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Spectral Geometry of the Jacobi Operator of Totally Real Submanifolds of QP^n

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Abstract. We calculate some invariants determined by the spectrum of the Jacobi operator J of n-dimensional totally real submanifolds of the quaternionic projective space QP^n and we use such invariants to characterize parallel submanifolds of QP^n.

1. Introduction

The Jacobi operator J is a second order elliptic operator associated to an isometric immersion of a compact Riemannian manifold M into a Riemannian manifold $\bar{M}$. J is defined on the space of smooth sections of the normal bundle $TM^{\perp}$ by the formula

$$J = D + \tilde{R} - \tilde{A} ,$$

where D is the rough Laplacian of the normal connection $\nabla^{\perp}$ on $TM^{\perp}$, $\tilde{R}$ and $\tilde{A}$ are linear transformations of $TM^{\perp}$ defined by means of a partial Ricci tensor of $\bar{M}$ and of the second fundamental form A, respectively. J appears in the formula which gives the second variation for the area function of a compact minimal submanifold (see [S]). For this reason, J is also called the *second variation operator*. Its spectrum, denoted by

$$\operatorname{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\} ,$$

is discrete, as a consequence of the compactness of M.

The Riemannian invariants determined by $\operatorname{spec}(M, J)$ have been calculated for several types of isometric immersions of submanifolds into real or complex space forms (see [D], [H], [Sh], [CP], [C]). Moreover, a similar study was made about spectral geometry determined by the Jacobi operator associated to the energy of a harmonic map ([CgY], [KPa], [KKiPa], [NiTV], [U], [Y]).

In this paper, we determine the first three terms of the asymptotic expansion for the partition function associated to the spectrum of the Jacobi operator of an n-dimensional totally

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real submanifold of the quaternionic projective space QP^n and we use the corresponding Riemannian spectral invariants to characterize n-dimensional totally real parallel submanifolds of QP^n.

The paper is organized in the following way. In Section 2, we shall recall some basic results about QP^n and n-dimensional totally real submanifolds of QP^n. In Section 3, we shall compute the first three terms of the asymptotic expansion for the partition function associated to spec(M, J), M being an n-dimensional totally real submanifold of QP^n. In Sections 4 and 5, we shall characterize totally real parallel submanifolds of QP^n, which are Einstein and conformally flat, respectively. In Section 6 we shall investigate the spectral rigidity of totally real parallel submanifolds of QP^n for small dimensions n.

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2. Totally real submanifolds of QP^n

Let $(\bar{M}, g)$ be a $4n$-dimensional quaternionic Riemannian manifold and V the three-dimensional vector bundle of tensors of type $(1, 1)$ with local basis of almost Hermitian structures I_1, I_2, I_3, satisfying

a) $I_1 I_2 = -I_2 I_1 = I_3,\ I_2 I_3 = -I_3 I_2 = I_1,\ I_3 I_1 = -I_1 I_3 = I_2,\ I_1^2 = I_2^2 = I_3^2 = -1$;

b) for any cross-section ξ of V, $\bar{\nabla}_X \xi$ is also a cross-section of V, where X is a vector field on M and $\bar{\nabla}$ is the Riemannian connection of $\bar{M}$.

If X is a unit vector on $\bar{M}$, the *quaternionic section* determined by X is the 4-plane $Q(X)$ spanned by X, $I_1 X$, $I_2 X$ and $I_3 X$. If $Q(X)$ and $Q(Y)$ are orthogonal, the plane spanned by X and Y is called a *totally real plane*. Any 2-plane in a quaternionic section is called a *quaternionic plane* and its sectional curvature is called *quaternionic sectional curvature*. A *quaternionic space form* is a quaternionic manifold of constant quaternionic sectional curvature. In particular, by QP^n we denote the $4n$-dimensional quaternionic projective space, equipped with the Riemannian metric $\bar{g}$ of constant quaternionic sectional curvature $c > 0$. Its curvature tensor $\bar{R}$, taken with the sign convention

$$\bar{R}(X, Y) = \bar{\nabla}_{[X,Y]} - [\bar{\nabla}_X, \bar{\nabla}_Y],$$

satisfies

$$\begin{aligned}\bar{R}(X, Y, Z, W) = \frac{c}{4}\{&\bar{g}(X, Z)\bar{g}(Y, W) - \bar{g}(Y, Z)\bar{g}(X, W)\\ &+ \bar{g}(I_1X, Z)\bar{g}(I_1Y, W) - \bar{g}(I_1Y, Z)\bar{g}(I_1X, W) + 2\bar{g}(I_1X, Y)\bar{g}(I_1Z, W)\\ &+ \bar{g}(I_2X, Z)\bar{g}(I_2Y, W) - \bar{g}(I_2Y, Z)\bar{g}(I_2X, W) + 2\bar{g}(I_2X, Y)\bar{g}(I_2Z, W)\\ &+ \bar{g}(I_3X, Z)\bar{g}(I_3Y, W) - \bar{g}(I_3Y, Z)\bar{g}(I_3X, W) + 2\bar{g}(I_3X, Y)\bar{g}(I_3Z, W)\}.\end{aligned}$$

Note that our convention for the sign of the curvature tensor is opposed to the one used by Simons in [S]. We refer to [I] for more details about quaternionic manifolds.

Next, let (M, g) be an n-dimensional Riemannian manifold, isometrically immersed into $(QP^n, \bar{g})$. By definition, M is a *totally real submanifold* of QP^n if each tangent 2-plane of M is mapped by the isometric immersion into a totally real plane of QP^n. We shall denote by ∇ and R the Levi Civita connection and the curvature tensor of M, respectively. The normal connection is given by

$$\begin{aligned}\nabla^{\perp} : TM \times TM^{\perp} &\longrightarrow TM^{\perp} \\ (X, \xi) &\longmapsto \nabla^{\perp}_X \xi ,\end{aligned}$$

where $\nabla^{\perp}_X \xi$ denotes the normal component of $\bar{\nabla}_X \xi$. The second fundamental form σ and the Weingarten operator A are respectively defined by

$$\sigma(X, Y) = \bar{\nabla}_X Y - \nabla_X Y , \quad A_\xi X = -\bar{\nabla}_X \xi + \nabla^{\perp}_X \xi$$

for all $X, Y \in TM$ and $\xi \in TM^{\perp}$. Moreover, $\bar{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$.

Let $R^{\perp}$ denote the curvature tensor associated to the normal connection $\nabla^{\perp}$. The curvature tensors R, $\bar{R}$ and $R^{\perp}$ satisfy the Gauss and the Ricci equations:

$$\begin{aligned}R(X, Y, Z, W) &= g(R(X, Y)Z, W) = \bar{R}(X, Y, Z, W) \\ &\quad + \bar{g}(\sigma(X, Z), \sigma(Y, W)) - \bar{g}(\sigma(Y, Z), \sigma(X, W)) , \\ R^{\perp}(X, Y, \xi, \eta) &= \bar{g}(R^{\perp}(X, Y)\xi, \eta) = \bar{R}(X, Y, \xi, \eta) - g([A_\xi, A_\eta]X, Y) ,\end{aligned}$$

where $[A_\xi, A_\eta] = A_\xi \circ A_\eta - A_\eta \circ A_\xi$ for all $X, Y, Z, W \in TM$ and $\xi, \eta \in TM^{\perp}$.

Let $\{e_1, \cdots, e_n, e_{I_1(1)} = I_1 e_1, \cdots, e_{I_1(n)} = I_1 e_n, e_{I_2(1)} = I_2 e_1, \cdots, e_{I_2(n)} = I_2 e_n, e_{I_3(1)} = I_3 e_1, \cdots, e_{I_3(n)} = I_3 e_n\}$ be a local orthonormal frame on QP^n such that, restricted to M, the vector fields $e_1, .., e_n$ are tangent to M. With respect to such frame field, we have

$$I_1 = \begin{pmatrix} 0 & -E & 0 & 0 \\ E & 0 & 0 & 0 \\ 0 & 0 & 0 & -E \\ 0 & 0 & E & 0 \end{pmatrix} , \quad I_2 = \begin{pmatrix} 0 & 0 & -E & 0 \\ 0 & 0 & 0 & E \\ E & 0 & 0 & 0 \\ 0 & -E & 0 & 0 \end{pmatrix} , \quad I_3 = \begin{pmatrix} 0 & 0 & 0 & -E \\ 0 & 0 & -E & 0 \\ 0 & E & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix} ,$$

where E is the $(n \times n)$-identity matrix. We shall use the following convention for the range of indices:

$A, B, C, D = 1, \cdots, n, I_1(1), \cdots, I_1(n), I_2(1), \cdots, I_2(n), I_3(1), \cdots, I_3(n)$;
$i, j, k, h = 1, \cdots, n$;
$\alpha, \beta = I_1(1), \cdots, I_1(n), I_2(1), \cdots, I_2(n), I_3(1), \cdots, I_3(n)$;
$\varphi, \psi = I_1, I_2, I_3$.

Putting $A_\alpha = A_{e_\alpha}$, $A_\alpha e_i = h^\alpha_{ij} e_j$ and $R^{\perp}_{ij\alpha\beta} = R^{\perp}(e_i, e_j, e_\alpha, e_\beta)$, the Gauss and Ricci equations become

$$R_{ijkh} = \frac{c}{4}(\delta_{ik}\delta_{jh} - \delta_{jk}\delta_{ih}) + \sum_\alpha (h^\alpha_{ik} h^\alpha_{jl} - h^\alpha_{jk} h^\alpha_{il}) \tag{2.1}$$

and

$$R^{\perp}_{ij\alpha\beta} = \frac{c}{4}((I_1)_{i\alpha}(I_1)_{j\beta} - (I_1)_{j\alpha}(I_1)_{i\beta} + (I_2)_{i\alpha}(I_2)_{j\beta} - (I_2)_{j\alpha}(I_2)_{i\beta} + (I_3)_{i\alpha}(I_3)_{j\beta} - (I_3)_{j\alpha}(I_3)_{i\beta}) - g([A_\alpha, A_\beta]e_i, e_j)\,. \tag{2.2}$$

Note that, for all i, j, k and φ, we have

$$h^{\varphi(i)}_{jk} = h^{\varphi(j)}_{ik} = h^{\varphi(k)}_{ij}\,.$$

The mean curvature vector is defined by

$$H = \mathrm{trace}(\sigma) = \sum_i \sigma(e_i, e_i) = \sum_{i,\varphi} \mathrm{tr} A_{\varphi(i)} e_{\varphi(i)} = \sum_\alpha \mathrm{tr} A_\alpha e_\alpha\,.$$

M is said to be *minimal* if $H = 0$, *totally geodesic* if $\sigma = 0$, *parallel* (or *with parallel second fundamental form*) if $\nabla'\sigma = 0$, where

$$(\nabla'_X \sigma)(Y, Z) = \nabla^{\perp}_X(\sigma(Y, Z)) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z)\,.$$

For the Ricci tensor ϱ of M, from (2.1) we easily obtain

$$\varrho_{ik} = \sum_j R_{ijkj} = \frac{c}{4}(n-1)\delta_{ik} + \sum_\alpha \{(\mathrm{tr} A_\alpha) h^\alpha_{ik}\} - \sum_\alpha h^\alpha_{il} h^\alpha_{kl} \tag{2.3}$$

and for the scalar curvature τ of M, we have

$$\tau = \sum_i \varrho_{ii} = n(n-1)\frac{c}{4} + ||H||^2 - ||\sigma||^2\,, \tag{2.4}$$

where $||\sigma||^2 = \sum \mathrm{tr} A^2_\alpha$ and $||H||^2 = \sum (\mathrm{tr} A_\alpha)^2$.

We also refer to [ChH] for more details. We now prove the following

LEMMA 2.1. *Let M be an n-dimensional totally real submanifold of QP^n. Then*

$$||R||^2 = c\tau - n(n-1)\frac{c^2}{8} - \sum_{\alpha,\beta} \mathrm{tr}[A_\alpha, A_\beta]^2\,. \tag{2.5}$$

If in addition M is minimal, then

$$||\varrho||^2 = 2(n-1)\frac{c}{4}\tau - n(n-1)^2\frac{c^2}{16} + \sum_{\alpha,\beta}(\mathrm{tr} A_\alpha A_\beta)^2\,, \tag{2.6}$$

$$\frac{1}{2}\Delta||\sigma||^2 = ||\nabla'\sigma||^2 - ||R||^2 - ||\varrho||^2 + (n+1)\frac{c}{4}\tau\,. \tag{2.7}$$

PROOF. (2.5) follows from (2.1), taking into account (2.4), once we note that

$$\sum_\alpha \left(\sum_{i,j,k,h} (h^\alpha_{ik} h^\alpha_{jh} - h^\alpha_{jk} h^\alpha_{ih}) \right)^2 = -\sum_{\alpha,\beta} \mathrm{tr}[A_\alpha, A_\beta]^2$$

and

$$\sum_{\alpha,i,j}(h^\alpha_{ii}h^\alpha_{jj}-(h^\alpha_{ij})^2)=||H||^2-||\sigma||^2\,.$$

Next, suppose that M is minimal. Then $\mathrm{tr}A_\alpha=0$ for all α and (2.3) reduces to

(2.8) $$\varrho_{ik}-\frac{c}{4}(n-1)\delta_{ik}=-\sum_{\alpha,l}h^\alpha_{il}h^\alpha_{kl}=-\sum_{i,k,\varphi}\mathrm{tr}A_{\varphi(i)}A_{\varphi(k)}\,.$$

In [ChH], the following formula was proved for any m-dimensional totally real submanifold of QP^n:

(2.9) $$\frac{1}{2}\Delta||\sigma||^2=||\nabla'\sigma||^2+\sum\sum_{\alpha,\beta}\mathrm{tr}[A_\alpha,A_\beta]^2-\sum_{\alpha,\beta}(\mathrm{tr}A_\alpha A_\beta)^2$$
$$+n\frac{c}{4}||\sigma||^2+\frac{c}{4}\sum_{i,\varphi}(\mathrm{tr}A^2_{\varphi(i)})\,.$$

Next, put $S_{\alpha,\beta}=\sum_{\alpha,\beta}\mathrm{tr}(A_\alpha A_\beta)$. Since $S_{\alpha,\beta}$ is a symmetric matrix, it can be diagonalized for a suitable choice of $\{e_\alpha\}$. Hence, we may assume that $\mathrm{tr}(A_\alpha A_\beta)=0$ for $\alpha\neq\beta$ (see also [CH, p. 198]). In particular, we then have

$$\sum_{i,k,\varphi}\mathrm{tr}A_{\varphi(i)}A_{\varphi(k)}=\sum_{\alpha,\beta}\mathrm{tr}A_\alpha A_\beta\,,\quad \sum_{i,k,\varphi}(\mathrm{tr}A_{\varphi(i)}A_{\varphi(k)})^2=\sum_{\alpha,\beta}(\mathrm{tr}A_\alpha A_\beta)^2\,.$$

Hence, (2.6) follows from (2.8). Moreover, since in our case $m=n$ and (2.4)–(2.6) hold, from (2.9) we get (2.7). □

3. Spectral invariants of the Jacobi operator

Let M be an n-dimensional Riemannian manifold immersed in a Riemannian manifold $\bar{M}$ of dimension $\bar{n}=n+r$. The normal bundle $TM^\perp$ is a real r-dimensional vector bundle on M, with inner product induced by the metric $\bar{g}$ of $\bar{M}$. Let D denote the so-called *rough Laplacian* associated to the normal connection $\nabla^\perp$ of $TM^\perp$, that is,

$$D\xi=-\nabla^\perp_{e_i}\nabla^\perp_{e_i}\xi+\nabla^\perp_{\nabla_{e_i}e_i}\xi\,,$$

where ξ is a section of $TM^\perp$. Next, let $\tilde{A}$ be the *Simons operator* defined in [S] by

$$\bar{g}(\tilde{A}\xi,\eta)=\mathrm{tr}(A_\xi\circ A_\eta)\,,$$

for $\xi,\eta\in TM^\perp$. Moreover, we consider the operator $\tilde{R}$ defined by

$$\tilde{R}(\xi)=-\sum_{i=1}^n(\bar{R}(e_i,\xi)e_i)^\perp\,,$$

where $(\bar{R}(e_i, \xi)e_i)^{\perp}$ denotes the normal component of $\bar{R}(e_i, \xi)e_i$.

The *Jacobi operator* (or *second variation operator*), acting on cross-sections of $TM^{\perp}$, is the second order elliptic differential operator J defined by (see [S] or [D])

$$J : TM^{\perp} \longrightarrow TM^{\perp}$$
$$\xi \longmapsto (D - \tilde{A} + \tilde{R})\xi .$$

When M is compact, we can define an inner product for cross-sections on $TM^{\perp}$, by

$$\langle \xi, \eta \rangle = \int_M \bar{g}(\xi, \eta) dv$$

and J is self-adjoint with respect to this product. Moreover, J is strongly elliptic and it has an infinite sequence of eigenvalues, with finite multiplicities, denoted by

$$\mathrm{spec}(M, J) = \{\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots + \uparrow \infty\} .$$

The partition function $Z(t) = \sum_{i=1}^{\infty} \exp(-\lambda_i t)$ has the asymptotic expansion

$$Z(t) \sim (4\pi t)^{-n/2}\{a_0(J) + a_1(J)t + a_2(J)t^2 + \cdots\} .$$

By Gilkey's results [G] (see also [D] and [H]), it follows that the coefficients a_0, a_1 and a_2 are given by the following

THEOREM 3.1 ([G]). *We have*

$$a_0 = r\mathrm{vol}(M) ,$$

$$a_1 = \frac{r}{6}\int_M \tau dv + \int_M \mathrm{tr}\tilde{E} dv ,$$

$$a_2 = \frac{r}{360}\int_M \{2||R||^2 - 2||\varrho||^2 + 5\tau^2\} dv + \frac{1}{360}\int_M \{-30||R^{\perp}||^2$$
$$+ \mathrm{tr}(60\tau\tilde{E} + 180\tilde{E}^2)\} dv ,$$

where $\tilde{E} = \tilde{A} - \tilde{R}$.

We now consider the case of an n-dimensional totally real submanifold of $QP^n(c)$ and we compute explicitly the coefficients a_0, a_1 and a_2 in terms of invariants depending on the curvature of M and its isometric immersion in QP^n.

PROPOSITION 3.2. *Let M be an n-dimensional totally real submanifold of QP^n. Then*

$$||R^{\perp}||^2 = ||R||^2 + n(n-1)c^2 , \tag{3.1}$$

$$\mathrm{tr}\tilde{E} = ||\sigma||^2 + \frac{3}{4}n(n+1)c , \tag{3.2}$$

$$\mathrm{tr}\tilde{E}^2 = \frac{3}{16}n(n+1)^2c^2 + (n+1)\frac{c}{2}||\sigma||^2 + \sum_{\alpha,\beta} \mathrm{tr}(A_\alpha A_\beta)^2 . \tag{3.3}$$

If in addition M is minimal, then

$$\operatorname{tr}\tilde{E}^2 = ||\varrho||^2 + (n+1)c||\sigma||^2 + \frac{1}{8}n(n^2+4n+1)c^2 . \tag{3.4}$$

PROOF. From (2.2) we get

$$||R^{\perp}||^2 = \sum_{i,j,\alpha,\beta} (R^{\perp}_{ij\alpha\beta})^2 = R_1 + R_2 + R_3 , \tag{3.5}$$

with

$$\begin{aligned}
R_1 &= \frac{c^2}{16}\sum\{(I_1)_{i\alpha}(I_1)_{j\beta} - (I_1)_{j\alpha}(I_1)_{i\beta} + (I_2)_{i\alpha}(I_2)_{j\beta} - (I_2)_{j\alpha}(I_2)_{i\beta} \\
&\quad + (I_3)_{i\alpha}(I_3)_{j\beta} - (I_3)_{j\alpha}(I_3)_{i\beta}\}^2 \\
&= \frac{c^2}{16}\sum\{(I_1)_{iI_1(k)}(I_1)_{jI_1(h)} - (I_1)_{jI_1(k)}(I_1)_{iI_1(h)} + (I_2)_{iI_2(k)}(I_2)_{jI_2(h)} \\
&\quad - (I_2)_{jI_2(k)}(I_2)_{iI_2(h)} + (I_3)_{iI_3(k)}(I_3)_{jI_3(h)} - (I_3)_{jI_3(k)}(I_3)_{iI_3(h)}\}^2 \\
&= \frac{c^2}{16}\sum((3\delta_{ik}\delta_{jh} - \delta_{ih}\delta_{jk}))^2 = \frac{9}{8}n(n-1)c^2, \\
R_2 &= \sum(g([A_\alpha, A_\beta]e_i, e_j))^2 = \sum_{\alpha,\beta}||[A_\alpha, A_\beta]||^2 = -\sum_{\alpha.\beta}\operatorname{tr}[A_\alpha, A_\beta]^2 ,
\end{aligned}$$

where we used the fact that $[A_\alpha, A_\beta]$ is skew-symmetric, and

$$\begin{aligned}
R_3 &= -\frac{c}{2}\sum\{(I_1)_{iI_1(k)}(I_1)_{jI_1(h)} - (I_1)_{jI_1(k)}(I_1)_{iI_1(h)} \\
&\quad + (I_2)_{iI_2(k)}(I_2)_{jI_2(h)} - (I_2)_{jI_2(k)}(I_2)_{iI_2(h)} \\
&\quad + (I_3)_{iI_3(k)}(I_3)_{jI_3(h)} - (I_3)_{jI_3(k)}(I_3)_{iI_3(h)}\}g([A_{\varphi(k)}, A_{\varphi(h)}]e_i, e_j) \\
&= -\frac{c}{2}\sum\{g([A_{\varphi(i)}, A_{\varphi(j)}]e_i, e_j) - g([A_{\varphi(j)}, A_{\varphi(i)}]e_i, e_j)\} \\
&= -c\sum g([A_{\varphi(i)}, A_{\varphi(j)}]e_i, e_j) \\
&= -c\sum\{g(A_{\varphi(j)}e_i, A_{\varphi(i)}e_j) - g(A_{\varphi(i)}e_i, A_{\varphi(j)}e_j)\} \\
&= ||\sigma||^2 - ||H||^2 .
\end{aligned}$$

Then (3.1) follows from (3.5), taking into account (2.3) and (2.4).

Next, using the Ricci equation (2.2), we easily obtain

$$\tilde{R}(\xi) = -(n+1)\frac{c}{4}\xi$$

and hence,

$$\mathrm{tr}\tilde{R} = -\frac{3}{4}n(n+1)c\,, \tag{3.6}$$

$$\mathrm{tr}\tilde{R}^2 = \frac{3}{16}n(n+1)^2c^2\,, \tag{3.7}$$

$$\mathrm{tr}\tilde{R}\circ\tilde{A} = -\frac{1}{4}(n+1)c\,\mathrm{tr}\tilde{A}\,. \tag{3.8}$$

Next, by the definition of $\tilde{A}$, we get

$$\mathrm{tr}\tilde{A} = \sum_{\alpha}\bar{g}(\tilde{A}e_\alpha, e_\alpha) = \sum_{i,\alpha}\bar{g}(A_\alpha e_i, A_\alpha e_i) = ||A||^2 = ||\sigma||^2 \tag{3.9}$$

and

$$\mathrm{tr}\tilde{A}^2 = \sum_{\alpha}\bar{g}(\tilde{A}e_\alpha, \tilde{A}e_\alpha) = \sum_{\alpha,\beta}(\bar{g}(A_\alpha, A_\beta))^2 = \sum_{\alpha,\beta}(\mathrm{tr}(A_\alpha A_\beta))^2\,. \tag{3.10}$$

Therefore, since $\mathrm{tr}\tilde{E} = \mathrm{tr}\tilde{A} - \mathrm{tr}\tilde{R}$ and $\mathrm{tr}\tilde{E}^2 = \mathrm{tr}(\tilde{A}^2 - 2\tilde{R}\circ\tilde{A} + \tilde{R}^2)$, from (3.6)–(3.10) we get (3.2) and (3.3).

Finally, if M is minimal, then we obtain (3.4) from (3.3), taking into account (2.6). □

Combining Theorem 3.1 and Proposition 3.2, we get

THEOREM 3.3. *On an n-dimensional totally real submanifold M of $QP^n(c)$, the first coefficients of the asymptotic expansion of the partition function of the Jacobi operator are given by*

$$a_0 = 3n\,\mathrm{vol}(M)\,, \tag{3.11}$$

$$\begin{aligned} a_1 &= \frac{n}{2}\int_M \tau\,dv + \int_M ||\sigma||^2 dv + \frac{3}{4}n(n+1)c\,\mathrm{vol}(M) \\ &= \frac{n-2}{2}\int_M \tau\,dv + \int_M ||H||^2 dv + n(n+1)c\,\mathrm{vol}(M)\,, \end{aligned} \tag{3.12}$$

$$\begin{aligned} a_2 = {} & \frac{n}{120}\int_M \{2||R||^2 - 2||\varrho||^2 + 5\tau^2\}dv + \frac{1}{120}\int_M \{-10||R||^2 \\ & -10n(n-1)c^2 + 20\tau(||\sigma||^2 + \frac{3}{4}n(n+1)c) \\ & +60[(n+1)\frac{c}{2}||\sigma||^2 + \frac{3}{16}n(n+1)^2c^2 + \mathrm{tr}\tilde{A}^2]\}dv\,. \end{aligned} \tag{3.13}$$

If in addition M is minimal, then

$$a_0 = 3n\,\mathrm{vol}(M)\,, \tag{3.14}$$

(3.15) $$a_1 = \frac{n-2}{2}\int_M \tau\, dv + n(n+1)c\ \mathrm{vol}(M)\,,$$

(3.16) $$a_2 = \frac{1}{120}\int_M \{2(n-5)||R||^2 - 2(n-30)||\varrho||^2 + 5(n-4)\tau^2\}dv$$
$$+k_1(n)c\int_M \tau\, dv + k_2(n)c^2\mathrm{vol}(M)\,,$$

where k_1 and k_2 are constants depending on n.

4. Totally real parallel Einstein submanifolds of QP^n

K. Tsukada [Ts] classified parallel submanifolds of $QP^n(c)$. In particular, he proved that if M_0 is a totally real parallel submanifold of $QP^n(c)$, then M_0 is either

(R-R) a totally real submanifold contained in a totally real totally geodesic submanifold of $QP^n(c)$, or

(R-C) a totally real submanifold contained in a totally complex totally geodesic submanifold of $QP^n(c)$

(see [Ts, Theorem 3.10]). In general, a totally real submanifold M_0 of $QP^n(c)$ has dimension $m \le n$. If M_0 is parallel and $\dim M_0 = n$, then M_0 is not of type (R-R), unless M_0 itself is a totally real totally geodesic submanifold of $QP^n(c)$, which is also of type (R-C). Therefore, M_0 is an n-dimensional totally real parallel submanifold M_0 of the quaternionic projective space $QP^n(c)$ if and only if it is an n-dimensional totally real parallel submanifold M_0 of the complex projective space $CP^n(c)$.

Totally real parallel submanifolds of $CP^n(c)$ have been classified by H. Naitoh [N]. We now synthesize some basic ideas of [N], referring to this paper for more details.

Let M be a simply connected Riemannian manifold, admitting a totally real parallel isometric immersion into $CP^n(c)$. In other words, M is the universal covering of a complete totally real submanifold M_0 embedded into $CP^n(c)$. If M as no Euclidean factor, then M is irreducible and of compact type [N, Section 4]. Note that, as it is well-known, an irreducible symmetric Riemannian manifold is Einstein. More explicitly, M must be one of the following:

(4.1) $$SO(n+1)/SO(n)\ (n\ge 2)\,,\quad SU(k)\,,\quad (k\ge 3)\,,$$
$$SU(k)/SO(k)\,,\quad (k\ge 3)\,,\quad SU(2k)/Sp(k)\,,\quad (k\ge 3)\,,\quad E_6/F_4\,,$$

the metric on M is determined uniquely by the constant c (the holomorphic sectional curvature of CP^n) and for each of these spaces there exists exactly one quotient which is a complete totally real submanifold M_0 embedded into $CP^n(c)$. Note that, since M is the universal covering of M_0, the Riemannian manifolds M and M_0 have the same Riemannian curvature invariants. Some of these invariants were computed explicitly for M in [C].

The embedded totally real parallel submanifolds M_0 of $CP^n(c)$ corresponding to the spaces listed in (4.1) could be deduced from Section 5 of [N], where the immersions were explicitly described. On the other hand, n-dimensional totally real submanifolds of $CP^n(c)$ correspond exactly, in the framework of symplectic geometry, to the so-called *Lagrangian submanifolds*. In order to study their Hamiltonian stability, compact minimal Lagrangian submanifolds of $CP^n(c)$, with parallel second fundamental form, were explicitly computed by A. Amarzaya and Y. Ohnita [AO]. Besides $\mathbf{R}P^n(\frac{c}{4})$, the totally geodesic one, and the flat torus T^n (which corrresponds to the Euclidean case), they are the followings:

$$SU(k)/Z_k\,, \quad SU(k)/SO(k)Z_k\,, \quad SU(2k)/Sp(k)Z_{2k}\,, \quad E_6/F_4Z_3\,.$$

If M admits a Euclidean factor and we suppose that M is Einstein, it is easy to show that the scalar curvature of M vanishes and so, M itself is Euclidean (we can refer to [C] for more details). In particular, if the corresponding embedded submanifold M_0 is compact, then M_0 is the n-dimensional flat torus, T^n.

Therefore, combining the results of [AO] and [C], we obtain the following table, which describes all n-dimensional compact totally real parallel Einstein submanifolds embedded into $CP^n(c)$ (and hence, of $QP^n(c)$).

It is easy to check that for two of such manifolds, having the same dimension, it never occurs that the pairs of Riemannian curvature invariants $(\tau, ||R||^2)$ coincide. Therefore, we have the following

THEOREM 4.1. *Each compact n-dimensional totally real parallel Einstein submanifold M_0 of $QP^n(c)$ is uniquely determined by the pair of Riemannian curvature invariants* $(\tau, ||R||^2)$.

Taking into account formulas (3.14)–(3.16) and Theorem 4.1, we can now prove the following

TABLE I

M	dim	τ	$\|\|R\|\|^2$
$\mathbf{R}P^n(\frac{c}{4})$	n	$\frac{n(n-1)}{4}c$	$\frac{n(n-1)}{8}c^2$
$SU(k)/Z_k$	k^2-1	$\frac{(k^2-1)}{4}c$	$\frac{(k^2-1)^2}{16}c^2$
$SU(k)/SO(k)Z_k$	$\frac{1}{2}(k-1)(k+2)$	$\frac{k^2(k-1)(k+2)}{32}c$	$\frac{k^3(k-1)(k+2)^2}{512}c^2$
$SU(2k)/Sp(k)Z_{2k}$	$(k-1)(2k+1)$	$\frac{k^2(k-1)(2k+1)}{4}c$	$\frac{k^3(k-1)^2(2k+1)}{16}c^2$
E_6/F_4Z_3	26	$\frac{637}{4}c$	$\frac{3185}{32}c^2$
T^n	n	0	0

THEOREM 4.2. *Each compact n-dimensional totally real parallel Einstein submanifold M_0 of $QP^n(c)$ is uniquely determined by its* spec(J).

PROOF. We treat the cases $n \neq 2, 5$, $n = 2$ and $n = 5$ separately.

a) If $n \neq 2, 5$, by Theorem 4.1, it is enough to prove that spec(J) determines the pair of Riemannian invariants $(\tau, ||R||^2)$ of M. In fact, suppose that spec$(M_0, J) =$ spec(M'_0, J), where M_0, M'_0 are n-dimensional compact totally real parallel Einstein submanifolds of $QP^n(c)$. Then, since $n \neq 2$, (3.14) and (3.15) imply that $\tau_0 = \tau'_0$. M_0, M'_0 being Einstein manifolds having the same dimension, it follows that $||\varrho_0||^2 = ||\varrho'_0||^2$. Thus, since $n \neq 5$, taking into account that $||R_0||^2$ and $||R'_0||^2$ are constant, from (3.16) we get $||R_0||^2 = ||R'_0||^2$.

b) If $n = 2$, from Table I we see that $M_0 = \mathbf{R}P^2(\frac{c}{4})$ or $M_0 = T^2$. Suppose that spec$(\mathbf{R}P^2(\frac{c}{4}), J) =$ spec(T^2, J). Then, in particular, $a_0(\mathbf{R}P^2(\frac{c}{4})) = a_0(T^2)$ and $a_2(\mathbf{R}P^2(\frac{c}{4})) = a_2(T^2)$, from which it follows easily that c vanishes, which cannot occur.

c) If $n = 5$, then $M_0 = \mathbf{R}P^5(\frac{c}{4})$, T^{15} or $SU(3)/SO(3)Z_3$. Suppose that spec(M_0, J) =spec(M'_0, J). Then, in particular, $a_0(M_0) = a_0(M'_0)$ and $a_1(M_0) = a_1(M'_0)$, from which it follows easily that $\tau_0 = \tau'_0$, which cannot occur, because, as it follows from Table I, for $\mathbf{R}P^5(\frac{c}{4})$, T^5 and $SU(3)/SO(3)Z_3$, we respectively have $\tau = 5c$, 0 and $\frac{45}{16}c$, with $c \neq 0$ □

We now characterize totally real parallel Einstein submanifolds M_0 of $QP^n(c)$, in the class of all totally real minimal submanifolds, by proving the following

THEOREM 4.3. *Let M be an n-dimensional compact totally real minimal submanifold of $QP^n(c)$. If* spec$(M, J) =$ spec(M_0, J), $5 < n \leq 17$, *then M is isometric to M_0.*

PROOF. Since spec(M, J)=spec(M_0, J), we have dim$M$=dimM_0=n and, from Theorem 3.3, we get

$$\text{vol}(M, g) = \text{vol}(M_0, g_0)\,, \tag{4.2}$$

$$\int_M \tau dv = \int_{M_0} \tau_0 dv\,, \quad \int_M ||\sigma||^2 dv = \int_{M_0} ||\sigma_0||^2 dv, \tag{4.3}$$

$$\begin{aligned}\int_M \{2(n-5)||R||^2 + 2(30-n)||\varrho||^2 + 5(n-4)\tau^2\}dv \\ = \int_{M_0} \{2(n-5)||R_0||^2 + 2(30-n)||\varrho_0||^2 + 5(n-4)\tau_0^2\}dv\,.\end{aligned} \tag{4.4}$$

Since τ_0 is constant and vol$(M) =$ vol(M_0), we have

$$\begin{aligned}\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv &= \int_M \tau^2 dv - 2\tau_0 \int_{M_0} \tau_0 dv + \int_{M_0} \tau_0^2 dv \\ &= \int_M (\tau - \tau_0)^2 dv \geq 0\end{aligned} \tag{4.5}$$

where the equality holds if and only if $\tau = \tau_0$.

Next, let $E = \varrho - \frac{\tau}{n} g$ denote the *Einstein curvature tensor* of (M, g). Since $||E||^2 = ||\varrho||^2 - \frac{\tau^2}{n}$ and $E_0 = 0$ because M_0 is an Einstein space, (4.4) becomes

$$\text{(4.6)} \qquad 2(n-5)\left(\int_M ||R||^2 dv - \int_{M_0} ||R_0||^2 dv\right) - 2(n-30)\int_M ||E||^2 dv + \frac{5n^2-22n+60}{2n}\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right) = 0\,.$$

Moreover, from (2.7) we also get

$$\frac{1}{2}\Delta||\sigma||^2 = ||\nabla'\sigma||^2 - ||R||^2 - ||E||^2 + \frac{1}{n}\tau^2 + (n+1)\frac{c}{4}\tau\,.$$

Integrating over M, we obtain

$$\text{(4.7)} \qquad \int_M ||\nabla'\sigma||^2 dv = \int_M ||R||^2 dv + \int_M ||E||^2 dv + \frac{1}{n}\int_M \tau^2 dv - (n+1)\frac{c}{4}\int_M \tau dv\,.$$

An analogous formula holds for M_0, with $\nabla'\sigma_0 = E_0 = 0$. Using (4.7) to calculate $\int_M ||R||^2 dv$, (4.6) becomes

$$\text{(4.8)} \qquad (n-5)\int_M ||\nabla'\sigma||^2 dv = \alpha(n)\int_M ||E||^2 dv + \beta(n)\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right),$$

where

$$\alpha(n) = 2n - 35\,,$$
$$\beta(n) = -\frac{5n^2-24n+70}{2n}\,.$$

If $5 < n \le 17$, then $n - 5 > 0$, while $\alpha(n), \beta(n) < 0$. Therefore, we get $\nabla'\sigma = 0$, $E = 0$ and $\tau = \tau_0$. Thus, M is an Einstein (compact) totally real parallel submanifold of $QP^n(c)$, with the same spec(J) of M_0. So, Theorem 4.2 implies that M is isometric to M_0. □

5. Totally real parallel conformally flat submanifolds of QP^n

In this section, by M_0 we shall denote an n-dimensional compact totally real parallel minimal submanifold of $CP^n(c)$, which is conformally flat. In other words, M_0 is one of the manifolds listed in the following Table II:

TABLE II

M_0	τ
$\mathbf{R}P^n(\frac{c}{4})$	$\frac{n(n-1)}{4}c$
T^n	0
$S^1 \times S^{n-1}$	$\frac{(n-2)(n^2-1)}{4n}c$

(see [E], [N], [CP]). As we noted in the previous Section 4, these are exactly the n-dimensional conformally flat totally real minimal parallel submanifolds of the quaternionic projective space $QP^n(c)$. We now prove the following

THEOREM 5.1. *Let M be an n-dimensional compact totally real minimal submanifold of $QP^n(c)$. If* $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$ *and* $18 \leq n \leq 33$, *then M is isometric to M_0.*

PROOF. The proof is similar to the one of Theorem 4.3. In particular, formulas (4.2)–(4.5) still hold. Here, we use the *conformal curvature tensor C* of (M, g) to rewrite (4.4). Since the curvature invariant $||R||^2$ is given by

$$||R||^2 = ||C||^2 + \frac{4}{n-2}||\varrho||^2 - \frac{2}{(n-1)(n-2)}\tau^2 , \tag{5.1}$$

from (4.4) we obtain

$$\begin{aligned}(n-5)\int_M ||C||^2 dv - \frac{n^2-36n+80}{n-2}\left(\int_M ||\varrho||^2 dv - \int_{M_0} ||\varrho_0||^2 dv\right) \\ + \frac{5n^3-35n^2+66n-20}{2(n-1)(n-2)}\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right) = 0 .\end{aligned} \tag{5.2}$$

Moreover, from (2.7) and (5.1), we also have

$$\frac{1}{2}\Delta||\sigma||^2 = ||\nabla'\sigma||^2 - ||C||^2 - \frac{n+2}{n-2}||\varrho||^2 + \frac{2}{(n-1)(n-2)}\tau^2 + (n+1)\frac{c}{4}\tau ,$$

from which, by integrating over M, we get

$$\begin{aligned}\int_M ||\nabla'\sigma||^2 dv = \int_M ||C||^2 dv + \frac{n+2}{n-2}\int_M ||\varrho||^2 dv \\ - \frac{2}{(n-1)(n-2)}\int_M \tau^2 dv - (n+1)\frac{c}{4}\int_M \tau dv\end{aligned}$$

and for M_0, since $\nabla'\sigma_0 = 0$ and $C_0 = 0$, we have

$$\frac{n+2}{n-2}\int_{M_0} ||\varrho_0||^2 dv = \frac{2}{(n-1)(n-2)}\int_{M_0} \tau_0^2 dv + (n+1)\frac{c}{4}\int_{M_0} \tau_0 dv .$$

Therefore,

$$\frac{n+2}{n-2}\Big(\int_M ||\varrho||^2 dv - \int_{M_0} ||\varrho_0||^2 dv\Big) = \int_M ||\nabla'\sigma||^2 dv - \int_M ||C||^2 dv \tag{5.3}$$
$$+\frac{2}{(n-1)(n-2)}\Big(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\Big).$$

Using (5.3), (5.2) becomes

$$\int_M ||\nabla'\sigma||^2 dv = a(n)\int_M ||C||^2 dv + b(n)\Big(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\Big), \tag{5.4}$$

where

$$a(n) = \frac{2n^2-39n+70}{n^2-36n+80}, \quad b(n) = \frac{5n^4-25n^3-6n^2+184n-200}{(n-1)(n-2)(n^2-36n+80)}.$$

It is easy to check that if $18 \le n \le 33$, then $a(n) < 0$ and $b(n) < 0$. Therefore, we get $\nabla'\sigma = 0$, $C = 0$ and $\tau = \tau_0$, that is, M is a compact conformally flat totally real minimal submanifold of $QP^n(c)$ with parallel second fundamental form. Therefore, M is isometric to one of the manifolds listed in Table II. Since $\tau = \tau_0$, we can conclude that M is isometric to M_0. □

Remark that the flat torus T^n is, at the same time, a conformally flat and an Einstein manifold. Therefore, combining Theorems 4.3 and 5.1, we get the following

COROLLARY 5.2. *In the class of all compact totally real minimal submanifolds of $QP^n(c)$, the flat torus T^n is characterized by its* spec(J) *when* $5 < n \le 33$.

Moreover, note that, using formulas (2.4), (3.14) and (3.15), it is easy to show that *in the class of all compact totally real minimal submanifolds of $QP^n(c)$, the real projective space $\mathbf{R}P^n(\frac{c}{4})$ is characterized by its spec(J) for all $n \ge 3$.*

6. Spectral rigidity of totally real submanifolds of small dimension

In this section, we characterize by means of spec(J) some special n-dimensional totally real submanifolds of QP^n when n is small.

Case of $n = 2$.

PROPOSITION 6.1. *Let M, M' be two compact totally real surfaces of $QP^2(c)$. If* spec(M, J) = spec(M', J), *then M is minimal if and only if M' is minimal.*

PROOF. Since $a_i(M) = a_i(M')$, using (3.11) and (3.12), we get easily

$$\int_M ||H||^2 dv = \int_{M'} ||H'||^2 dv,$$

from which the conclusion follows at once. □

Case of $n = 3$.

As it is well-known, the conformal curvature tensor C vanishes on any three-dimensional Riemannian manifold. Moreover, note that formula (5.4) holds for all $n \neq 2$. Therefore, it is easy to prove the following

THEOREM 6.2. *Let M be a compact minimal totally real submanifold of $QP^n(c)$. If* $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$, *where $n = 3$ and M_0 is a compact parallel totally real submanifold, then M is isometric to M_0.*

PROOF. We first remark that, according to Naitoh's classification, if M_0 is a three-dimensional compact totally real parallel submanifolds of $CP^3(c)$ (and hence, of $QP^3(c)$), then M_0 is $\mathbf{R}P^3(\frac{c}{4})$, T^3 or $S^1 \times S^2(k)$, with $k = c/3$. Suppose now $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$. Since $n = 3$, formula (5.4) becomes

$$\int_M ||\nabla'\sigma||^2 dv = b(3)\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right) = -\frac{14}{19}\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right),$$

from which it follows $\nabla'\sigma = \tau - \tau_0 = 0$. Thus, since M is parallel and $\tau = \tau_0$, we can conclude that M is isometric to M_0. □

Case of $n = 4$.

THEOREM 6.3. *Let M, M_0 be compact minimal totally real submanifolds of $QP^n(c)$, with M_0 parallel and either Einstein or conformally flat. If $n = 4$ and* $\mathrm{spec}(M, J) = \mathrm{spec}(M_0, J)$, *then*

$$\chi(M) \geq \chi(M_0)$$

and the equality holds if and only if M is isometric to M_0.

PROOF. The Gauss-Bonnet formula for any 4-dimensional compact manifold M is given by

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{||R||^2 - 4||\varrho||^2 + \tau^2\} dv . \tag{6.1}$$

Suppose first that M_0 is Einstein. Using $||\varrho||^2 = ||E||^2 + \tau^2/4$, (6.1) becomes

$$\chi(M) = \frac{1}{32\pi^2}\int_M \{||R||^2 - 4||E||^2\} dv , \quad \chi(M_0) = \frac{1}{32\pi^2}\int_{M_0} ||R_0||^2 dv . \tag{6.2}$$

On the other hand, since $a_i(M, J) = a_i(T^2, J)$, using formulas (3.14), (3.15), (3.16) (for $n = 4$), we obtain

$$\int_M ||R||^2 dv - \int_{M_0} ||R_0||^2 dv = 26\int_M ||E||^2 dv + \frac{13}{2}\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right). \tag{6.3}$$

Using (6.2) and (6.3), we then get

$$(32\pi^2)\{\chi(M) - \chi(M_0)\} = 22\int_M ||E||^2 dv + \frac{13}{2}\left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv\right). \tag{6.4}$$

Therefore, $\chi(M) \geq \chi(M_0)$ since $||E||^2 \geq 0$ and $\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \geq 0$.

In particular, if $\chi(M) = \chi(M_0)$, then (6.4) yields $E = \tau - \tau_0 = 0$. Moreover, by (6.3) it follows $\int_M ||R||^2 dv = \int_{M_0} ||R_0||^2 dv$ and hence, by (4.7), $\int_M ||\nabla' \sigma||^2 dv = \int_{M_0} ||\nabla' \sigma_0||^2 dv = 0$. So, M is also parallel and, as in the proof of Theorem 4.3, we can conclude that M is isometric to M_0.

If M_0 is conformally flat, the proof is similar. Using the conformal curvature tensor C, we eventually get

$$(6.5) \qquad (32\pi^2)\{\chi(M) - \chi(M_0)\} = \frac{11}{12} \int_M ||C||^2 dv + \frac{13}{36} \left(\int_M \tau^2 dv - \int_{M_0} \tau_0^2 dv \right) \geq 0,$$

where the equality holds if and only if $C = \tau - \tau_0 = 0$. Moreover, M is parallel and the conclusion then follows as in the proof of Theorem 5.1 □

In particular, from Theorem 6.3 we obtain at once the following

COROLLARY 6.4. *In the class of all* 4*-dimensional compact minimal totally real submanifolds of* $QP^4(c)$*, of non-positive Euler number,* T^4 *and* $S^1 \times S^3(k)$*, with* $k = 5c/16$*, are completely determined by their* spec(J)*.*

Case of $n = 5$.

Using the same methods of the proof of Theorem 4.3, we can easily prove the following result for five-dimensional totally real submanifolds of $QP^5(c)$.

PROPOSITION 6.5. *Let* M *be an* n*-dimensional compact totally real minimal submanifold of* $QP^n(c)$*. If* spec$(M, J) =$ spec(M_0, J)*, with* M_0 *parallel and Einstein and* $n = 5$*, then* M *is also Einstein and* $\tau = \tau_0$*.*

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