transformations of TM^\perp defined by means of a partial Ricci tensor of \bar{M} and of the second fundamental form A , respectively. J appears in the formula which gives the second variation for the area function of a compact minimal submanifold (see [S]). For this reason, J is also called the *second variation operator*. Its spectrum, denoted by

$$\text{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\},$$

is discrete, as a consequence of the compactness of M .

The Riemannian invariants determined by $\text{spec}(M, J)$ have been calculated for several types of isometric immersions of submanifolds into real or complex space forms (see [D], [H], [Sh], [CP], [C]). Moreover, a similar study was made about spectral geometry determined by the Jacobi operator associated to the energy of a harmonic map ([CgY], [KPa], [KKiPa], [NiTV], [U], [Y]).

In this paper, we determine the first three terms of the asymptotic expansion for the partition function associated to the spectrum of the Jacobi operator of an n -dimensional totally

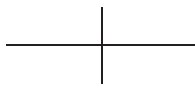
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real submanifold of the quaternionic projective space QP^n and we use the corresponding Riemannian spectral invariants to characterize n -dimensional totally real parallel submanifolds of QP^n .

The paper is organized in the following way. In Section 2, we shall recall some basic results about QP^n and n -dimensional totally real submanifolds of QP^n . In Section 3, we shall compute the first three terms of the asymptotic expansion for the partition function associated to $\text{spec}(M, J)$, M being an n -dimensional totally real submanifold of QP^n . In Sections 4 and 5, we shall characterize totally real parallel submanifolds of QP^n , which are Einstein and conformally flat, respectively. In Section 6 we shall investigate the spectral rigidity of totally real parallel submanifolds of QP^n for small dimensions n .

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2. Totally real submanifolds of QP^n

Let (\bar{M}, g) be a $4n$ -dimensional quaternionic Riemannian manifold and V the three-dimensional vector bundle of tensors of type $(1, 1)$ with local basis of almost Hermitian structures I_1, I_2, I_3 , satisfying

- a) $I_1 I_2 = -I_2 I_1 = I_3, I_2 I_3 = -I_3 I_2 = I_1, I_3 I_1 = -I_1 I_3 = I_2, I_1^2 = I_2^2 = I_3^2 = -1$;
- b) for any cross-section ξ of V , $\bar{\nabla}_X \xi$ is also a cross-section of V , where X is a vector field on M and $\bar{\nabla}$ is the Riemannian connection of \bar{M} .

If X is a unit vector on \bar{M} , the *quaternionic section* determined by X is the 4-plane $Q(X)$ spanned by $X, I_1 X, I_2 X$ and $I_3 X$. If $Q(X)$ and $Q(Y)$ are orthogonal, the plane spanned by X and Y is called a *totally real plane*. Any 2-plane in a quaternionic section is called a *quaternionic plane* and its sectional curvature is called *quaternionic sectional curvature*. A *quaternionic space form* is a quaternionic manifold of constant quaternionic sectional curvature. In particular, by QP^n we denote the $4n$ -dimensional quaternionic projective space, equipped with the Riemannian metric \bar{g} of constant quaternionic sectional curvature $c > 0$. Its curvature tensor \bar{R} , taken with the sign convention

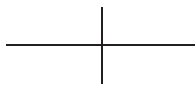
$$\bar{R}(X, Y) = \bar{\nabla}_{[X, Y]} - [\bar{\nabla}_X, \bar{\nabla}_Y],$$

satisfies

$$\begin{aligned} \bar{R}(X, Y, Z, W) = & \frac{c}{4} \{ \bar{g}(X, Z) \bar{g}(Y, W) - \bar{g}(Y, Z) \bar{g}(X, W) \\ & + \bar{g}(I_1 X, Z) \bar{g}(I_1 Y, W) - \bar{g}(I_1 Y, Z) \bar{g}(I_1 X, W) + 2 \bar{g}(I_1 X, Y) \bar{g}(I_1 Z, W) \\ & + \bar{g}(I_2 X, Z) \bar{g}(I_2 Y, W) - \bar{g}(I_2 Y, Z) \bar{g}(I_2 X, W) + 2 \bar{g}(I_2 X, Y) \bar{g}(I_2 Z, W) \\ & + \bar{g}(I_3 X, Z) \bar{g}(I_3 Y, W) - \bar{g}(I_3 Y, Z) \bar{g}(I_3 X, W) + 2 \bar{g}(I_3 X, Y) \bar{g}(I_3 Z, W) \}. \end{aligned}$$

Note that our convention for the sign of the curvature tensor is opposed to the one used by Simons in [S]. We refer to [I] for more details about quaternionic manifolds.





Next, let (M, g) be an n -dimensional Riemannian manifold, isometrically immersed into (QP^n, \bar{g}) . By definition, M is a *totally real submanifold* of QP^n if each tangent 2-plane of M is mapped by the isometric immersion into a totally real plane of QP^n . We shall denote by ∇ and R the Levi Civita connection and the curvature tensor of M , respectively. The normal connection is given by

$$\begin{aligned} \nabla^\perp : TM \times TM^\perp &\longrightarrow TM^\perp \\ (X, \xi) &\longmapsto \nabla_X^\perp \xi, \end{aligned}$$

where $\nabla_X^\perp \xi$ denotes the normal component of $\bar{\nabla}_X \xi$. The second fundamental form σ and the Weingarten operator A are respectively defined by

$$\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y, \quad A_\xi X = -\bar{\nabla}_X \xi + \nabla_X^\perp \xi$$

for all $X, Y \in TM$ and $\xi \in TM^\perp$. Moreover, $\bar{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$.

Let R^\perp denote the curvature tensor associated to the normal connection ∇^\perp . The curvature tensors R, \bar{R} and R^\perp satisfy the Gauss and the Ricci equations:

$$\begin{aligned} R(X, Y, Z, W) &= g(R(X, Y)Z, W) = \bar{R}(X, Y, Z, W) \\ &\quad + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(Y, Z), \sigma(X, W)), \\ R^\perp(X, Y, \xi, \eta) &= \bar{g}(R^\perp(X, Y)\xi, \eta) = \bar{R}(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y), \end{aligned}$$

where $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$ for all $X, Y, Z, W \in TM$ and $\xi, \eta \in TM^\perp$.

Let $\{e_1, \dots, e_n, e_{I_1(1)} = I_1 e_1, \dots, e_{I_1(n)} = I_1 e_n, e_{I_2(1)} = I_2 e_1, \dots, e_{I_2(n)} = I_2 e_n, e_{I_3(1)} = I_3 e_1, \dots, e_{I_3(n)} = I_3 e_n\}$ be a local orthonormal frame on QP^n such that, restricted to M , the vector fields e_1, \dots, e_n are tangent to M . With respect to such frame field, we have

$$I_1 = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix}, \quad I_2 = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix}, \quad I_3 = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix},$$

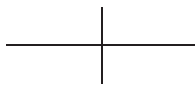
where E is the $(n \times n)$ -identity matrix. We shall use the following convention for the range of indices:

- $A, B, C, D = 1, \dots, n, I_1(1), \dots, I_1(n), I_2(1), \dots, I_2(n), I_3(1), \dots, I_3(n);$
- $i, j, k, h = 1, \dots, n;$
- $\alpha, \beta = I_1(1), \dots, I_1(n), I_2(1), \dots, I_2(n), I_3(1), \dots, I_3(n);$
- $\varphi, \psi = I_1, I_2, I_3.$

Putting $A_\alpha = A_{e_\alpha}$, $A_\alpha e_i = h_{ij}^\alpha e_j$ and $R_{ij\alpha\beta}^\perp = R^\perp(e_i, e_j, e_\alpha, e_\beta)$, the Gauss and Ricci equations become

$$(2.1) \quad R_{ijkh} = \frac{c}{4}(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) + \sum_{\alpha} (h_{ik}^\alpha h_{jl}^\alpha - h_{jk}^\alpha h_{il}^\alpha)$$





and

$$(2.2) \quad R_{ij\alpha\beta}^\perp = \frac{c}{4}((I_1)_{i\alpha}(I_1)_{j\beta} - (I_1)_{j\alpha}(I_1)_{i\beta} + (I_2)_{i\alpha}(I_2)_{j\beta} - (I_2)_{j\alpha}(I_2)_{i\beta} + (I_3)_{i\alpha}(I_3)_{j\beta} - (I_3)_{j\alpha}(I_3)_{i\beta}) - g([A_\alpha, A_\beta]e_i, e_j).$$

Note that, for all i, j, k and φ , we have

$$h_{jk}^{\varphi(i)} = h_{ik}^{\varphi(j)} = h_{ij}^{\varphi(k)}.$$

The mean curvature vector is defined by

$$H = \text{trace}(\sigma) = \sum_i \sigma(e_i, e_i) = \sum_{i,\varphi} \text{tr} A_{\varphi(i)} e_{\varphi(i)} = \sum_\alpha \text{tr} A_\alpha e_\alpha.$$

M is said to be *minimal* if $H = 0$, *totally geodesic* if $\sigma = 0$, *parallel* (or with *parallel second fundamental form*) if $\nabla' \sigma = 0$, where

$$(\nabla'_X \sigma)(Y, Z) = \nabla_X^\perp(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z).$$

For the Ricci tensor ϱ of M , from (2.1) we easily obtain

$$(2.3) \quad \varrho_{ik} = \sum_j R_{ijkj} = \frac{c}{4}(n-1)\delta_{ik} + \sum_\alpha \{(\text{tr} A_\alpha)h_{ik}^\alpha\} - \sum_\alpha h_{il}^\alpha h_{kl}^\alpha$$

and for the scalar curvature τ of M , we have

$$(2.4) \quad \tau = \sum_i \varrho_{ii} = n(n-1)\frac{c}{4} + \|H\|^2 - \|\sigma\|^2,$$

where $\|\sigma\|^2 = \sum \text{tr} A_\alpha^2$ and $\|H\|^2 = \sum (\text{tr} A_\alpha)^2$.

We also refer to [ChH] for more details. We now prove the following

LEMMA 2.1. *Let M be an n -dimensional totally real submanifold of $Q P^n$. Then*

$$(2.5) \quad \|R\|^2 = c\tau - n(n-1)\frac{c^2}{8} - \sum_{\alpha,\beta} \text{tr}[A_\alpha, A_\beta]^2.$$

If in addition M is minimal, then

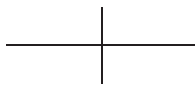
$$(2.6) \quad \|\varrho\|^2 = 2(n-1)\frac{c}{4}\tau - n(n-1)^2\frac{c^2}{16} + \sum_{\alpha,\beta} (\text{tr} A_\alpha A_\beta)^2,$$

$$(2.7) \quad \frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|R\|^2 - \|\varrho\|^2 + (n+1)\frac{c}{4}\tau.$$

PROOF. (2.5) follows from (2.1), taking into account (2.4), once we note that

$$\sum_\alpha \left(\sum_{i,j,k,h} (h_{ik}^\alpha h_{jh}^\alpha - h_{jk}^\alpha h_{ih}^\alpha) \right)^2 = - \sum_{\alpha,\beta} \text{tr}[A_\alpha, A_\beta]^2$$





and

$$\sum_{\alpha, i, j} (h_{ii}^\alpha h_{jj}^\alpha - (h_{ij}^\alpha)^2) = \|H\|^2 - \|\sigma\|^2.$$

Next, suppose that M is minimal. Then $\text{tr} A_\alpha = 0$ for all α and (2.3) reduces to

$$(2.8) \quad Q_{ik} - \frac{c}{4}(n-1)\delta_{ik} = -\sum_{\alpha, l} h_{il}^\alpha h_{kl}^\alpha = -\sum_{i, k, \varphi} \text{tr} A_{\varphi(i)} A_{\varphi(k)}.$$

In [ChH], the following formula was proved for any m -dimensional totally real submanifold of $Q P^n$:

$$(2.9) \quad \frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 + \sum_{\alpha, \beta} \sum \text{tr} [A_\alpha, A_\beta]^2 - \sum_{\alpha, \beta} (\text{tr} A_\alpha A_\beta)^2 + n \frac{c}{4} \|\sigma\|^2 + \frac{c}{4} \sum_{i, \varphi} (\text{tr} A_{\varphi(i)}^2).$$

Next, put $S_{\alpha, \beta} = \sum_{\alpha, \beta} \text{tr} (A_\alpha A_\beta)$. Since $S_{\alpha, \beta}$ is a symmetric matrix, it can be diagonalized for a suitable choice of $\{e_\alpha\}$. Hence, we may assume that $\text{tr} (A_\alpha A_\beta) = 0$ for $\alpha \neq \beta$ (see also [CH, p. 198]). In particular, we then have

$$\sum_{i, k, \varphi} \text{tr} A_{\varphi(i)} A_{\varphi(k)} = \sum_{\alpha, \beta} \text{tr} A_\alpha A_\beta, \quad \sum_{i, k, \varphi} (\text{tr} A_{\varphi(i)} A_{\varphi(k)})^2 = \sum_{\alpha, \beta} (\text{tr} A_\alpha A_\beta)^2.$$

Hence, (2.6) follows from (2.8). Moreover, since in our case $m = n$ and (2.4)–(2.6) hold, from (2.9) we get (2.7). □

3. Spectral invariants of the Jacobi operator

Let M be an n -dimensional Riemannian manifold immersed in a Riemannian manifold \bar{M} of dimension $\bar{n} = n + r$. The normal bundle TM^\perp is a real r -dimensional vector bundle on M , with inner product induced by the metric \bar{g} of \bar{M} . Let D denote the so-called *rough Laplacian* associated to the normal connection ∇^\perp of TM^\perp , that is,

$$D\xi = -\nabla_{e_i}^\perp \nabla_{e_i}^\perp \xi + \nabla_{\nabla_{e_i}^\perp \xi}^\perp \xi,$$

where ξ is a section of TM^\perp . Next, let \tilde{A} be the *Simons operator* defined in [S] by

$$\bar{g}(\tilde{A}\xi, \eta) = \text{tr}(A_\xi \circ A_\eta),$$

for $\xi, \eta \in TM^\perp$. Moreover, we consider the operator \tilde{R} defined by

$$\tilde{R}(\xi) = -\sum_{i=1}^n (\bar{R}(e_i, \xi)e_i)^\perp,$$



where $(\bar{R}(e_i, \xi)e_i)^\perp$ denotes the normal component of $\bar{R}(e_i, \xi)e_i$.

The *Jacobi operator* (or *second variation operator*), acting on cross-sections of TM^\perp , is the second order elliptic differential operator J defined by (see [S] or [D])

$$\begin{aligned} J : TM^\perp &\longrightarrow TM^\perp \\ \xi &\longmapsto (D - \tilde{A} + \tilde{R})\xi. \end{aligned}$$

When M is compact, we can define an inner product for cross-sections on TM^\perp , by

$$\langle \xi, \eta \rangle = \int_M \bar{g}(\xi, \eta) dv$$

and J is self-adjoint with respect to this product. Moreover, J is strongly elliptic and it has an infinite sequence of eigenvalues, with finite multiplicities, denoted by

$$\text{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \leq \dots + \uparrow \infty\}.$$

The partition function $Z(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)$ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-n/2} \{a_0(J) + a_1(J)t + a_2(J)t^2 + \dots\}.$$

By Gilkey's results [G] (see also [D] and [H]), it follows that the coefficients a_0 , a_1 and a_2 are given by the following

THEOREM 3.1 ([G]). *We have*

$$a_0 = r \text{vol}(M),$$

$$a_1 = \frac{r}{6} \int_M \tau dv + \int_M \text{tr} \tilde{E} dv,$$

$$\begin{aligned} a_2 = & \frac{r}{360} \int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} dv + \frac{1}{360} \int_M \{-30\|R^\perp\|^2 \\ & + \text{tr}(60\tau \tilde{E} + 180\tilde{E}^2)\} dv, \end{aligned}$$

where $\tilde{E} = \tilde{A} - \tilde{R}$.

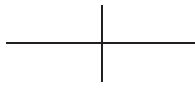
We now consider the case of an n -dimensional totally real submanifold of $QP^n(c)$ and we compute explicitly the coefficients a_0 , a_1 and a_2 in terms of invariants depending on the curvature of M and its isometric immersion in QP^n .

PROPOSITION 3.2. *Let M be an n -dimensional totally real submanifold of QP^n . Then*

$$(3.1) \quad \|R^\perp\|^2 = \|R\|^2 + n(n-1)c^2,$$

$$(3.2) \quad \text{tr} \tilde{E} = \|\sigma\|^2 + \frac{3}{4}n(n+1)c,$$

$$(3.3) \quad \text{tr} \tilde{E}^2 = \frac{3}{16}n(n+1)^2c^2 + (n+1)\frac{c}{2}\|\sigma\|^2 + \sum_{\alpha, \beta} \text{tr}(A_\alpha A_\beta)^2.$$



If in addition M is minimal, then

$$(3.4) \quad \text{tr} \tilde{E}^2 = \|\varrho\|^2 + (n+1)c\|\sigma\|^2 + \frac{1}{8}n(n^2 + 4n + 1)c^2.$$

PROOF. From (2.2) we get

$$(3.5) \quad \|R^\perp\|^2 = \sum_{i,j,\alpha,\beta} (R_{ij\alpha\beta}^\perp)^2 = R_1 + R_2 + R_3,$$

with

$$\begin{aligned} R_1 &= \frac{c^2}{16} \sum \{(I_1)_{i\alpha}(I_1)_{j\beta} - (I_1)_{j\alpha}(I_1)_{i\beta} + (I_2)_{i\alpha}(I_2)_{j\beta} - (I_2)_{j\alpha}(I_2)_{i\beta} \\ &\quad + (I_3)_{i\alpha}(I_3)_{j\beta} - (I_3)_{j\alpha}(I_3)_{i\beta}\}^2 \\ &= \frac{c^2}{16} \sum \{(I_1)_{iI_1(k)}(I_1)_{jI_1(h)} - (I_1)_{jI_1(k)}(I_1)_{iI_1(h)} + (I_2)_{iI_2(k)}(I_2)_{jI_2(h)} \\ &\quad - (I_2)_{jI_2(k)}(I_2)_{iI_2(h)} + (I_3)_{iI_3(k)}(I_3)_{jI_3(h)} - (I_3)_{jI_3(k)}(I_3)_{iI_3(h)}\}^2 \\ &= \frac{c^2}{16} \sum ((3\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}))^2 = \frac{9}{8}n(n-1)c^2, \\ R_2 &= \sum (g([A_\alpha, A_\beta]e_i, e_j))^2 = \sum_{\alpha,\beta} \|[A_\alpha, A_\beta]\|^2 = - \sum_{\alpha,\beta} \text{tr}[A_\alpha, A_\beta]^2, \end{aligned}$$

where we used the fact that $[A_\alpha, A_\beta]$ is skew-symmetric, and

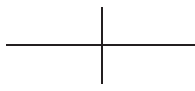
$$\begin{aligned} R_3 &= -\frac{c}{2} \sum \{(I_1)_{iI_1(k)}(I_1)_{jI_1(h)} - (I_1)_{jI_1(k)}(I_1)_{iI_1(h)} \\ &\quad + (I_2)_{iI_2(k)}(I_2)_{jI_2(h)} - (I_2)_{jI_2(k)}(I_2)_{iI_2(h)} \\ &\quad + (I_3)_{iI_3(k)}(I_3)_{jI_3(h)} - (I_3)_{jI_3(k)}(I_3)_{iI_3(h)}\} g([A_{\varphi(k)}, A_{\varphi(h)}]e_i, e_j) \\ &= -\frac{c}{2} \sum \{g([A_{\varphi(i)}, A_{\varphi(j)}]e_i, e_j) - g([A_{\varphi(j)}, A_{\varphi(i)}]e_i, e_j)\} \\ &= -c \sum g([A_{\varphi(i)}, A_{\varphi(j)}]e_i, e_j) \\ &= -c \sum \{g(A_{\varphi(j)}e_i, A_{\varphi(i)}e_j) - g(A_{\varphi(i)}e_i, A_{\varphi(j)}e_j)\} \\ &= \|\sigma\|^2 - \|H\|^2. \end{aligned}$$

Then (3.1) follows from (3.5), taking into account (2.3) and (2.4).

Next, using the Ricci equation (2.2), we easily obtain

$$\tilde{R}(\xi) = -(n+1)\frac{c}{4}\xi$$





and hence,

$$(3.6) \quad \operatorname{tr} \tilde{R} = -\frac{3}{4}n(n+1)c,$$

$$(3.7) \quad \operatorname{tr} \tilde{R}^2 = \frac{3}{16}n(n+1)^2c^2,$$

$$(3.8) \quad \operatorname{tr} \tilde{R} \circ \tilde{A} = -\frac{1}{4}(n+1)c \operatorname{tr} \tilde{A}.$$

Next, by the definition of \tilde{A} , we get

$$(3.9) \quad \operatorname{tr} \tilde{A} = \sum_{\alpha} \bar{g}(\tilde{A}e_{\alpha}, e_{\alpha}) = \sum_{i, \alpha} \bar{g}(A_{\alpha}e_i, A_{\alpha}e_i) = \|A\|^2 = \|\sigma\|^2$$

and

$$(3.10) \quad \operatorname{tr} \tilde{A}^2 = \sum_{\alpha} \bar{g}(\tilde{A}e_{\alpha}, \tilde{A}e_{\alpha}) = \sum_{\alpha, \beta} (\bar{g}(A_{\alpha}, A_{\beta}))^2 = \sum_{\alpha, \beta} (\operatorname{tr}(A_{\alpha}A_{\beta}))^2.$$

Therefore, since $\operatorname{tr} \tilde{E} = \operatorname{tr} \tilde{A} - \operatorname{tr} \tilde{R}$ and $\operatorname{tr} \tilde{E}^2 = \operatorname{tr}(\tilde{A}^2 - 2\tilde{R} \circ \tilde{A} + \tilde{R}^2)$, from (3.6)–(3.10) we get (3.2) and (3.3).

Finally, if M is minimal, then we obtain (3.4) from (3.3), taking into account (2.6). \square

Combining Theorem 3.1 and Proposition 3.2, we get

THEOREM 3.3. *On an n -dimensional totally real submanifold M of $QP^n(c)$, the first coefficients of the asymptotic expansion of the partition function of the Jacobi operator are given by*

$$(3.11) \quad a_0 = 3n \operatorname{vol}(M),$$

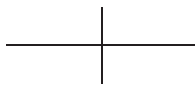
$$(3.12) \quad a_1 = \frac{n}{2} \int_M \tau dv + \int_M \|\sigma\|^2 dv + \frac{3}{4}n(n+1)c \operatorname{vol}(M) \\ = \frac{n-2}{2} \int_M \tau dv + \int_M \|H\|^2 dv + n(n+1)c \operatorname{vol}(M),$$

$$(3.13) \quad a_2 = \frac{n}{120} \int_M \{2\|R\|^2 - 2\|\varrho\|^2 + 5\tau^2\} dv + \frac{1}{120} \int_M \{-10\|R\|^2 \\ - 10n(n-1)c^2 + 20\tau(\|\sigma\|^2 + \frac{3}{4}n(n+1)c) \\ + 60[(n+1)\frac{c}{2}\|\sigma\|^2 + \frac{3}{16}n(n+1)^2c^2 + \operatorname{tr} \tilde{A}^2]\} dv.$$

If in addition M is minimal, then

$$(3.14) \quad a_0 = 3n \operatorname{vol}(M),$$





$$(3.15) \quad a_1 = \frac{n-2}{2} \int_M \tau dv + n(n+1)c \operatorname{vol}(M),$$

$$(3.16) \quad a_2 = \frac{1}{120} \int_M \{2(n-5)\|R\|^2 - 2(n-30)\|\varrho\|^2 + 5(n-4)\tau^2\} dv \\ + k_1(n)c \int_M \tau dv + k_2(n)c^2 \operatorname{vol}(M),$$

where k_1 and k_2 are constants depending on n .

4. Totally real parallel Einstein submanifolds of QP^n

K. Tsukada [Ts] classified parallel submanifolds of $QP^n(c)$. In particular, he proved that if M_0 is a totally real parallel submanifold of $QP^n(c)$, then M_0 is either

(R-R) a totally real submanifold contained in a totally real totally geodesic submanifold of $QP^n(c)$, or

(R-C) a totally real submanifold contained in a totally complex totally geodesic submanifold of $QP^n(c)$

(see [Ts, Theorem 3.10]). In general, a totally real submanifold M_0 of $QP^n(c)$ has dimension $m \leq n$. If M_0 is parallel and $\dim M_0 = n$, then M_0 is not of type (R-R), unless M_0 itself is a totally real totally geodesic submanifold of $QP^n(c)$, which is also of type (R-C). Therefore, M_0 is an n -dimensional totally real parallel submanifold M_0 of the quaternionic projective space $QP^n(c)$ if and only if it is an n -dimensional totally real parallel submanifold M_0 of the complex projective space $CP^n(c)$.

Totally real parallel submanifolds of $CP^n(c)$ have been classified by H. Naitoh [N]. We now synthesize some basic ideas of [N], referring to this paper for more details.

Let M be a simply connected Riemannian manifold, admitting a totally real parallel isometric immersion into $CP^n(c)$. In other words, M is the universal covering of a complete totally real submanifold M_0 embedded into $CP^n(c)$. If M has no Euclidean factor, then M is irreducible and of compact type [N, Section 4]. Note that, as it is well-known, an irreducible symmetric Riemannian manifold is Einstein. More explicitly, M must be one of the following:

$$(4.1) \quad SO(n+1)/SO(n) \ (n \geq 2), \quad SU(k), \quad (k \geq 3), \\ SU(k)/SO(k), \quad (k \geq 3), \quad SU(2k)/Sp(k), \quad (k \geq 3), \quad E_6/F_4,$$

the metric on M is determined uniquely by the constant c (the holomorphic sectional curvature of CP^n) and for each of these spaces there exists exactly one quotient which is a complete totally real submanifold M_0 embedded into $CP^n(c)$. Note that, since M is the universal covering of M_0 , the Riemannian manifolds M and M_0 have the same Riemannian curvature invariants. Some of these invariants were computed explicitly for M in [C].

The embedded totally real parallel submanifolds M_0 of $CP^n(c)$ corresponding to the spaces listed in (4.1) could be deduced from Section 5 of [N], where the immersions were



explicitly described. On the other hand, n -dimensional totally real submanifolds of $CP^n(c)$ correspond exactly, in the framework of symplectic geometry, to the so-called *Lagrangian submanifolds*. In order to study their Hamiltonian stability, compact minimal Lagrangian submanifolds of $CP^n(c)$, with parallel second fundamental form, were explicitly computed by A. Amarzaya and Y. Ohnita [AO]. Besides $RP^n(\frac{c}{4})$, the totally geodesic one, and the flat torus T^n (which corresponds to the Euclidean case), they are the followings:

$$SU(k)/Z_k, \quad SU(k)/SO(k)Z_k, \quad SU(2k)/Sp(k)Z_{2k}, \quad E_6/F_4Z_3.$$

If M admits a Euclidean factor and we suppose that M is Einstein, it is easy to show that the scalar curvature of M vanishes and so, M itself is Euclidean (we can refer to [C] for more details). In particular, if the corresponding embedded submanifold M_0 is compact, then M_0 is the n -dimensional flat torus, T^n .

Therefore, combining the results of [AO] and [C], we obtain the following table, which describes all n -dimensional compact totally real parallel Einstein submanifolds embedded into $CP^n(c)$ (and hence, of $QP^n(c)$).

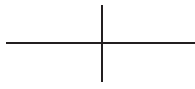
It is easy to check that for two of such manifolds, having the same dimension, it never occurs that the pairs of Riemannian curvature invariants $(\tau, \|R\|^2)$ coincide. Therefore, we have the following

THEOREM 4.1. *Each compact n -dimensional totally real parallel Einstein submanifold M_0 of $QP^n(c)$ is uniquely determined by the pair of Riemannian curvature invariants $(\tau, \|R\|^2)$.*

Taking into account formulas (3.14)–(3.16) and Theorem 4.1, we can now prove the following

TABLE I

| M | dim | τ | $\ R\ ^2$ |
|----------------------|-------------------------|-----------------------------|----------------------------------|
| $RP^n(\frac{c}{4})$ | n | $\frac{n(n-1)}{4}c$ | $\frac{n(n-1)}{8}c^2$ |
| $SU(k)/Z_k$ | $k^2 - 1$ | $\frac{(k^2-1)}{4}c$ | $\frac{(k^2-1)^2}{16}c^2$ |
| $SU(k)/SO(k)Z_k$ | $\frac{1}{2}(k-1)(k+2)$ | $\frac{k^2(k-1)(k+2)}{32}c$ | $\frac{k^3(k-1)(k+2)^2}{512}c^2$ |
| $SU(2k)/Sp(k)Z_{2k}$ | $(k-1)(2k+1)$ | $\frac{k^2(k-1)(2k+1)}{4}c$ | $\frac{k^3(k-1)^2(2k+1)}{16}c^2$ |
| E_6/F_4Z_3 | 26 | $\frac{637}{4}c$ | $\frac{3185}{32}c^2$ |
| T^n | n | 0 | 0 |



THEOREM 4.2. *Each compact n -dimensional totally real parallel Einstein submanifold M_0 of $QP^n(c)$ is uniquely determined by its $\text{spec}(J)$.*

PROOF. We treat the cases $n \neq 2, 5$, $n = 2$ and $n = 5$ separately.

a) If $n \neq 2, 5$, by Theorem 4.1, it is enough to prove that $\text{spec}(J)$ determines the pair of Riemannian invariants $(\tau, \|R\|^2)$ of M . In fact, suppose that $\text{spec}(M_0, J) = \text{spec}(M'_0, J)$, where M_0, M'_0 are n -dimensional compact totally real parallel Einstein submanifolds of $QP^n(c)$. Then, since $n \neq 2$, (3.14) and (3.15) imply that $\tau_0 = \tau'_0$. M_0, M'_0 being Einstein manifolds having the same dimension, it follows that $\|\varrho_0\|^2 = \|\varrho'_0\|^2$. Thus, since $n \neq 5$, taking into account that $\|R_0\|^2$ and $\|R'_0\|^2$ are constant, from (3.16) we get $\|R_0\|^2 = \|R'_0\|^2$.

b) If $n = 2$, from Table I we see that $M_0 = \mathbf{RP}^2(\frac{c}{4})$ or $M_0 = T^2$. Suppose that $\text{spec}(\mathbf{RP}^2(\frac{c}{4}), J) = \text{spec}(T^2, J)$. Then, in particular, $a_0(\mathbf{RP}^2(\frac{c}{4})) = a_0(T^2)$ and $a_2(\mathbf{RP}^2(\frac{c}{4})) = a_2(T^2)$, from which it follows easily that c vanishes, which cannot occur.

c) If $n = 5$, then $M_0 = \mathbf{RP}^5(\frac{c}{4}), T^{15}$ or $SU(3)/SO(3)Z_3$. Suppose that $\text{spec}(M_0, J) = \text{spec}(M'_0, J)$. Then, in particular, $a_0(M_0) = a_0(M'_0)$ and $a_1(M_0) = a_1(M'_0)$, from which it follows easily that $\tau_0 = \tau'_0$, which cannot occur, because, as it follows from Table I, for $\mathbf{RP}^5(\frac{c}{4}), T^5$ and $SU(3)/SO(3)Z_3$, we respectively have $\tau = 5c, 0$ and $\frac{45}{16}c$, with $c \neq 0$ \square

We now characterize totally real parallel Einstein submanifolds M_0 of $QP^n(c)$, in the class of all totally real minimal submanifolds, by proving the following

THEOREM 4.3. *Let M be an n -dimensional compact totally real minimal submanifold of $QP^n(c)$. If $\text{spec}(M, J) = \text{spec}(M_0, J)$, $5 < n \leq 17$, then M is isometric to M_0 .*

PROOF. Since $\text{spec}(M, J) = \text{spec}(M_0, J)$, we have $\dim M = \dim M_0 = n$ and, from Theorem 3.3, we get

$$(4.2) \quad \text{vol}(M, g) = \text{vol}(M_0, g_0),$$

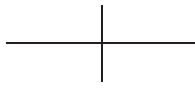
$$(4.3) \quad \int_M \tau dv = \int_{M_0} \tau_0 dv, \quad \int_M \|\sigma\|^2 dv = \int_{M_0} \|\sigma_0\|^2 dv,$$

$$(4.4) \quad \int_M \{2(n-5)\|R\|^2 + 2(30-n)\|\varrho\|^2 + 5(n-4)\tau^2\} dv \\ = \int_{M_0} \{2(n-5)\|R_0\|^2 + 2(30-n)\|\varrho_0\|^2 + 5(n-4)\tau_0^2\} dv.$$

Since τ_0 is constant and $\text{vol}(M) = \text{vol}(M_0)$, we have

$$(4.5) \quad \int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv = \int_M \tau^2 dv - 2\tau_0 \int_{M_0} \tau_0 dv + \int_{M_0} \tau_0^2 dv \\ = \int_M (\tau - \tau_0)^2 dv \geq 0$$





where the equality holds if and only if $\tau = \tau_0$.

Next, let $E = \varrho - \frac{\tau}{n}g$ denote the *Einstein curvature tensor* of (M, g) . Since $\|E\|^2 = \|\varrho\|^2 - \frac{\tau^2}{n}$ and $E_0 = 0$ because M_0 is an Einstein space, (4.4) becomes

$$(4.6) \quad 2(n-5) \left(\int_M \|R\|^2 dv - \int_{M_0} \|R_0\|^2 dv \right) - 2(n-30) \int_M \|E\|^2 dv \\ + \frac{5n^2 - 22n + 60}{2n} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) = 0.$$

Moreover, from (2.7) we also get

$$\frac{1}{2} \Delta \|\sigma\|^2 = \|\nabla' \sigma\|^2 - \|R\|^2 - \|E\|^2 + \frac{1}{n} \tau^2 + (n+1) \frac{c}{4} \tau.$$

Integrating over M , we obtain

$$(4.7) \quad \int_M \|\nabla' \sigma\|^2 dv = \int_M \|R\|^2 dv + \int_M \|E\|^2 dv \\ + \frac{1}{n} \int_M \tau^2 dv - (n+1) \frac{c}{4} \int_M \tau dv.$$

An analogous formula holds for M_0 , with $\nabla' \sigma_0 = E_0 = 0$. Using (4.7) to calculate $\int_M \|R\|^2 dv$, (4.6) becomes

$$(4.8) \quad (n-5) \int_M \|\nabla' \sigma\|^2 dv = \alpha(n) \int_M \|E\|^2 dv + \beta(n) \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right),$$

where

$$\alpha(n) = 2n - 35, \\ \beta(n) = -\frac{5n^2 - 24n + 70}{2n}.$$

If $5 < n \leq 17$, then $n-5 > 0$, while $\alpha(n), \beta(n) < 0$. Therefore, we get $\nabla' \sigma = 0$, $E = 0$ and $\tau = \tau_0$. Thus, M is an Einstein (compact) totally real parallel submanifold of $QP^n(c)$, with the same $\text{spec}(J)$ of M_0 . So, Theorem 4.2 implies that M is isometric to M_0 . \square

5. Totally real parallel conformally flat submanifolds of QP^n

In this section, by M_0 we shall denote an n -dimensional compact totally real parallel minimal submanifold of $CP^n(c)$, which is conformally flat. In other words, M_0 is one of the manifolds listed in the following Table II:



TABLE II

| M_0 | τ |
|------------------------------|----------------------------|
| $\mathbf{R}P^n(\frac{c}{4})$ | $\frac{n(n-1)}{4}c$ |
| T^n | 0 |
| $S^1 \times S^{n-1}$ | $\frac{(n-2)(n^2-1)}{4n}c$ |

(see [E], [N], [CP]). As we noted in the previous Section 4, these are exactly the n -dimensional conformally flat totally real minimal parallel submanifolds of the quaternionic projective space $QP^n(c)$. We now prove the following

THEOREM 5.1. *Let M be an n -dimensional compact totally real minimal submanifold of $QP^n(c)$. If $\text{spec}(M, J) = \text{spec}(M_0, J)$ and $18 \leq n \leq 33$, then M is isometric to M_0 .*

PROOF. The proof is similar to the one of Theorem 4.3. In particular, formulas (4.2)–(4.5) still hold. Here, we use the *conformal curvature tensor* C of (M, g) to rewrite (4.4). Since the curvature invariant $\|R\|^2$ is given by

$$(5.1) \quad \|R\|^2 = \|C\|^2 + \frac{4}{n-2}\|\varrho\|^2 - \frac{2}{(n-1)(n-2)}\tau^2,$$

from (4.4) we obtain

$$(5.2) \quad (n-5) \int_M \|C\|^2 dv - \frac{n^2 - 36n + 80}{n-2} \left(\int_M \|\varrho\|^2 dv - \int_{M_0} \|\varrho_0\|^2 dv \right) \\ + \frac{5n^3 - 35n^2 + 66n - 20}{2(n-1)(n-2)} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) = 0.$$

Moreover, from (2.7) and (5.1), we also have

$$\frac{1}{2}\Delta\|\sigma\|^2 = \|\nabla'\sigma\|^2 - \|C\|^2 - \frac{n+2}{n-2}\|\varrho\|^2 + \frac{2}{(n-1)(n-2)}\tau^2 + (n+1)\frac{c}{4}\tau,$$

from which, by integrating over M , we get

$$\int_M \|\nabla'\sigma\|^2 dv = \int_M \|C\|^2 dv + \frac{n+2}{n-2} \int_M \|\varrho\|^2 dv \\ - \frac{2}{(n-1)(n-2)} \int_M \tau^2 dv - (n+1)\frac{c}{4} \int_M \tau dv$$

and for M_0 , since $\nabla'\sigma_0 = 0$ and $C_0 = 0$, we have

$$\frac{n+2}{n-2} \int_{M_0} \|\varrho_0\|^2 dv = \frac{2}{(n-1)(n-2)} \int_{M_0} \tau_0^2 dv + (n+1)\frac{c}{4} \int_{M_0} \tau_0 dv.$$

Therefore,

$$(5.3) \quad \frac{n+2}{n-2} \left(\int_M \|\varrho\|^2 dv - \int_{M_0} \|\varrho_0\|^2 dv \right) = \int_M \|\nabla' \sigma\|^2 dv - \int_M \|C\|^2 dv \\ + \frac{2}{(n-1)(n-2)} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right).$$

Using (5.3), (5.2) becomes

$$(5.4) \quad \int_M \|\nabla' \sigma\|^2 dv = a(n) \int_M \|C\|^2 dv + b(n) \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right),$$

where

$$a(n) = \frac{2n^2 - 39n + 70}{n^2 - 36n + 80}, \quad b(n) = \frac{5n^4 - 25n^3 - 6n^2 + 184n - 200}{(n-1)(n-2)(n^2 - 36n + 80)}.$$

It is easy to check that if $18 \leq n \leq 33$, then $a(n) < 0$ and $b(n) < 0$. Therefore, we get $\nabla' \sigma = 0$, $C = 0$ and $\tau = \tau_0$, that is, M is a compact conformally flat totally real minimal submanifold of $QP^n(c)$ with parallel second fundamental form. Therefore, M is isometric to one of the manifolds listed in Table II. Since $\tau = \tau_0$, we can conclude that M is isometric to M_0 . \square

Remark that the flat torus T^n is, at the same time, a conformally flat and an Einstein manifold. Therefore, combining Theorems 4.3 and 5.1, we get the following

COROLLARY 5.2. *In the class of all compact totally real minimal submanifolds of $QP^n(c)$, the flat torus T^n is characterized by its $\text{spec}(J)$ when $5 < n \leq 33$.*

Moreover, note that, using formulas (2.4), (3.14) and (3.15), it is easy to show that *in the class of all compact totally real minimal submanifolds of $QP^n(c)$, the real projective space $RP^n(\frac{c}{4})$ is characterized by its $\text{spec}(J)$ for all $n \geq 3$.*

6. Spectral rigidity of totally real submanifolds of small dimension

In this section, we characterize by means of $\text{spec}(J)$ some special n -dimensional totally real submanifolds of QP^n when n is small.

Case of $n = 2$.

PROPOSITION 6.1. *Let M, M' be two compact totally real surfaces of $QP^2(c)$. If $\text{spec}(M, J) = \text{spec}(M', J)$, then M is minimal if and only if M' is minimal.*

PROOF. Since $a_i(M) = a_i(M')$, using (3.11) and (3.12), we get easily

$$\int_M \|H\|^2 dv = \int_{M'} \|H'\|^2 dv,$$

from which the conclusion follows at once. \square

Case of $n = 3$.

As it is well-known, the conformal curvature tensor C vanishes on any three-dimensional Riemannian manifold. Moreover, note that formula (5.4) holds for all $n \neq 2$. Therefore, it is easy to prove the following

THEOREM 6.2. *Let M be a compact minimal totally real submanifold of $QP^n(c)$. If $\text{spec}(M, J) = \text{spec}(M_0, J)$, where $n = 3$ and M_0 is a compact parallel totally real submanifold, then M is isometric to M_0 .*

PROOF. We first remark that, according to Naitoh's classification, if M_0 is a three-dimensional compact totally real parallel submanifolds of $CP^3(c)$ (and hence, of $QP^3(c)$), then M_0 is $\mathbf{R}P^3(\frac{c}{4})$, T^3 or $S^1 \times S^2(k)$, with $k = c/3$. Suppose now $\text{spec}(M, J) = \text{spec}(M_0, J)$. Since $n = 3$, formula (5.4) becomes

$$\int_M \|\nabla'\sigma\|^2 dv = b(3) \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) = -\frac{14}{19} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right),$$

from which it follows $\nabla'\sigma = \tau - \tau_0 = 0$. Thus, since M is parallel and $\tau = \tau_0$, we can conclude that M is isometric to M_0 . \square

Case of $n = 4$.

THEOREM 6.3. *Let M, M_0 be compact minimal totally real submanifolds of $QP^n(c)$, with M_0 parallel and either Einstein or conformally flat. If $n = 4$ and $\text{spec}(M, J) = \text{spec}(M_0, J)$, then*

$$\chi(M) \geq \chi(M_0)$$

and the equality holds if and only if M is isometric to M_0 .

PROOF. The Gauss-Bonnet formula for any 4-dimensional compact manifold M is given by

$$(6.1) \quad \chi(M) = \frac{1}{32\pi^2} \int_M \{ \|R\|^2 - 4\|Q\|^2 + \tau^2 \} dv.$$

Suppose first that M_0 is Einstein. Using $\|Q\|^2 = \|E\|^2 + \tau^2/4$, (6.1) becomes

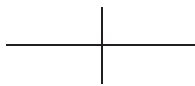
$$(6.2) \quad \chi(M) = \frac{1}{32\pi^2} \int_M \{ \|R\|^2 - 4\|E\|^2 \} dv, \quad \chi(M_0) = \frac{1}{32\pi^2} \int_{M_0} \|R_0\|^2 dv.$$

On the other hand, since $a_i(M, J) = a_i(T^2, J)$, using formulas (3.14), (3.15), (3.16) (for $n = 4$), we obtain

$$(6.3) \quad \int_M \|R\|^2 dv - \int_{M_0} \|R_0\|^2 dv = 26 \int_M \|E\|^2 dv + \frac{13}{2} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right).$$

Using (6.2) and (6.3), we then get

$$(6.4) \quad (32\pi^2) \{ \chi(M) - \chi(M_0) \} = 22 \int_M \|E\|^2 dv + \frac{13}{2} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right).$$



Therefore, $\chi(M) \geq \chi(M_0)$ since $\|E\|^2 \geq 0$ and $\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \geq 0$.

In particular, if $\chi(M) = \chi(M_0)$, then (6.4) yields $E = \tau - \tau_0 = 0$. Moreover, by (6.3) it follows $\int_M \|R\|^2 dv = \int_{M_0} \|R_0\|^2 dv$ and hence, by (4.7), $\int_M \|\nabla' \sigma\|^2 dv = \int_{M_0} \|\nabla' \sigma_0\|^2 dv = 0$. So, M is also parallel and, as in the proof of Theorem 4.3, we can conclude that M is isometric to M_0 .

If M_0 is conformally flat, the proof is similar. Using the conformal curvature tensor C , we eventually get

$$(6.5) \quad (32\pi^2)\{\chi(M) - \chi(M_0)\} = \frac{11}{12} \int_M \|C\|^2 dv + \frac{13}{36} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) \geq 0,$$

where the equality holds if and only if $C = \tau - \tau_0 = 0$. Moreover, M is parallel and the conclusion then follows as in the proof of Theorem 5.1 \square

In particular, from Theorem 6.3 we obtain at once the following

COROLLARY 6.4. *In the class of all 4-dimensional compact minimal totally real submanifolds of $QP^4(c)$, of non-positive Euler number, T^4 and $S^1 \times S^3(k)$, with $k = 5c/16$, are completely determined by their $\text{spec}(J)$.*

Case of $n = 5$.

Using the same methods of the proof of Theorem 4.3, we can easily prove the following result for five-dimensional totally real submanifolds of $QP^5(c)$.

PROPOSITION 6.5. *Let M be an n -dimensional compact totally real minimal submanifold of $QP^n(c)$. If $\text{spec}(M, J) = \text{spec}(M_0, J)$, with M_0 parallel and Einstein and $n = 5$, then M is also Einstein and $\tau = \tau_0$.*

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