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Hamiltonian Minimal Lagrangian Cones in $\mathbf{C}^m$

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Abstract. We give a correspondence among a Hamiltonian minimal Lagrangian cone in $\mathbf{C}^m$, a Legendrian minimal Legendrian submanifold in the unit sphere $S^{2m-1}(1)$ and a Hamiltonian minimal Lagrangian submanifold in the complex projective space $\mathbf{C}P^{m-1}$. As an application of this result, we prove that a Hamiltonian minimal Lagrangian cone in $\mathbf{C}^m$ such that the first Betti number of its link is 0 must be a special Lagrangian cone. Moreover, we construct Hamiltonian minimal (non-minimal) Lagrangian cones in $\mathbf{C}^3$ with a toroidal link, which are parametrized by a triple of relatively prime positive integers (p, q, r), and discuss their Hamiltonian stabilities.

1. Introduction

The notion of *special Lagrangian submanifolds* in $\mathbf{C}^m$ was introduced by Harvey and Lawson [2] as examples of calibrated submanifolds. Recently, motivated by physical problems mainly concerned with mirror symmetry (see e.g. [9]), special Lagrangian m-folds in $\mathbf{C}^m$ have been studied by many mathematicians intensively. In particular, Haskins [3] constructed many explicit examples of special Lagrangian cones in $\mathbf{C}^3$. His results were generalized and extended by Joyce [4] to higher dimensional case.

In the meantime, Oh [5] introduced the notion of *Hamiltonian minimal* (H-minimal) Lagrangian submanifolds and initiated the study of volume minimizations of Lagrangian submanifolds under Hamiltonian deformations in Kähler manifolds. A Lagrangian submanifold in a Kähler manifold is said to be *H-minimal* if the volume is stationary for compactly supported smooth variations arising from Hamiltonian deformations.

Inspired by these developments, we start the study of H-minimal Lagrangian cones in $\mathbf{C}^m$ as a generalization of the notion of special Lagrangian cones in $\mathbf{C}^m$. In the study of special Lagrangian cones in $\mathbf{C}^m$, a correspondence among a special Lagrangian cone in $\mathbf{C}^m$, a minimal Legendrian submanifold in $S^{2m-1}(1)$ and a minimal Lagrangian submanifold in $\mathbf{C}P^{m-1}$ is fundamental and used effectively by Haskins in [3].

In this paper, first of all, we establish a correspondence between an H-minimal Lagrangian cone in $\mathbf{C}^m$ and an *L-minimal Legendrian submanifold* (see Section 3) in the unit sphere $S^{2m-1}(1)$ (Theorem 4.1). Using this theorem and the classification of H-minimal

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Lagrangian cones in $\mathbf{C}^2$ obtained by Schoen and Wolfson [8, Section 7], we determine all L-minimal Legendrian closed curves in $S^3(1)$. Moreover, we give a correspondence between an L-minimal Legendrian submanifold in a Sasakian manifold and an H-minimal Lagrangian submanifold in the corresponding Kähler manifold (Theorem 4.4). We apply these results to proving that an H-minimal Lagrangian cone in $\mathbf{C}^m$ such that the first Betti number of its link is 0 must be a special Lagrangian cone. Therefore, to obtain examples of H-minimal Lagrangian cones which are not minimal, we must find cones of which links have non-zero first Betti numbers. We shall construct H-minimal (non-minimal) Lagrangian cones in $\mathbf{C}^m$ with a toroidal link. Finally, we examine the structure of these examples in detail in the case where $m = 3$ and prove that there exists infinitely many Hamiltonian unstable H-minimal Lagrangian cones with a flat torus link in $\mathbf{C}^3$.

2. H-minimal Lagrangian submanifolds in a Kähler manifold

In this section, we review some facts about H-minimal Lagrangian submanifolds in a Kähler manifold. Let $(N^m, J, \langle , \rangle, \omega)$ be a connected complex m-dimensional Kähler manifold with complex structure J and Kähler form ω. Let L^m be a real m-dimensional connected manifold. In this paper all manifolds, maps, etc. are supposed to be of class C^∞. An immersion $l : L^m \to N^m$ is said to be *Lagrangian* if $l^*\omega = 0$. This condition is equivalent to the condition that

$$J(l_*X) \perp l_*Y$$

for any vector fields $X, Y \in \Gamma(TL)$ on L, that is, we have

$$T_{l(p)}N = l_*T_pL \oplus J(l_*T_pL)$$

for any $p \in L$.

From now on, we assume that L^m is compact without boundary. Then we have the linear isomorphism

$$\tilde{\omega} : \Gamma(T^\perp L) \to \Omega^1(L)\,,$$

$$V \mapsto l^*(V \lrcorner \omega) =: \alpha_V$$

where $T^\perp L$ denotes the normal bundle of the immersion l. This map is also isometric, i.e., we have

$$\langle V, W \rangle = \langle \alpha_V, \alpha_W \rangle$$

for any $V, W \in \Gamma(T^\perp L)$.

DEFINITION 2.1 ([5]). A variational vector field $V \in \Gamma(T^\perp L)$ of l is called *Hamiltonian variation* when α_V is exact.

This condition implies that the infinitesimal deformation of l with variational vector field V preserves the Lagrangian constraint. But the converse is not true in general.

DEFINITION 2.2 ([5]). Let N be a Kähler manifold. A Lagrangian immersion $l : L \to N$ is said to be *H-minimal* if it satisfies

$$\frac{d}{dt}\mathrm{Vol}(l_t(L))\bigg|_{t=0} = 0$$

for all smooth deformations $\phi = \{l_t\}_{-\varepsilon \leq t \leq \varepsilon}$ of $l = l_0$ with Hamiltonian variation vector field V.

The following two facts are used in Sections 4 and 5.

LEMMA 2.3 ([5]). *Let (N^m, ω) be a Kähler manifold. A Lagrangian immersion $l : L \to N$ is H-minimal if and only if the mean curvature vector H of l satisfies*

$$\delta\alpha_H = 0\,,$$

where δ is the codifferential operator on L with respect to the induced metric.

COROLLARY 2.4 ([5]). *Assume that (N^m, ω) is a Kähler-Einstein manifold and let $l : L \to N$ be an H-minimal Lagrangian immersion. If $H^1(L; \mathbf{R}) = 0$, then $l : L \to N$ must be minimal.*

Since we will treat Lagrangian cones in $\mathbf{C}^m$ later, we must consider the case where a Lagrangian submanifold L is non-compact. In this case, an immersed Lagrangian submanifold is said to be *minimal* (resp. *H-minimal*) if the volume is critical for compactly supported smooth (resp. Hamiltonian) variations. Then we have the same criterion as Lemma 2.3 states for any open subset U of L such that its closure is compact in L.

3. L-minimal Legendrian submanifolds in a Sasakian manifold

Let $(M^{2m-1}, \langle,\rangle; \eta, \xi, \varphi)$ be a connected $(2m-1)$-dimensional Sasakian manifold with contact form η, characteristic vector field ξ and structure tensor field φ. Let L^{m-1} be an $(m-1)$-dimensional connected manifold. An embedding $i : L^{m-1} \to M^{2m-1}$ is said to be *Legendrian* if $i^*\eta = 0$. This condition implies that

$$i_*X \perp \xi$$

for any vector field $X \in \Gamma(TL)$ on L, that is, we have

$$i_*T_pL \subset \mathcal{H}_{i(p)}, \forall p \in L\,,$$

where $\mathcal{H}$ denotes the horizontal subbundle $(\mathbf{R}\xi)^\perp$ of TM. By Reckziegel's results (see [7]), the Legendrian embedding i has the following properties:

(i) i is *anti-invariant*, i.e., for any $p \in L$, we have

$$i_*T_pL \perp \varphi(i_*T_pL)\,.$$

(ii) The second fundamental form $\tilde{B}$ of i takes its values in $\mathcal{H}$.

(iii) If $\pi : M \to N$ is a canonical fibration of M, then $\pi \circ i : L \to N$ is a Lagrangian immersion into the Kähler manifold N and its second fundamental form is given by $\pi_* \tilde{B}$.

From now on, we assume that L^{m-1} is compact without boundary. Then we have the following linear isomorphism

$$\chi : \Gamma(T^{\perp}L) \to C^{\infty}(L) \oplus \Omega^1(L)\,,$$
$$V \mapsto (\eta(V), \tilde{\alpha}_V)$$

where $\tilde{\alpha}_V := -i^*(V \lrcorner d\eta)$ and $T^{\perp}L$ denotes the normal bundle of the embedding i. Moreover, we have

$$\langle V, W\rangle = \eta(V)\eta(W) + \langle \tilde{\alpha}_V, \tilde{\alpha}_W\rangle$$

for any $V, W \in \Gamma(T^{\perp}L)$ (cf. [6, Lemma 3.2]).

DEFINITION 3.1. A variational vector field $V \in \Gamma(i^*(TM))$ of i is called *Legendrian variation* if $V^{\perp} = \chi^{-1}(\eta(V^{\perp}), d(\eta(V^{\perp})))$.

This condition means that the infinitesimal deformation of i with variational vector field V preserves the Legendrian constraint.

Next, we introduce the notion of *Legendrian minimal* (abbreviated as *L-minimal*) Legendrian submanifolds in a Sasakian manifold.

DEFINITION 3.2. Let (M^{2m-1}, η) be a Sasakian manifold. A Legendrian embedding $i : L \to M$ is said to be *L-minimal* if it satisfies

$$\frac{d}{dt}\mathrm{Vol}(i_t(L))\Big|_{t=0} = 0$$

for all smooth deformations $\phi = \{i_t\}_{-\varepsilon \le t \le \varepsilon}$ of $i = i_0$ with Legendrian variation vector field V.

In virtue of the first variation formula for volume, the above condition is equivalent to the condition that

$$\int_L \langle \tilde{H}, V\rangle dvol = 0$$

for all Legendrian variations $V \in \Gamma(i^*(TM))$. Here $\tilde{H}$ denotes the mean curvature vector field of i defined by $\tilde{H} :=$ trace $\tilde{B}$.

The Euler-Lagrange equation of this variational problem is given as follows.

LEMMA 3.3. *Let (M^{2m-1}, η) be a Sasakian manifold. Then a Legendrian embedding $i : L \to (M, \eta)$ is L-minimal if and only if the mean curvature vector $\tilde{H}$ of i satisfies that*

$$\delta\tilde{\alpha}_{\tilde{H}} = 0\,,$$

where $\tilde{\alpha}_{\tilde{H}} = -i^*(\tilde{H} \lrcorner d\eta)$ and δ is the codifferential operator on L with respect to the induced metric from M.

PROOF. By definition, a Legendrian embedding $i: L \to M$ is L-minimal if and only if

$$0 = \int_L \langle \tilde{H}, V\rangle dvol$$
$$= \int_L \eta(\tilde{H})\eta(V^{\perp}) dvol + \int_L \langle \tilde{\alpha}_{\tilde{H}}, \tilde{\alpha}_{V^{\perp}}\rangle dvol$$

for all Legendrian variations V. Here $V^{\perp}$ denotes the normal component of V. Since i is Legendrian, we obtain $\eta(\tilde{H}) = 0$ by the fact (ii). And the Legendrian constraint of the normal variation $V^{\perp}$ implies that $\tilde{\alpha}_{V^{\perp}} = d\left(\eta(V^{\perp})\right)$. Hence, we have

$$0 = \int_L \langle \tilde{\alpha}_{\tilde{H}}, df\rangle dvol = \int_L (\delta\tilde{\alpha}_{\tilde{H}}) f dvol$$

for all smooth functions $f \in C^{\infty}(L)$ on L, which is equivalent to

$$\delta\tilde{\alpha}_{\tilde{H}} = 0 . \qquad \square$$

4. H-minimal Lagrangian cones in $\mathbf{C}^m$

In this section, we introduce the notion of *H-minimal Lagrangian cones* in the complex Euclidean space $\mathbf{C}^m$. We note that this object is first investigated by Schoen and Wolfson [8] in the case where $m = 2$.

Let L^n be an n-dimensional closed connected manifold. Let $i : L^n \to S^{2m-1}(1)$ be an embedding of L into the unit sphere in $\mathbf{C}^m$. We abuse the notation L also to denote the embedded submanifold $i(L)$ in $S^{2m-1}(1)$. The *cone* over L is defined as

$$C(L) = \{tx | t \in [0, \infty), x \in L\} .$$

The intersection L of $C = C(L)$ and $S^{2m-1}(1)$ is called the *link* of the cone C. In this paper the link of a cone is supposed to be oriented. An *H-minimal Lagrangian cone* in $\mathbf{C}^m$ is defined as a Lagrangian cone in $\mathbf{C}^m$ such that its volume is stationary for all compactly supported Hamiltonian variations which leave the cone vertex fixed.

THEOREM 4.1. *Let $i : L^{m-1} \to S^{2m-1}(1) \subset \mathbf{C}^m$ be an embedding. Then $C(L)$ is an H-minimal Lagrangian cone in $\mathbf{C}^m$ if and only if L is an L-minimal Legendrian submanifold in $S^{2m-1}(1)$.*

PROOF. Let $\vec{x}$ denote the position vector at a point $x \in S^{2m-1}(1)$. Let J be the standard complex structure on $\mathbf{C}^m$. Then $\xi_x := -J\vec{x}\ \left(x \in S^{2m-1}(1)\right)$ forms a Killing vector field ξ on $S^{2m-1}(1)$ and $\mathcal{H} := \xi^{\perp}$ defines a Sasakian structure (η, ξ, φ) on $S^{2m-1}(1)$ (see [1, p. 30–32]). Then we see that $C(L) - \{0\}$ is Lagrangian in $\mathbf{C}^m$ if and only if L is Legendrian in $S^{2m-1}(1)$.

Next, we relate the mean curvature vector $\tilde{H}_t$ of tL in the sphere $S^{2m-1}(t)$ of radius t to the mean curvature vector $H_{C(L)}$ of $C(L)$ in $\mathbf{C}^m$ at tx. This is accomplished by following Simons' argument (see [10, Proposition 6.1.1]) precisely. We fix our notation. We denote the Levi-Civita connections of tL^{m-1}, $S^{2m-1}(t)$ and $\mathbf{C}^m$ by ∇, ∇^S and $\bar{\nabla}$, respectively. The second fundamental form and the mean curvature vector field of tL in $S^{2m-1}(t)$ are denoted by $\tilde{B}_t$ and $\tilde{H}_t$, respectively. The second fundamental form of $S^{2m-1}(t)$ in $\mathbf{C}^m$ is denoted by $\bar{B}_t$. For simplicity we denote $\tilde{H}_1$ by $\tilde{H}$. Let ν denote the unit radial vector field on $C(L)$ corresponding to the coordinate t. Then ν_x is an outer unit normal vector at $x \in S^{2m-1}(t)$. Since ν_x is the velocity vector at x of the radial geodesic through $x \in \mathbf{C}^m - \{0\}$, we have

$$\bar{\nabla}_\nu \nu = 0 .$$

Let $\{e_1, \cdots, e_{m-1}\}$ be an orthonormal basis of $T_xL, x \in L \subset S^{2m-1}(1)$. Extend them to a local orthonormal frame $\{E_1, \cdots, E_{m-1}\}$ on L such that $\nabla_{E_i} E_j(x) = 0$ for any i, j. By parallel translation in $\mathbf{C}^m$, extend them up and down along radial geodesics, then we get a local orthonormal frame $\{E_1(x,t), \cdots, E_{m-1}(x,t), \nu_{tx}\}$ on $C(L) - \{0\}$. At $tx \in tL \subset S^{2m-1}(t)$, we have

$$\begin{aligned}
\tilde{H}_t = \sum_{i=1}^{m-1} \tilde{B}_t(E_i, E_i) &= \sum_{i=1}^{m-1} (\nabla^S_{E_i} E_i - \nabla_{E_i} E_i) \\
&= \sum_{i=1}^{m-1} (\bar{\nabla}_{E_i} E_i - \bar{B}_t(E_i, E_i)) \\
&= \sum_{i=1}^{m-1} \left(\bar{\nabla}_{E_i} E_i + \frac{1}{t} \langle E_i, E_i \rangle \nu_{tx} \right) \\
&= \sum_{i=1}^{m-1} \bar{\nabla}_{E_i} E_i + \frac{m-1}{t} \nu_{tx} .
\end{aligned}$$

Hence, we have

$$\bar{\nabla}_\nu \nu + \sum_{i=1}^{m-1} \bar{\nabla}_{E_i} E_i = \tilde{H}_t - \frac{m-1}{t} \nu . \tag{1}$$

Because the normal component of the left hand side is nothing but the mean curvature vector $H_{C(L)}$ of $C(L) \subset \mathbf{C}^m$ at tx, equation (1) yields

$$H_{C(L)} = \tilde{H}_t .$$

Consider the 1-form $\alpha_{H_{C(L)}} = \langle JH_{C(L)}, \cdot \rangle$ on $C(L) - \{0\}$. We calculate the codifferential of $\alpha_{H_{C(L)}}$. Since $H_{C(L)} = \tilde{H}_t = \frac{1}{t}\tilde{H}$, we have

$$\begin{aligned}
\delta \alpha_{H_{C(L)}} &= - \sum_{i=1}^{m-1} E_i(\alpha_{H_{C(L)}}(E_i)) - \nu(\alpha_{H_{C(L)}}(\nu)) \\
&= - \sum_{i=1}^{m-1} E_i \langle JH_{C(L)}, E_i \rangle - \nu \langle JH_{C(L)}, \nu \rangle
\end{aligned}$$

$$= \frac{1}{t}\sum_{i=1}^{m-1} E_i\langle \tilde{H}, \varphi E_i\rangle - \nu\left(\frac{1}{t}\langle \varphi\tilde{H}, \nu\rangle\right)$$

$$= \frac{1}{t}\sum_{i=1}^{m-1} E_i(d\eta(\tilde{H}, E_i))$$

$$= \frac{1}{t}\delta\tilde{\alpha}_{\tilde{H}}\,, \tag{2}$$

where the term $\langle\varphi\tilde{H}, \nu\rangle$ vanishes because $\varphi\tilde{H} \in \mathcal{H}$ and therefore $\varphi\tilde{H}$ is perpendicular to ν.

By Oh's criterion (Lemma 2.3), Lemma 3.3 and equation (2) above, we conclude that $C(L)$ is H-minimal if and only if L is L-minimal. □

Here, we want to explain examples of H-minimal Lagrangian cones in $\mathbf{C}^2$ given by Schoen and Wolfson [8, Section 7]. In the process of their study of area minimization problem among Lagrangian surfaces in a Kähler surface, they completely classified admissible singular points of area minimizing Lagrangian surfaces. These singularities are locally modelled by H-minimal Lagrangian cones in $\mathbf{C}^2$ (see [8, Theorem 7.1]).

THEOREM 4.2 (Schoen-Wolfson). *All H-minimal Lagrangian cones $C(\gamma)$ in $\mathbf{C}^2$ are the cones over the curves γ below.*

$$\gamma(s) = \frac{1}{\sqrt{p+q}}\left(\sqrt{q}e^{\sqrt{-1}\sqrt{\frac{p}{q}}s}, \sqrt{-1}\sqrt{p}e^{-\sqrt{-1}\sqrt{\frac{q}{p}}s}\right), \tag{3}$$

where s is an arc length parameter and $0 \leq s \leq 2\pi\sqrt{pq}$. They are parameterized by a pair of relatively prime positive integers (p, q).

So Theorem 4.1 yields the following immediately.

COROLLARY 4.3. *All L-minimal Legendrian closed curves γ in $S^3(1)$ are given by the equation (3). They are torus knots of type (p, q).*

Next, we give a correspondence between an L-minimal Legendrian submanifold in a Sasakian manifold and an H-minimal Lagrangian submanifold in the corresponding Kähler manifold under the canonical fibration.

THEOREM 4.4. *Let $\pi : M \to N$ be a canonical fibration of a Sasakian manifold (M^{2m-1}, η). If $i : L \to M$ is a Legendrian embedding of closed $(m-1)$-dimensional manifold L into M, then $\pi\circ i : L \to N$ is a Lagrangian immersion. Moreover, the Legendrian embedding i is L-minimal if and only if the Lagrangian immersion $\pi\circ i$ is H-minimal.*

PROOF. The former part of this theorem is a well-known result and is nothing but the fact (iii) in Section 3. We shall prove the latter part. By Lemma 3.3, the Legendrian embedding i is L-minimal if and only if its mean curvature vector $\tilde{H}$ satisfies that

$$\delta\tilde{\alpha}_{\tilde{H}} = 0\,.$$

The complex structure and the Kähler form of N are denoted by J and ω, respectively. For horizontal vector fields $X, Y \in \Gamma(\mathcal{H})$, we have

$$\begin{aligned} d\eta(X,Y) &= \langle X, \varphi Y\rangle \\ &= -\langle \varphi X, Y\rangle \\ &= -\langle \pi_*\varphi X, \pi_* Y\rangle_N \\ &= -\langle J\pi_* X, \pi_* Y\rangle_N \\ &= -\omega(\pi_* X, \pi_* Y) \\ &= -\pi^*\omega(X,Y)\,, \end{aligned}$$

where $\langle\cdot,\cdot\rangle_N$ denotes the Kähler metric on N. Using this formula and the fact (iii), we have

$$\begin{aligned} \tilde{\alpha}_{\tilde{H}} &= -i^*(\tilde{H} \lrcorner d\eta) \\ &= -d\eta(\tilde{H}, i_*(\cdot)) \\ &= \pi^*\omega(\tilde{H}, i_*(\cdot)) \\ &= \omega(\pi_*\tilde{H}, (\pi\circ i)_*(\cdot)) \\ &= \omega(H, (\pi\circ i)_*(\cdot)) \\ &= (\pi\circ i)^*(H \lrcorner \omega) \\ &= \alpha_H\,, \end{aligned}$$

where H denotes the mean curvature vector field of $\pi\circ i$ and $\alpha_H \in \Omega^1(L)$ is defined for the Lagrangian immersion $\pi\circ i : L \to N$.

Thus, by Lemma 2.3, we complete the proof. □

5. The comparison of H-minimal Lagrangian cones with special Lagrangian cones in $\mathbf{C}^m$

Harvey and Lawson proved in [2] that the notion of special Lagrangian submanifolds in $\mathbf{C}^m$ is equivalent (up to the action by unitary groups) to oriented minimal Lagrangian submanifolds. Of course, minimal Lagrangian submanifolds are H-minimal, but the converse is not true in general. In this section, we give a sufficient condition for an H-minimal Lagrangian cone in $\mathbf{C}^m$ to be minimal, i.e., a special Lagrangian cone. This is an easy consequence of our main theorems.

PROPOSITION 5.1. *Let C be an H-minimal Lagrangian cone in $\mathbf{C}^m$. Let L be the link of C, i.e., $L = C \cap S^{2m-1}(1)$. If $H^1(L;\mathbf{R}) = 0$, then C must be a special Lagrangian cone.*

PROOF. By Theorem 4.1, the link L is an L-minimal Legendrian submanifold in $S^{2m-1}(1)$. Let $\pi : S^{2m-1}(1) \to \mathbf{C}P^{m-1}(4)$ be the Hopf fibration. Here $\mathbf{C}P^{m-1}(4)$ denotes the $(m-1)$-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 4. This is a canonical fibration of $S^{2m-1}(1)$ with the standard Sasakian structure. By Theorem 4.4, $\pi(L)$ is a closed H-minimal immersed Lagrangian submanifold in $\mathbf{C}P^{m-1}(4)$. Since $H^1(L;\mathbf{R}) = 0$, $\pi(L)$ must be minimal by Corollary 2.4. Hence, L and $C = C(L)$ must be also minimal by the equations $\pi_* \tilde{H} = H$ and $H_{C(L)} = \tilde{H}$. □

REMARK 5.2. As an easy application of the above proposition and the well-known pinching result of minimal Lagrangian 2-sphere in $\mathbf{C}P^2$ obtained by Yau (see [11, Theorem 7]), we can weaken the assumption of a theorem by Haskins (see [3, Theorem B]):

Let C be an H-minimal Lagrangian cone in $\mathbf{C}^3$ such that its link is a 2-sphere. Then C must be a special Lagrangian 3-plane.

6. Examples: H-minimal Lagrangian cones in $\mathbf{C}^m$ with a toroidal link

As Proposition 5.1 implies, to obtain examples of H-minimal Lagrangian cones which are not minimal, we must find cones of which links have non-vanishing first cohomology groups with real coefficients. In this section, we construct many examples of H-minimal Lagrangian cones in $\mathbf{C}^m$ such that their links are flat tori. These examples are higher dimensional analogues of Schoen-Wolfson's examples.

Let $p_1, p_2, \cdots, p_m$ be positive integers such that $p_1 \le p_2 \le \cdots \le p_m$ and $\gcd(p_1, p_2, \cdots, p_m) = 1$. We consider the following m-dimensional torus in $\mathbf{C}^m$:

$$S^1\left(\sqrt{\frac{p_1}{\sum_{i=1}^m p_i}}\right) \times \cdots \times S^1\left(\sqrt{\frac{p_m}{\sum_{i=1}^m p_i}}\right) \subset S^{2m-1}(1)\,.$$

We denote by $(z_1, \cdots, z_m)$ the standard coordinates on $\mathbf{C}^m$. Then we can parameterize it as

$$z_1 = \sqrt{\frac{p_1}{\sum_{i=1}^m p_i}}\, e^{\sqrt{-1}\sqrt{\frac{\sum_{i=1}^m p_i}{p_1}}\theta_1}, \cdots, z_m = \sqrt{\frac{p_m}{\sum_{i=1}^m p_i}}\, e^{\sqrt{-1}\sqrt{\frac{\sum_{i=1}^m p_i}{p_m}}\theta_m}, \tag{4}$$

where each $\theta_j\,(j = 1, \cdots, m)$ is a real parameter and

$$0 \le \theta_j \le 2\pi\sqrt{\frac{p_j}{\sum_{i=1}^m p_i}}\,.$$

On this torus with coordinates $(\theta_1, \cdots, \theta_m)$, the direction of the characteristic vector field (which generates the Hopf action) is represented by

$$\xi := \left(\sqrt{\frac{p_1}{\sum_{i=1}^m p_i}}, \cdots, \sqrt{\frac{p_m}{\sum_{i=1}^m p_i}}\right).$$

The codimension one torus of the above torus, which is perpendicular to the direction ξ and through the point $(\sqrt{p_1/(\sum_{i=1}^m p_i)}, \cdots, \sqrt{p_m/(\sum_{i=1}^m p_i)})$ on $\mathbf{C}^m$, is represented as follows:

$$T^{m-1}_{(p_1,\cdots,p_m)} := \left\{ (z_1, \cdots, z_m) \text{ satisfying (4)} \left| \sum_{i=1}^m \sqrt{p_i}\theta_i = \frac{2\pi l}{\sqrt{\sum_{i=1}^m p_i}} \quad (l \in \mathbf{Z}) \right. \right\}$$

By definition, $T^{m-1}_{(p_1,\cdots,p_m)}$ is an $(m-1)$-dimensional Legendrian flat torus in $S^{2m-1}(1)$. The image of $T^{m-1}_{(p_1,\cdots,p_m)}$ by the Hopf fibration $\pi : S^{2m-1}(1) \to \mathbf{C}P^{m-1}$ is a Lagrangian flat torus in $\mathbf{C}P^{m-1}$.

REMARK 6.1. Here we consider a moment map $\mu : \mathbf{C}P^{m-1} \to \mathbf{R}^{m-1}$

$$[z_1 : \cdots : z_m] \longmapsto \left(\frac{|z_2|^2}{\sum_{i=1}^m |z_i|^2}, \cdots, \frac{|z_m|^2}{\sum_{i=1}^m |z_i|^2} \right),$$

which associates with the following isometric T^{m-1}-action on $\mathbf{C}P^{m-1}$:

$$\left(e^{\sqrt{-1}t_1}, \cdots, e^{\sqrt{-1}t_{m-1}}\right) \cdot [z_1 : \cdots : z_m] = [z_1 : e^{\sqrt{-1}t_1} z_2 : \cdots : e^{\sqrt{-1}t_{m-1}} z_m] .$$

Then the image of $\pi(T^{m-1}_{(p_1,\cdots,p_m)})$ by this moment map μ corresponds to an inner rational point of the following closed convex polytope in $\mathbf{R}^{m-1}$:

$$\left\{ (q_1, \cdots, q_{m-1}) \in \mathbf{R}^{m-1} \left| \sum_{i=1}^{m-1} q_i \leq 1; \; q_i \geq 0, \; i = 1, \cdots, m-1 \right. \right\}$$

More precisely,

$$\mu(\pi\left(T^{m-1}_{(p_1,\cdots,p_m)}\right)) = \left(\frac{p_2}{\sum_{i=1}^m p_i}, \cdots, \frac{p_m}{\sum_{i=1}^m p_i} \right) \in \mathbf{Q}^{m-1} .$$

Since it is well-known that $\pi(T^{m-1}_{(p_1,\cdots,p_m)})$ is an H-minimal Lagrangian torus in $\mathbf{C}P^{m-1}$, we obtain an H-minimal Lagrangian cone $C(T^{m-1}_{(p_1,\cdots,p_m)})$ in $\mathbf{C}^m$ by Theorems 4.1 and 4.4.

If $(p_2/(\sum_{i=1}^m p_i), \cdots, p_m/(\sum_{i=1}^m p_i)) \neq (1/m, \cdots, 1/m)$, then $\pi(T^{m-1}_{(p_1,\cdots,p_m)})$ is not minimal in $\mathbf{C}P^{m-1}$.

If $(p_2/(\sum_{i=1}^m p_i), \cdots, p_m/(\sum_{i=1}^m p_i)) = (1/m, \cdots, 1/m)$, then we have $(p_1, \cdots, p_m) = (1, \cdots, 1)$ and $\pi(T^{m-1}_{(1,\cdots,1)})$ is minimal in $\mathbf{C}P^{m-1}$. This is the so-called Clifford torus in $\mathbf{C}P^{m-1}$.

Therefore, we obtain many examples of non-minimal H-minimal Lagrangian cones in $\mathbf{C}^m$ of which links are flat tori.

7. A family of infinitely many Hamiltonian unstable H-minimal Lagrangian cones in $\mathbf{C}^3$

In this section, we examine the structure of H-minimal Lagrangian cones in $\mathbf{C}^3$ with a flat torus link. Let us review the description of the preceeding section:

$$C(T^2_{(p,q,r)}) := \left\{ (tz_1, tz_2, tz_3) \in \mathbf{C}^3 \;\middle|\; t \geq 0\,, \quad z_1 = \sqrt{\frac{p}{p+q+r}} e^{\sqrt{-1}\sqrt{\frac{p+q+r}{p}}\theta_1}\,, \right.$$
$$z_2 = \sqrt{\frac{q}{p+q+r}} e^{\sqrt{-1}\sqrt{\frac{p+q+r}{q}}\theta_2}\,, \quad z_3 = \sqrt{\frac{r}{p+q+r}} e^{\sqrt{-1}\sqrt{\frac{p+q+r}{r}}\theta_3}\,,$$
$$\left. \sqrt{p}\theta_1 + \sqrt{q}\theta_2 + \sqrt{r}\theta_3 = \frac{2\pi l}{\sqrt{p+q+r}} \quad (l \in \mathbf{Z}) \right\},$$

where p, q, r are positive integers, $p \leq q \leq r$ and $\gcd(p, q, r) = 1$. We call it H-minimal Lagrangian cone *of type* (p, q, r).

From now on, we concentrate our study on H-minimal Lagrangian cones of type $(1, q, r)$. It seems to me that the following parameterization of general case (p, q, r) is difficult.

We write down the second variation formula for H-minimal Lagrangian cone of type $(1, q, r)$ by using Oh's formula (Theorem 7.3). So we must calculate the second fundamental form and the mean curvature vector of it. First of all, we shall decide the structure of the lattice of the link $T^2_{(1,q,r)}$ and give a parameterization along its lattice.

Consider $\mathbf{R}^3$ with the coordinate $(\theta_1, \theta_2, \theta_3)$ above. Then we see that one edge of the fundamental domain of $T^2_{(1,q,r)}$ lies in the (θ_1, θ_2)-plane and another in the plane defined by $\theta_2 = 2\pi\sqrt{q/(1+q+r)}$. A simple calculation shows that $\mathbf{Z}$-basis of the lattice $\Gamma = \mathbf{Z}v_1 \oplus \mathbf{Z}v_2$ of $T^2_{(1,q,r)}$ is given by the following two vectors:

$$v_1 = 2\pi\sqrt{\frac{q(1+q)}{1+q+r}}\left(\sqrt{\frac{q}{1+q}}, -\sqrt{\frac{1}{1+q}}, 0\right),$$
$$v_2 = 2\pi\sqrt{\frac{r(1+r)}{1+q+r}}\left(-\sqrt{\frac{r}{1+r}}, 0, \sqrt{\frac{1}{1+r}}\right).$$

Any point on $T^2_{(1,q,r)} = \mathbf{R}^2/\Gamma$ is represented by the coordinate $(\theta_1, \theta_2, \theta_3)$ as

$$\left(\sqrt{\frac{q}{1+q}}, -\sqrt{\frac{1}{1+q}}, 0\right)s_1 + \left(-\sqrt{\frac{r}{1+r}}, 0, \sqrt{\frac{1}{1+r}}\right)s_2\,,$$

where s_1 and s_2 are arc length parameters and satisfy that

$$0 \leq s_1 \leq 2\pi\sqrt{\frac{q(1+q)}{1+q+r}}\,, \quad 0 \leq s_2 \leq 2\pi\sqrt{\frac{r(1+r)}{1+q+r}}\,.$$

Therefore, we have the following parameterization of $C(T^2_{(1,q,r)}) \subset \mathbf{C}^3$:

$$z_1 = \sqrt{\frac{1}{1+q+r}}\; t\; e^{\sqrt{-1}\sqrt{\frac{1+q+r}{1}}\left(\sqrt{\frac{q}{1+q}}s_1 - \sqrt{\frac{r}{1+r}}s_2\right)},$$

$$z_2 = \sqrt{\frac{q}{1+q+r}}\; t\; e^{-\sqrt{-1}\sqrt{\frac{1+q+r}{q}}\left(\sqrt{\frac{1}{1+q}}s_1\right)},$$

$$z_3 = \sqrt{\frac{r}{1+q+r}}\; t\; e^{\sqrt{-1}\sqrt{\frac{1+q+r}{r}}\left(\sqrt{\frac{1}{1+r}}s_2\right)}.$$

We denote the position vector of $C\big(T^2_{(1,q,r)}\big)$ in $\mathbf{C}^3$ by $P(t, s_1, s_2)$. Put

$$e_1 := \frac{\partial P}{\partial t}, \quad e_2 := \frac{\partial e_1}{\partial s_1}\left(= \frac{1}{t}\frac{\partial P}{\partial s_1}\right), \quad e_3 := \frac{\frac{\partial e_1}{\partial s_2} - \langle \frac{\partial e_1}{\partial s_2}, e_2\rangle e_2}{\left\| \frac{\partial e_1}{\partial s_2} - \langle \frac{\partial e_1}{\partial s_2}, e_2\rangle e_2 \right\|},$$

where $\langle \cdot, \cdot \rangle$ (resp. $\|\cdot\|$) denotes the standard inner product (resp. norm) on $\mathbf{C}^3 \cong \mathbf{R}^6$. Then $\{e_1, e_2, e_3\}$ is an orthonormal frame on $C(T^2_{(1,q,r)})$. An easy calculation shows that

$$e_1 = \frac{1}{\sqrt{1+q+r}} \times \left(e^{\sqrt{-1}\sqrt{\frac{1+q+r}{1}}\left(\sqrt{\frac{q}{1+q}}s_1 - \sqrt{\frac{r}{1+r}}s_2\right)}, \sqrt{q}e^{-\sqrt{-1}\sqrt{\frac{1+q+r}{q(1+q)}}s_1}, \sqrt{r}e^{\sqrt{-1}\sqrt{\frac{1+q+r}{r(1+r)}}s_2} \right),$$

$$e_2 = \frac{\sqrt{-1}}{\sqrt{1+q}}\left(\sqrt{q}e^{\sqrt{-1}\sqrt{\frac{1+q+r}{1}}\left(\sqrt{\frac{q}{1+q}}s_1 - \sqrt{\frac{r}{1+r}}s_2\right)}, -e^{-\sqrt{-1}\sqrt{\frac{1+q+r}{q(1+q)}}s_1}, 0 \right),$$

$$e_3 = \frac{\sqrt{-1}}{\sqrt{(1+q)(1+q+r)}} \times \left(-\sqrt{r}e^{\sqrt{-1}\sqrt{\frac{1+q+r}{1}}\left(\sqrt{\frac{q}{1+q}}s_1 - \sqrt{\frac{r}{1+r}}s_2\right)}, -\sqrt{qr}e^{-\sqrt{-1}\sqrt{\frac{1+q+r}{q(1+q)}}s_1}, (1+q)e^{\sqrt{-1}\sqrt{\frac{1+q+r}{r(1+r)}}s_2} \right),$$

and e_3 is represented by $\frac{\partial P}{\partial s_1}$ and $\frac{\partial P}{\partial s_2}$ as

$$e_3 = \frac{1}{t}\left(\sqrt{\frac{qr}{1+q+r}}\frac{\partial P}{\partial s_1} + \sqrt{\frac{(1+q)(1+r)}{1+q+r}}\frac{\partial P}{\partial s_2} \right).$$

With respect to the orthonormal frame $\{e_1, e_2, e_3\}$, the second fundamental form of H-minimal Lagrangian cone of type $(1, q, r)$ is as follows.

LEMMA 7.1. *Let B be the second fundamental form of $C\big(T^2_{(1,q,r)}\big)$ in $\mathbf{C}^3$. Then*

$$B_{e_1,e_1} = B_{e_1,e_2} = B_{e_1,e_3} = 0,$$

$$B_{e_2,e_2} = \frac{1}{t}\left\{ \sqrt{\frac{1+q+r}{q(1+q)}}(q-1)Je_2 - \sqrt{\frac{r}{1+q}}Je_3 \right\},$$

$$B_{e_2,e_3} = -\frac{1}{t}\sqrt{\frac{r}{1+q}}Je_2\,,$$
$$B_{e_3,e_3} = \frac{1}{t}\frac{1+q-r}{\sqrt{(1+q)r}}Je_3\,,$$

where J denotes the standard complex structure on $\mathbf{C}^3$.

Hence, the mean curvature vector field H of $C(T^2_{(1,q,r)})$ in $\mathbf{C}^3$ is given by

$$H = \frac{1}{t}\left\{\sqrt{\frac{1+q+r}{q(1+q)}}(q-1)Je_2 + \frac{1+q-2r}{\sqrt{(1+q)r}}Je_3\right\}. \tag{5}$$

Here, we want to explain the notion of *Hamiltonian stability* of H-minimal Lagrangian submanifolds.

DEFINITION 7.2. Let N be a Kähler manifold. Suppose that a Lagrangian embedding $l : L \to N$ is H-minimal. Then the Lagrangian embedding l (or Lagrangian submanifold L) is said to be Hamiltonian stable if it satisfies that

$$\frac{d^2}{dt^2}\mathrm{Vol}(l_t(L))\bigg|_{t=0} \geq 0$$

for all smooth deformations $\phi = \{l_t\}_{-\varepsilon\leq t\leq\varepsilon}$ of $l = l_0$ with Hamiltonian variation vector field V.

The next theorem is the second variation formula for H-minimal Lagrangian submanifolds possibly with boundary (see [5, Theorem 3.4] and [8, Theorem 6.4]).

THEOREM 7.3 (Oh). *Let $l : L \to N$ be an H-minimal Lagrangian submanifold of a Kähler manifold N. Let $\phi : (-\varepsilon, \varepsilon) \times L \to N$ be a smooth Hamiltonian variation of $l = l_0$ with normal variation vector field V, which leaves the boundary fixed. Then*

$$\frac{d^2}{dt^2}\mathrm{Vol}(l_t(L))\bigg|_{t=0} = \int_L \{\langle\delta\alpha_V, \delta\alpha_V\rangle - \mathrm{Ric}(V, V) + \langle V, H\rangle^2 - 2\langle V, B_{JH,JV}\rangle\}dvol\,,$$

where $\alpha_V = l^*(V \lrcorner\omega)$ is a closed 1-form on L, δ is the adjoint of d on L and Ric *is the Ricci curvature of N.*

Next, we apply the second variation formula to Hamiltonian variations of the H-minimal Lagrangian cone $C\big(T^2_{(1,q,r)}\big)$. For simplicity $C(T^2_{(1,q,r)})$ is denoted by $C_{1,q,r}$. The following proposition is easily verified by Lemma 7.1, equation (5) and Theorem 7.3.

PROPOSITION 7.4. *Let V be a compactly supported Hamiltonian variation, $V = J\nabla f$, $f \in C^\infty_c(C_{1,q,r})$, which leaves the cone vertex fixed. Then the second variation with* *variation vector field V is given by*

$$\frac{d^2}{dt^2}\mathrm{Vol}(l_t(L))\bigg|_{t=0} = \int_{C_{1,q,r}} (\Delta f)^2 - \frac{1}{t^4}\frac{(1+q)(1+r)}{1+q+r}$$
$$\times \left\{\frac{1-2q+r}{q}\left(\frac{\partial f}{\partial s_1}\right)^2 - \frac{2(3qr-q-r-1)}{\sqrt{qr(1+q)(1+r)}}\frac{\partial f}{\partial s_1}\frac{\partial f}{\partial s_2} + \frac{1+q-2r}{r}\left(\frac{\partial f}{\partial s_2}\right)^2\right\} dvol\,. \tag{6}$$

Now we are ready to state and prove our main theorem in this section.

THEOREM 7.5. *Let r be a positive integer and $r \geq 2$. Then the H-minimal Lagrangian cone of type $(1, 1, r)$ in $\mathbf{C}^3$ is unstable for Hamiltonian variations fixing a neighborhood of the cone vertex.*

REMARK 7.6. The minimal Lagrangian cone of type $(1, 1, 1)$ is nothing but a special Lagrangian cone in $\mathbf{C}^3$. Hence, $C_{1,1,1}$ is strictly (Hamiltonian) stable.

PROOF OF THEOREM 7.5. Our proof is based on Schoen-Wolfson's method, but the calculation is slightly complicated since in our case the link is not a curve but a surface.

Consider the following function on $C_{1,1,r}$ which is constant with respect to s_2:

$$f(t, s_1, s_2) = \zeta(t)\cos\left(\sqrt{\frac{2+r}{2}}\, s_1\right), \quad 0 \leq s_1 \leq 2\pi\sqrt{\frac{2}{2+r}}\,,$$

where $\zeta(t) \in C_c^\infty(\mathbf{R}_+)$ is defined as follows:

$$\zeta(t) = \begin{cases} \delta(t)\,, & 0 \leq t \leq \varepsilon \\ t\,, & \varepsilon \leq t \leq 1 \\ \eta(t)\,, & 1 \leq t \end{cases}$$

where η satisfies

- η has support on $[1, 2]$;
- $\eta(1) = 1$, $d\eta/dt(1) = 1$, $d^2\eta/dt^2(1) = 0$;
- $0 \leq \eta \leq c$, $|d\eta/dt| \leq c$, $|d^2\eta/dt^2| \leq c$ for some constant $c > 0$;

and δ satisfies

- δ has support on $[\varepsilon/2, \varepsilon]$;
- $\delta(\varepsilon) = \varepsilon$, $d\delta/dt(\varepsilon) = 1$, $d^2\delta/dt^2(\varepsilon) = 0$;
- $0 \leq \delta \leq \varepsilon$, $|d\delta/dt| \leq 4$, $|d^2\delta/dt^2| \leq 4/\varepsilon$;

and ε is a real parameter satisfying $0 < \varepsilon < 1$.

The Laplace-Beltrami operator on $C_{1,1,r}$ is given by

$$\Delta = \frac{\partial^2}{\partial t^2} + \frac{2}{t}\frac{\partial}{\partial t} + \frac{1}{t^2}\frac{2(1+r)}{2+r}\left\{\frac{\partial^2}{\partial s_1^2} + \sqrt{\frac{2r}{1+r}}\frac{\partial^2}{\partial s_1 \partial s_2} + \frac{\partial^2}{\partial s_2^2}\right\}.$$

Hence, we have

$$\begin{aligned}\Delta f &= \Delta\zeta \cos\left(\sqrt{\frac{2+r}{2}}s_1\right) + \zeta\,\Delta\cos\left(\sqrt{\frac{2+r}{2}}s_1\right)\\ &= \left(\frac{\partial^2\zeta}{\partial t^2} + \frac{2}{t}\frac{\partial\zeta}{\partial t} - \frac{(1+r)\zeta}{t^2}\right)\cos\left(\sqrt{\frac{2+r}{2}}s_1\right).\end{aligned}$$

By the second variation formula (6),

$$\begin{aligned}\frac{d^2}{dt^2}\mathrm{Vol}(l_t(L))\Big|_{t=0} = \int_{C_{1,1,r}} \Bigg\{ &\left(\frac{\partial^2\zeta}{\partial t^2} + \frac{2}{t}\frac{\partial\zeta}{\partial t} - \frac{(1+r)\zeta}{t^2}\right)^2 \cos^2\left(\sqrt{\frac{2+r}{2}}s_1\right)\\ &- t^{-4}(r^2-1)\zeta^2\sin^2\left(\sqrt{\frac{2+r}{2}}s_1\right)\Bigg\}dvol\,.\end{aligned}$$

Since the volume element is calculated as

$$dvol = \sqrt{\frac{2+r}{2(1+r)}}\,t\,dt\,ds_1 ds_2\,,$$

we have

$$\begin{aligned}&\frac{d^2}{dt^2}\mathrm{Vol}(l_t(L))\Big|_{t=0}\\ &= \pi\sqrt{2r}\int_0^\infty \Bigg\{\left(\frac{\partial^2\zeta}{\partial t^2} + \frac{2}{t}\frac{\partial\zeta}{\partial t} - \frac{(1+r)\zeta}{t^2}\right)^2 \int_0^{2\pi\sqrt{\frac{2}{2+r}}} \cos^2\left(\sqrt{\frac{2+r}{2}}s_1\right)ds_1\\ &\qquad - t^{-4}(r^2-1)\zeta^2 \int_0^{2\pi\sqrt{\frac{2}{2+r}}} \sin^2\left(\sqrt{\frac{2+r}{2}}s_1\right)ds_1\Bigg\}t\,dt\\ &= 2\pi^2\sqrt{\frac{r}{2+r}}\int_0^\infty\left\{\left(\frac{\partial^2\zeta}{\partial t^2} + \frac{2}{t}\frac{\partial\zeta}{\partial t} - \frac{(1+r)\zeta}{t^2}\right)^2 - t^{-4}(r^2-1)\zeta^2\right\}t\,dt\,. \qquad (7)\end{aligned}$$

Using the function ζ defined above, (7) becomes three integrals:

$$\begin{aligned}&2\pi^2\sqrt{\frac{r}{2+r}}\int_{\frac{\varepsilon}{2}}^{\varepsilon}\left\{\left(\frac{\partial^2\delta}{\partial t^2} + \frac{2}{t}\frac{\partial\delta}{\partial t} - \frac{(1+r)\delta}{t^2}\right)^2 - t^{-4}(r^2-1)\delta^2\right\}t\,dt\\ &+ 2\pi^2\sqrt{\frac{r}{2+r}}\int_{\varepsilon}^{1}(2-2r)\frac{dt}{t}\\ &+ 2\pi^2\sqrt{\frac{r}{2+r}}\int_1^2\left\{\left(\frac{\partial^2\eta}{\partial t^2} + \frac{2}{t}\frac{\partial\eta}{\partial t} - \frac{(1+r)\eta}{t^2}\right)^2 - t^{-4}(r^2-1)\eta^2\right\}t\,dt\,.\end{aligned}$$

It is easy to check that the absolute value of the first integral is bounded by a constant which depends on r but is independent of ε. The third integral is obviously bounded. The second integral equals

$$4\pi^2\sqrt{\frac{r}{2+r}}(r-1)\log\varepsilon\,.$$

Therefore, for sufficiently small ε, the right hand side of equation (7) is negative. □

8. Open questions

Finally, we pose some problems related to this paper. We proved in Section 7 that H-minimal Lagrangian cones in $\mathbf{C}^3$ of type $(1, 1, r)$ $(r \geq 2)$ are Hamiltonian unstable. However, the following problem still remains:

PROBLEM A. Determine all Hamiltonian stable H-minimal Lagrangian cones of type (p, q, r) in $\mathbf{C}^3$.

We will investigate this problem in the future.

Another question we want to raise is concerned with the construction of H-minimal Lagrangian cones in $\mathbf{C}^3$ other than of type (p, q, r). Each cone of type (p, q, r) possesses T^2-symmetry, so it is natural to consider the following problem:

PROBLEM B. Construct new examples of H-minimal Lagrangian cones in $\mathbf{C}^3$ with S^1-symmetries.

This problem was treated by Haskins [3] in the case of special Lagrangian cones.

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References

[1] D. E. BLAIR, *Contact manifolds in Riemannian geometry*, Lecture Notes in Mathematics **509**, (1976) Springer,

[2] R. HARVEY and H. B. LAWSON, Calibrated geometries, Acta Math. **148** (1982), 47–157.

[3] M. HASKINS, Special Lagrangian cones, Amer. J. Math. **126** (2004), 845–871.

[4] D. JOYCE, Special Lagrangian m-folds in $\mathbf{C}^m$ with symmetries, Duke Math. J. **115** (2002), 1–51.

[5] Y.G. OH, Volume minimization of Lagrangian submanifolds under Hamiltonian deformations, Math. Z. **212** (1993), 175-192.

[6] H. ONO, Second variation and Legendrian stabilities of minimal Legendrian submanifolds in Sasakian manifolds, preprint.

[7] H. RECKZIEGEL, A correspondence between horizontal submanifolds of Sasakian manifolds and totally real submanifolds of Kähler manifolds, *Topics in differential geometry*, *Vol. I*, *II* (Debrecen, 1984), North-Holland, (1988), 1063–1081.

[8] R. SCHOEN and J. WOLFSON, Minimizing area among lagrangian surfaces: The mapping problem, J. Diff. Geom. **58** (2001), 1–86.

[9] A. STROMINGER, S.T. YAU and E. ZASLOW, Mirror symmetry is T-duality, Nucl. Phys. B **479** (1996), 243–259.

[10] J. SIMONS, Minimal varieties in Riemannian manifolds, Ann. Math. **88** (1968), 62–105.

[11] S. T. YAU, Submanifolds with constant mean curvature I, Amer. J. Math. **96**(1974), 346–366.

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