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On the Reduced Lefschetz Module and the Centric p-Radical Subgroups

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Abstract. The purpose of this paper is to show that the reduced Lefschetz module of the G-poset $\mathcal{B}_p^{cen}(G)$ consisting of all centric p-radical subgroups of a finite group G is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module where $\mathcal{X}$ is a family of p-subgroups of the normalizers of non-centric p-radical subgroups of G. As corollary, we have a lower bound of the p-power of the reduced Euler characteristic $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$.

1. Introduction

A non-trivial p-subgroup U of a finite group G is *p-radical* (resp. *p-centric*) if $O_p(N_G(U)) = U$ (resp. if any p-element in $C_G(U)$ lies in U). Denote by $\mathcal{B}_p(G)$ (resp. $\mathcal{B}_p^{cen}(G)$) the set of all p-radicals of G (resp. the set of all elements in $\mathcal{B}_p(G)$ which are p-centrics of G). Recall that the subgroup family $\mathcal{B}_p(G)$ is regarded as a G-poset with respect to the inclusion-relation together with G-conjugate action, and is also viewed as a G-simplicial complex defined by the inclusion-chains as simplices. Then $\mathcal{B}_p(G)$ is now a standard G-poset (or G-complex) in "Subgroup Complexes of Finite Groups" (see e.g. [1, Chapter 6]). But the importance of the subcomplex $\mathcal{B}_p^{cen}(G)$ also appeared in the study of p-local geometry of a finite group, group cohomology, or even modular representation theory. In particular, concerning group geometry, a lot of known important p-local geometries for sporadic simple groups can be realized as the smaller $\mathcal{B}_p^{cen}(G)$ rather than the whole $\mathcal{B}_p(G)$ (see [9, 11]). Furthermore it is known concerning group cohomology that $\mathcal{B}_p^{cen}(G)$ induces the alternating sum decomposition of the cohomology $H^*(G)_p$ (cf. [6, Theorem 9.1]) as well as $\mathcal{B}_p(G)$ does (cf. [14, Theorem A]).

However one of the troubles to treat $\mathcal{B}_p^{cen}(G)$ is that this is not homotopy equivalent to $\mathcal{B}_p(G)$ in general. For example, since the reduced Lefschetz module $\tilde{L}_G(\mathcal{B}_p(G))$, called the generalized Steinberg module for G, is a projective virtual $\mathbb{Z}_p[G]$-module, its dimension $\tilde{\chi}(\mathcal{B}_p(G))$ is divisible by the highest p-power $|G|_p$ of the order of G. But this is usually no longer true for $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$, and in fact there are a lot of works on the calculation of $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$ by hand for sporadic simple groups G. Among those works, S. D. Smith [10,

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p. 306] explained the reason, using [12, Theorem 2.1] of Thévenaz on the reduced Lefschetz module, that why the value $\tilde{\chi}(\Delta)_2 = 2^4$ is less than $|M_{12}|_2 = 2^6$ where Δ is the 2-local geometry for the Mathieu simple group M_{12} of degree 12. This observation actually inspired us to study the general phenomenon of the reduced Euler characteristic $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$ and the reduced Lefschetz module $\tilde{L}_G(\mathcal{B}_p^{cen}(G))$. In this paper, we will show in particular that $\tilde{L}_G(\mathcal{B}_p^{cen}(G))$ is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module where $\mathcal{X}$ is a family of p-subgroups of the normalizers of non-centric p-radical subgroups of G. As corollary, we have a lower bound of the p-power of $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$. Our method to study the reduced Lefschetz module is based on an idea developed by Thévenaz [12].

In Section 2, we will recall some definitions and basic results on the representation ring, the Burnside ring, and the Lefschetz invariant $\tilde{\Lambda}_G(\mathcal{P})$ and module $\tilde{L}_G(\mathcal{P})$ of a G-poset $\mathcal{P}$. In Section 3, we will examine some properties of $\tilde{\Lambda}_G(\mathcal{P})$ and $\tilde{L}_G(\mathcal{P})$; that will be applied in Section 4 to the collection $\mathcal{B}_p^{cen}(G)$ of p-radical and p-centric subgroups of G.

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2. Preliminaries

In this section, we will recall some definitions and basic results on the representation ring, the Burnside ring, and the (reduced) Lefschetz invariant and module. For the details, refer to [1, Chapter 6] or [12].

2.1. The representation ring and the Burnside ring. Let G be a finite group, p a prime divisor of the order of G, and $\mathbb{Z}_p$ the p-adic integers. First we recall the representation ring $A(G)$ of $\mathbb{Z}_p[G]$-modules. This is the $\mathbb{Q}$-vector space generated by isomorphism classes $[M]$ of finitely generated $\mathbb{Z}_p[G]$-modules M, with relations

$$[M] + [N] := [M \oplus N] \quad \text{and} \quad [M] \cdot [N] := [M \otimes_{\mathbb{Z}_p} N]\,,$$

where $\oplus$ the direct sum and $\otimes_{\mathbb{Z}_p}$ the tensor product. Thus classes of indecomposable finitely generated $\mathbb{Z}_p[G]$-modules form a $\mathbb{Q}$-basis of $A(G)$. Given a family $\mathcal{X}$ of subgroups of G closed under taking G-conjugation and forming subgroups, we denote by $A(G, \mathcal{X})$ an ideal of the ring $A(G)$ spanned by summands of sums of relatively H-projective $\mathbb{Z}_p[G]$-modules for some $H \in \mathcal{X}$. Recall that the definition of a relatively H-projective $\mathbb{Z}_p[G]$-module is that it is a direct summand of $\mathrm{Ind}_H^G W$ induced from a $\mathbb{Z}_p[H]$-module W. An element in $A(G, \mathcal{X})$ is called an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module. In particular, if $\mathcal{X} = \{\{1\}\}$ the trivial subgroup of G then an element in $A(G, \mathcal{X})$ is called a projective virtual $\mathbb{Z}_p[G]$-module.

On the other hand, the Burnside ring $B(G)$ of G is also the $\mathbb{Q}$-vector space generated by equivalence classes $[X]$ of finite G-sets X, with relations

$$[X] + [Y] := [X \uplus Y] \quad \text{and} \quad [X] \cdot [Y] := [X \times Y]\,,$$

where $\uplus$ the disjoint union and $\times$ the Cartesian product. Thus classes of transitive finite G-sets (that is, those of cosets G/H for all subgroups H of G) form a $\mathbb{Q}$-basis of $B(G)$. Recall that for each subgroup H of G, there is a primitive idempotent $e_{G,H}$ of $B(G)$ characterized by

$$\sharp\{x \in e_{G,H} \mid gx = x \ (\forall g \in K)\} = \begin{cases} 1 & \text{if } K \text{ is conjugate to } H \\ 0 & \text{otherwise} \end{cases}$$

for a subgroup K of G (cf. [12, p.124]). Then $\{e_{G,H} \mid H \leq G\}$ gives the set of all primitive idempotents in $B(G)$. The following is a well-known formula of $e_{G,H}$ (cf. [5, Theorem 3.1] or [16, Theorem 3.1]):

LEMMA 1 (Gluck [5], Yoshida [16]). *For a subgroup H of G, we have*

$$e_{G,H} = \frac{1}{|N_G(H)|} \sum_{K \leq H} |K|\mu(K, H)[G/K]$$

in $B(G)$ where μ is the Möbius function on the lattice of all subgroups of G.

Next we consider a natural homomorphism

$$r : B(G) \to A(G)$$

defined by $r([G/H]) := [\mathrm{Ind}_H^G \mathbb{Z}_p]$; the permutation representation on a basis $G/H = \{gH \mid g \in G\}$. Let $\mathcal{H}(G) := \{H \leq G \mid H/O_p(H) \text{ is cyclic}\}$. A subgroup in $\mathcal{H}(G)$ is called *cyclic mod-p* of G. The significance of $\mathcal{H}(G)$ can be found in [3, Section 4] as follows:

LEMMA 2 (Conlon [3]). *For a subgroup H of G, $r(e_{G,H}) = 0$ if and only if $H \notin \mathcal{H}(G)$.*

REMARK 1. For a subgroup H of G, $r(e_{G,H})$ is a linear combination of $r([G/K]) = [\mathrm{Ind}_K^G \mathbb{Z}_p] = [\mathrm{Ind}_H^G(\mathrm{Ind}_K^H \mathbb{Z}_p)]$ for $K \leq H$ by Lemma 1. Thus $r(e_{G,H})$ is the sum of relatively H-projective $\mathbb{Z}_p[G]$-modules in $A(G)$ (see the definition of the relative projectivity above).

2.2. The Lefschetz invariant and module. Let $\mathcal{P}$ be a G-poset; that is, a finite poset with an order preserving G-action. The order G-complex of $\mathcal{P}$ is a simplicial complex whose q-simplices are chains $x_0 < x_1 < \cdots < x_q$ ($x_i \in \mathcal{P}$), and on which G acts naturally. Throughout this paper, we do not distinguish between a G-poset $\mathcal{P}$ and its order G-complex denoted by the same notation $\mathcal{P}$. Note that an order G-complex $\mathcal{P}$ is admissible; namely if $g \in G$ fixes a simplex $x_0 < \cdots < x_q$ in $\mathcal{P}$ then all vertices x_i ($i = 0, \cdots, q$) are fixed by g. Now let $\mathcal{P}_q$ be the set of all q-simplices in $\mathcal{P}$. Then $\mathcal{P}_q$ is a finite G-set, and define elements in $B(G)$ as follows:

$$\Lambda_G(\mathcal{P}) := \sum_{q=0}^{\dim(\mathcal{P})} (-1)^q \mathcal{P}_q\,, \quad \tilde{\Lambda}_G(\mathcal{P}) := \Lambda_G(\mathcal{P}) - 1\,,$$

where the dimension $\dim(\mathcal{P})$ is the maximal length of chains in $\mathcal{P}$, and $1 \in B(G)$ is the trivial G-set. The element $\Lambda_G(\mathcal{P})$ (resp. $\tilde{\Lambda}_G(\mathcal{P})$) is called the *Lefschetz* (resp. *reduced Lefschetz*) *invariant* of $\mathcal{P}$. Recall that using a basis $\{e_{G,H} \mid H \leq G\}$ of $B(G)$, $\Lambda_G(\mathcal{P})$ and $\tilde{\Lambda}_G(\mathcal{P})$ can be expressed as follows (cf. [12, Proposition 1.2]):

$$(1) \qquad \Lambda_G(\mathcal{P}) = \sum_{H \in \mathrm{Sgp}(G)/G} \chi(\mathcal{P}^H) e_{G,H}\,, \quad \tilde{\Lambda}_G(\mathcal{P}) = \sum_{H \in \mathrm{Sgp}(G)/G} \tilde{\chi}(\mathcal{P}^H) e_{G,H}\,,$$

where $\mathrm{Sgp}(G)$ the set of all subgroups of G, $\mathrm{Sgp}(G)/G$ the G-conjugacy classes of $\mathrm{Sgp}(G)$, $\mathcal{P}^H = \{x \in \mathcal{P} \mid hx = x \ (\forall h \in H)\}$ the set of fixed points by H, and $\chi(\mathcal{P}^H)$ (resp. $\tilde{\chi}(\mathcal{P}^H)$) the Euler (resp. reduced Euler) characteristic of $\mathcal{P}^H$. Therefore $\Lambda_G(\mathcal{P})$ depends only on the family $\{\chi(\mathcal{P}^H) \mid H \in \mathrm{Sgp}(G)/G\}$ of the Euler characteristics, and $\tilde{\Lambda}_G(\mathcal{P})$ as well. Finally the *Lefschetz* (resp. *reduced Lefschetz*) *module* of $\mathcal{P}$ over $\mathbb{Z}_p$ is defined as an element $L_G(\mathcal{P}) := r(\Lambda_G(\mathcal{P}))$ (resp. $\tilde{L}_G(\mathcal{P}) := r(\tilde{\Lambda}_G(\mathcal{P}))$) in $A(G)$.

3. Some properties of $\tilde{\Lambda}_G(\mathcal{P})$ and $\tilde{L}_G(\mathcal{P})$

In this section, we will study the reduced Lefschetz invariant and module of a certain G-subposet of a G-poset applying a result of Thévenaz [12]. Let X be a finite H-set for a subgroup H of G. Then a finite G-set $G \times_H X$ is defined as the quotient $(G \times X)/\sim$ with respect to $(gh, x) \sim (g, hx)$ for $h \in H$, and set $\mathrm{Ind}_H^G[X] := [G \times_H X]$ in $B(G)$. In particular, we have that $\mathrm{Ind}_H^G[H/S] = [G/S]$ for $S \leq H \leq G$. The following result [12, Corollary 3.4] of Thévenaz will be used in our investigation.

LEMMA 3 (Thévenaz [12]). *Let $\mathcal{P}$ and $\mathcal{Q}$ be G-posets, and $f : \mathcal{Q} \to \mathcal{P}$ an order-preserving G-map. Then we have*

$$\tilde{\Lambda}_G(\mathcal{P}) = \tilde{\Lambda}_G(\mathcal{Q}) + \sum_{x \in \mathcal{P}/G} \mathrm{Ind}_{G_x}^G(\tilde{\Lambda}_{G_x}(f^{-1}(\mathcal{P}_{\leq x})) \cdot \tilde{\Lambda}_{G_x}(\mathcal{P}_{>x}))$$

in $B(G)$, where G_x the stabilizer of x in G, and $\mathcal{P}/G$ is the G-conjugacy classes of $\mathcal{P}$. Note that $\mathcal{P}_{\leq x} = \{y \in \mathcal{P} \mid y \leq x\}$, $\mathcal{P}_{>x} = \{y \in \mathcal{P} \mid y > x\}$, and $f^{-1}(\mathcal{P}_{\leq x})$ are G_x-posets.

From now on, we will consider the following situation on G-posets $\mathcal{P}$ and $\mathcal{Q}$.

HYPOTHESIS (**H**). $\mathcal{Q}$ is a G-subposet of $\mathcal{P}$ such that $\mathcal{Q}_{\leq x} = \emptyset$ for each x in $(\mathcal{P} \setminus \mathcal{Q})$.

REMARK 2. Recall that a subset R of a poset $\mathcal{P}$ is called "closed" if $x \leq y$ for $x \in \mathcal{P}$ and $y \in R$ implies $x \in R$; which is mentioned by Quillen [7, p. 103]. (Note that this condition is also called "order ideal" in the more combinatorial literature.) He demonstrated in [7, Corollary 1.8] that a closed subset in the product $\mathcal{P} \times \mathcal{Q}$ provides some technique of showing the homotopy equivalence between posets $\mathcal{P}$ and $\mathcal{Q}$. This technique of Quillen is further developed by Smith-Yoshiara in [11, p. 332]. Here we mention that the result [12, Corollary 3.4] above is also based on the idea of "closed sets in products". Indeed, Thévenaz considers in [12, Theorem 3.3] (from which the Corollary is obtained) a poset $\mathcal{P} +_R \mathcal{Q}$ defined by a certain closed subset R of $\mathcal{P} \times \mathcal{Q}$, though there R is called an “ideal relation” in $\mathcal{P} \times \mathcal{Q}$. Furthermore the referee indicated to the author that Hypothesis (**H**) should be also related to “closed sets”. Indeed, the condition (**H**) is rephrased that $(\mathcal{P} \setminus \mathcal{Q})$ is a closed subset of $\mathcal{P}$.

PROPOSITION 1. *Let $\mathcal{P}$ and $\mathcal{Q}$ be G-posets satisfying Hypothesis* (**H**). *Then we have*

1. $$\tilde{\Lambda}_G(\mathcal{P}) = \tilde{\Lambda}_G(\mathcal{Q}) - \sum_{x \in (\mathcal{P} \setminus \mathcal{Q})/G} \left(\sum_{S \in \mathrm{Sgp}(G_x)/G_x} \alpha(x, S) e_{G,S} \right),$$
   where $\alpha(x, S) := \tilde{\chi}((\mathcal{P}_{>x})^S)|N_G(S) : N_{G_x}(S)|$ *an integer.*
2. $$\tilde{L}_G(\mathcal{P}) = \tilde{L}_G(\mathcal{Q}) - \sum_{x \in (\mathcal{P} \setminus \mathcal{Q})/G} \left(\sum_{S \in \mathcal{H}(G_x)/G_x} \alpha(x, S) r(e_{G,S}) \right).$$

PROOF. 1. Let $\phi : \mathcal{Q} \hookrightarrow \mathcal{P}$ be the inclusion G-map. Then for each $x \in \mathcal{P}$, we have that $\phi^{-1}(\mathcal{P}_{\leq x}) = \mathcal{Q}_{\leq x}$. If $x \in \mathcal{Q}$ then $\phi^{-1}(\mathcal{P}_{\leq x})$ has the unique maximal element x, and hence is G_x-contractible. On the other hand, if $x \in (\mathcal{P} \setminus \mathcal{Q})$ then $\phi^{-1}(\mathcal{P}_{\leq x})$ is empty by Hypothesis (**H**). Thus we have that

$$\phi^{-1}(\mathcal{P}_{\leq x}) = \begin{cases} G_x\text{-contractible} & \text{if } x \in \mathcal{Q} \\ \emptyset & \text{if } x \in (\mathcal{P} \setminus \mathcal{Q}), \end{cases}$$

and this follows that

$$\tilde{\Lambda}_{G_x}(\phi^{-1}(\mathcal{P}_{\leq x})) = \begin{cases} 0 & \text{if } x \in \mathcal{Q} \\ -1 & \text{if } x \in (\mathcal{P} \setminus \mathcal{Q}). \end{cases}$$

Note that the above numbers 0 and -1 are the multiples of 0 and -1 by the trivial G_x-set; which is the sum of all primitive idempotents in $B(G_x)$. Hence by Lemma 3, we get that

$$\tilde{\Lambda}_G(\mathcal{P}) = \tilde{\Lambda}_G(\mathcal{Q}) - \sum_{x \in (\mathcal{P} \setminus \mathcal{Q})/G} \mathrm{Ind}_{G_x}^G (\tilde{\Lambda}_{G_x}(\mathcal{P}_{>x})). \tag{2}$$

On the other hand, since

$$\tilde{\Lambda}_{G_x}(\mathcal{P}_{>x}) = \sum_{S \in \mathrm{Sgp}(G_x)/G_x} \tilde{\chi}((\mathcal{P}_{>x})^S) e_{G_x,S}$$

by (1) in Section 2.2, we have that

$$\mathrm{Ind}_{G_x}^G (\tilde{\Lambda}_{G_x}(\mathcal{P}_{>x})) = \sum_{S \in \mathrm{Sgp}(G_x)/G_x} \tilde{\chi}((\mathcal{P}_{>x})^S)\, \mathrm{Ind}_{G_x}^G\, e_{G_x,S}. \tag{3}$$

Recall that

$$\mathrm{Ind}_{G_x}^G\, e_{G_x,S} = \mathrm{Ind}_{G_x}^G \left[\frac{1}{|N_{G_x}(S)|} \sum_{K \leq S} |K| \mu(K, S)[G_x/K] \right]$$

$$= \frac{1}{|N_{G_x}(S)|} \sum_{K \le S} |K|\mu(K, S) \operatorname{Ind}_{G_x}^{G}[G_x/K]$$
$$= \frac{|N_G(S) : N_{G_x}(S)|}{|N_G(S)|} \sum_{K \le S} |K|\mu(K, S)[G/K]$$
$$= |N_G(S) : N_{G_x}(S)| e_{G,S} \quad \text{(cf. [16, Lemma 3.5(1)])}.$$

Note that $\operatorname{Ind}_{G_x}^{G}[G_x/K] = [G/K]$ as mentioned in the first paragraph of this section. Thus (3) above can be expressed as

$$\operatorname{Ind}_{G_x}^{G}(\tilde{\Lambda}_{G_x}(\mathcal{P}_{>x})) = \sum_{S \in \mathrm{Sgp}(G_x)/G_x} \tilde{\chi}((\mathcal{P}_{>x})^S)|N_G(S) : N_{G_x}(S)| e_{G,S}\,. \tag{4}$$

Therefore combining (2) with (4), the first assertion is proved.

2. Applying the homomorphism $r : B(G) \to A(G)$ defined in Section 2.1, we have that

$$\tilde{L}_G(\mathcal{P}) = \tilde{L}_G(\mathcal{Q}) - \sum_{x \in (\mathcal{P} \backslash \mathcal{Q})/G} \left(\sum_{S \in \mathrm{Sgp}(G_x)/G_x} \alpha(x, S) r(e_{G,S}) \right).$$

But $r(e_{G,S}) = 0$ if $S \notin \mathcal{H}(G) \cap \mathrm{Sgp}(G_x) = \mathcal{H}(G_x)$ by Lemma 2. The proof is complete. □

PROPOSITION 2. *Let $\mathcal{P}$ and $\mathcal{Q}$ be G-posets satisfying Hypothesis* (**H**)*. If $\tilde{L}_G(\mathcal{P})$ is a projective virtual $\mathbb{Z}_p[G]$-module then $\tilde{L}_G(\mathcal{Q})$ is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module where*

$$\mathcal{X} = \{X \le O_p(S) \mid S \in \mathcal{H}(G_x),\ x \in (\mathcal{P} \backslash \mathcal{Q}),\ \tilde{\chi}((\mathcal{P}_{>x})^S) \ne 0\}.$$

PROOF. By Proposition 1(2), we have that

$$\tilde{L}_G(\mathcal{P}) = \tilde{L}_G(\mathcal{Q}) - \sum_{x \in (\mathcal{P} \backslash \mathcal{Q})/G} \left(\sum_{S \in \mathcal{H}(G_x)/G_x} \alpha(x, S) r(e_{G,S}) \right).$$

Recall that by definition, $\alpha(x, S) \ne 0$ if and only if $\tilde{\chi}((\mathcal{P}_{>x})^S) \ne 0$. Now $r(e_{G,S})$ is the sum of relatively S-projective $\mathbb{Z}_p[G]$-modules in $A(G)$ as mentioned in Remark 1 of Section 2. But since S is cyclic mod-p, we have that $p \nmid |S : O_p(S)|$ by definition. Thus $r(e_{G,S})$ is the sum of relatively $O_p(S)$-projective $\mathbb{Z}_p[G]$-modules in $A(G)$.

Now standard and routine arguments show that $\mathcal{X}$ is closed under G-conjugation and subgroups. Indeed, first we recall that $\bigcup_{x \in (\mathcal{P} \backslash \mathcal{Q})} \mathcal{H}(G_x)$ and $(\mathcal{P} \backslash \mathcal{Q})$ are invariant by G-conjugation. If $S \in \mathcal{H}(G_x)$ and $x \in (\mathcal{P} \backslash \mathcal{Q})$ then for $g \in G$, we have that $\mathcal{D} := (\mathcal{P}_{>g^{-1}x})^{g^{-1}Sg} = \{g^{-1}z \mid z \in (\mathcal{P}_{>x})^S\}$; which is isomorphic to $(\mathcal{P}_{>x})^S$ as poset. This implies that $\tilde{\chi}(\mathcal{D}) = \tilde{\chi}((\mathcal{P}_{>x})^S)$, and thus if $\tilde{\chi}((\mathcal{P}_{>x})^S) \ne 0$ then so is $\tilde{\chi}(\mathcal{D})$. (Note also that $g^{-1}G_x g = G_{g^{-1}x}$.) This means that $\mathcal{X}$ is closed under G-conjugation. Since it is clear that $\mathcal{X}$ is closed under forming subgroups, the proof is complete. □

REMARK 3. For $x \in (\mathcal{P}\backslash\mathcal{Q})$ and $S \in \mathcal{H}(G_x)$, consider a poset $\hat{P} := (\mathcal{P}_{\geq x})^S = (\mathcal{P}_{>x})^S \cup \{x\}$ having the unique minimal element x. The reduced Euler characteristic of $\hat{P}\backslash\{x\}$ appeared in Proposition 2 can be expressed as follows (cf. [16, Lemma 2.4]):

$$\tilde{\chi}((\mathcal{P}_{>x})^S) = \tilde{\chi}(\hat{P}\backslash\{x\}) = -\sum_{y\in\hat{P}} \mu(x, y)$$

where μ is the Möbius function on the poset $\hat{P}$.

COROLLARY 1. *Keep the assumption of Proposition* 2. *Let $p^n = |G|_p$ the p-part of the order of G, and*

$$\begin{aligned} p^{d_1} &= \max\{|X| \mid X \in \mathcal{X}\} \\ &= \max\{|O_p(S)| \mid S \in \mathcal{H}(G_x),\ x \in (\mathcal{P}\backslash\mathcal{Q}),\ \tilde{\chi}((\mathcal{P}_{>x})^S) \neq 0\}\,. \end{aligned}$$

Then $\tilde{\chi}(\mathcal{Q})$ is divisible by p^{n-d_1}.

PROOF. Recall that $\tilde{\chi}(\mathcal{Q}) = \dim(\tilde{L}_G(\mathcal{Q}))$ by their definitions, and that $\tilde{L}_G(\mathcal{Q})$ is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module by Proposition 3. Then the assertion clearly holds from the fact that if p^a divides the index $|G : H|$ of a subgroup H of G then p^a divides the dimension of any H-projective $\mathbb{Z}_p[G]$-module. □

However, it is hard in general to determine the subgroup family $\mathcal{X}$ or even $\mathcal{H}(G_x)$. Thus we hope to know, from group theoretical properties of G_x for x in $(\mathcal{P}\backslash\mathcal{Q})$, a positive integer d near d_1. To do this, for each x in $(\mathcal{P}\backslash\mathcal{Q})$, let

$$\{\langle z_{x,1}\rangle, \cdots, \langle z_{x,l_x}\rangle, \langle z_{x,l_x+1}\rangle, \cdots, \langle z_{x,m_x}\rangle\}$$

be a complete set of G_x-conjugate classes of all subgroups of order p in G_x such that $(\mathcal{P}_{>x})^{\langle z_{x,i}\rangle}$ $(i = 1, \cdots, l_x)$ is contractible, and $(\mathcal{P}_{>x})^{\langle z_{x,j}\rangle}$ $(j = l_x + 1, \cdots, m_x)$ is not.

LEMMA 4. *For $x \in (\mathcal{P}\backslash\mathcal{Q})$ and $S \in \mathcal{H}(G_x)$ with $O_p(S) \neq 1$, we have the following.*

1. *If S contains a subgroup G_x-conjugate to $\langle z_{x,i}\rangle$ for $i = 1, \cdots, l_x$, then $\tilde{\chi}((\mathcal{P}_{>x})^S) = 0$.*
2. *If $O_p(S) \in \mathcal{X}$ (see Proposition 2 for $\mathcal{X}$) then any subgroup of order p in $O_p(S)$ is G_x-conjugate to $\langle z_{x,j}\rangle$ for some $j = l_x + 1, \cdots, m_x$.*

This was shown in the proof of Theorems A and A' in [14, p. 148], but we will give a sketch of the proof following [14].

PROOF. 1. Up to conjugacy, we may assume that $\langle z_{x,i}\rangle \leq O_p(S) \leq S$, and then there exists a normal chain $\langle z_{x,i}\rangle = K_1 \lhd K_2 \lhd \cdots \lhd K_{l-1} \lhd K_l = O_p(S)$. Now since $(\mathcal{P}_{>x})^{K_1}$ is contractible by our choice of $\langle z_{x,i}\rangle$, it is $\mathbb{Z}_p$-acyclic (*i.e.* $\tilde{H}_*((\mathcal{P}_{>x})^{K_1}, \mathbb{Z}_p) = 0$). Applying P. A. Smith's theorem [2, Theorem 10.5(b) in VII] on fixed-points subcomplex, $(\mathcal{P}_{>x})^{K_2} = ((\mathcal{P}_{>x})^{K_1})^{K_2}$ is also $\mathbb{Z}_p$-acyclic. (Note that K_{i+1} acts on $(\mathcal{P}_{>x})^{K_i}$ for $i = 1, \cdots, l-1$.) Repeating this process, we have that $(\mathcal{P}_{>x})^{O_p(S)}$ is $\mathbb{Z}_p$-acyclic, and hence $\mathbb{Q}$-acyclic. On the other hand, since $S/O_p(S)$ is a cyclic group $\langle gO_p(S)\rangle$ for some $g \in S$, we have that $(\mathcal{P}_{>x})^S = ((\mathcal{P}_{>x})^{O_p(S)})^g$. Thus by the "Lefschetz trace formula", $\tilde{\chi}((\mathcal{P}_{>x})^S) = \tilde{\chi}(((\mathcal{P}_{>x})^{O_p(S)})^g) = Tr(g, \tilde{L}_S((\mathcal{P}_{>x})^{O_p(S)})) = 0$. The last equality is due to the $\mathbb{Q}$-acyclicity of $(\mathcal{P}_{>x})^{O_p(S)}$.

2. Straightforward from the first assertion and the definition of $\mathcal{X}$. The proof is complete. □

For each x in $(\mathcal{P}\backslash\mathcal{Q})$, let R_x be a p-subgroup of G_x of maximal order such that any subgroup of order p in R_x is G_x-conjugate to $\langle z_{x,j}\rangle$ for some $j = l_x + 1, \cdots, m_x$. Then by Lemma 4(2), we have that $|O_p(S)| \leq |R_x|$ for any $S \in \mathcal{H}(G_x)$ with $O_p(S) \in \mathcal{X}$. Therefore

COROLLARY 2. *Keep the assumption of Proposition 2. Let $p^n = |G|_p$ and $p^d = \max\{|R_x| \mid x \in (\mathcal{P}\backslash\mathcal{Q})\}$ where R_x is defined as above. Then $\tilde{\chi}(\mathcal{Q})$ is divisible by p^{n-d}.*

4. The centric p-radical subgroups

In this section, we will apply the result in Section 3 to the G-poset consisting of all (centric) p-radical subgroups. Let $\mathcal{P}$ be a G-poset of p-subgroups of G, that is, a poset of p-subgroups closed under G-conjugation. A p-subgroup U of G is *p-centric* if any p-element in $C_G(U)$ lies in U. Denote by $\mathcal{P}^{cen}$ the set of all p-centrics in $\mathcal{P}$, which is invariant by G-conjugation.

LEMMA 5. *For each U in $(\mathcal{P} \setminus \mathcal{P}^{cen})$, we have $(\mathcal{P}^{cen})_{\leq U} = \emptyset$.*

PROOF. Since U is not p-centric, there exists a p-element x in $C_G(U) \setminus U$. Suppose now that there exists R in $(\mathcal{P}^{cen})_{\leq U}$, *i.e.* $R \leq U$. Then x centralizes R, but not in R itself from our choice of x. But this contradicts that R is p-centric. Therefore $(\mathcal{P}^{cen})_{\leq U} = \emptyset$. □

This lemma tells us that $\mathcal{P}$ and $\mathcal{P}^{cen}$ satisfy Hypothesis (**H**); namely we can say that the non-centric collection $(\mathcal{P} \setminus \mathcal{P}^{cen})$ is a "closed subset" of $\mathcal{P}$ as discussed in Remark 2 of Section 3. Now we have the following by Proposition 2:

PROPOSITION 3. *Let $\mathcal{P}$ be a G-poset of p-subgroups of G. Suppose that $\tilde{L}_G(\mathcal{P})$ is a projective virtual $\mathbb{Z}_p[G]$-module. Then $\tilde{L}_G(\mathcal{P}^{cen})$ is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module where*

$$\mathcal{X} = \{X \leq O_p(S) \mid S \in \mathcal{H}(G_U),\ U \in (\mathcal{P}\backslash\mathcal{P}^{cen}),\ \tilde{\chi}((\mathcal{P}_{>U})^S) \neq 0\}\,.$$

A non-trivial p-subgroup U of G is *p-radical* if $O_p(N_G(U)) = U$, and denote by $\mathcal{B}_p(G)$ the set of all p-radicals of G. The property "p-radical" is invariant by G-conjugation, and thus $\mathcal{B}_p(G)$ is a G-poset of p-subgroups. Denote by $\mathcal{B}_p^{cen}(G)$ the set of all p-centrics in $\mathcal{B}_p(G)$.

REMARK 4. 1. It is worth mentioning that $O_p(G)$ is the unique minimal p-radical subgroup of G by definition, so if $O_p(G)$ is added in $\mathcal{B}_p(G)$ then the poset $\mathcal{B}_p(G)$ is always contractible. Thus it may be suitable to exclude $O_p(G)$ from $\mathcal{B}_p(G)$; that is, the poset $\mathcal{B}_p(G)$ may be defined as the set of all p-subgroups U having the property that $O_p(N_G(U)) = U$

but except for $O_p(G)$. But this definition is the same as the earlier one when we are in the situation like "G is simple" for example.

2. It is also worth while to mention that Dwyer showed in [4, Section 4] that the collection $\mathcal{B}_p^{cen}(G)$ gives a "sharp subgroup decomposition" of the homology of the classifying space BG of a finite group G. Indeed, he considered the collection of all non-trivial p-subgroups of G from which all non-centric members are eliminated, and further subcollection obtained from that by eliminating all non-radical members. (Note that a p-radical subgroup of G is called in [4] "p-stubborn".) Thus the resulting collection is nothing else but $\mathcal{B}_p^{cen}(G)$ in our notation. Furthermore Grodal [6, Theorem 9.1] obtained the "normalizer-sharpness" for $\mathcal{B}_p^{cen}(G)$ as well.

PROPOSITION 4. 1. *$\tilde{L}_G(\mathcal{B}_p^{cen}(G))$ is an $\mathcal{X}$-projective virtual $\mathbb{Z}_p[G]$-module where*

$$\mathcal{X} = \{X \le O_p(S) \mid S \in \mathcal{H}(G_U),\ U \in (\mathcal{B}_p(G)\backslash\mathcal{B}_p^{cen}(G)),\ \tilde{\chi}((\mathcal{B}_p(G)_{>U})^S) \neq 0\}\,.$$

2. *Let $p^n = |G|_p$ and*

$$p^{d_1} = \max\{|O_p(S)| \mid S \in \mathcal{H}(G_U),\ U \in (\mathcal{B}_p(G)\backslash\mathcal{B}_p^{cen}(G)),\ \tilde{\chi}((\mathcal{B}_p(G)_{>U})^S) \neq 0\}\,.$$

Then $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$ is divisible by p^{n-d_1}.

PROOF. 1. Since the G-poset $\mathcal{S}_p(G)$ consisting of all non-trivial p-subgroups of G is G-homotopy equivalent to $\mathcal{B}_p(G)$ (cf. [1, Theorem 6.6.6]), we have that $\tilde{L}_G(\mathcal{B}_p(G)) = \tilde{L}_G(\mathcal{S}_p(G))$ in $A(G)$. But it is well-known that the generalized Steinberg module $\tilde{L}_G(\mathcal{S}_p(G))$ for G is virtual projective (cf. [1, Theorem 6.7.2]). Thus the assertion follows from Proposition 3.

2. The same proof as in that of Corollary 1. $\square$

As in Corollary 2, we will consider the p-part of $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$ using the information of p-subgroups of G_U for each U in $(\mathcal{B}_p(G)\backslash\mathcal{B}_p^{cen}(G))$. Let

$$\{\langle z_{U,1}\rangle, \cdots, \langle z_{U,l_U}\rangle, \langle z_{U,l_U+1}\rangle, \cdots, \langle z_{U,m_U}\rangle\}\,,$$

for such U, be a complete set of G_U-conjugate classes of all subgroups of order p in G_U such that $(\mathcal{B}_p(G)_{>U})^{\langle z_{U,i}\rangle}$ $(i = 1, \cdots, l_U)$ is contractible, and $(\mathcal{B}_p(G)_{>U})^{\langle z_{U,j}\rangle}$ $(j = l_U + 1, \cdots, m_U)$ is not. Furthermore let R_U be a p-subgroup of G_U of maximal order such that any subgroup of order p in R_U is G_U-conjugate to $\langle z_{U,j}\rangle$ for some $j = l_U + 1, \cdots, m_U$. Then we have the following as in Corollary 2:

COROLLARY 3. *Let $p^n = |G|_p$ and $p^d = \max\{|R_U| \mid U \in (\mathcal{B}_p(G)\backslash\mathcal{B}_p^{cen}(G))\}$ where R_U is defined as above. Then $\tilde{\chi}(\mathcal{B}_p^{cen}(G))$ is divisible by p^{n-d}.*

We have discussed in this section taking the particular posets $\mathcal{B}_p(G)$ and $\mathcal{B}_p^{cen}(G)$, and examined the values of their reduced Euler characteristics. Here we will show the readers, who are not familiar with p-radical subgroups, that what they look like by exhibiting 2-radical

TABLE 1. $\mathcal{B}_2(Co_1)$ ([8, Table 1]).

R		$N_G(R)/R$	R		$N_G(R)/R$
V_1	$\cong 2^2$	$(3 \times G_2(4))2$	U_O	$\cong 2_+^{1+8}$	$O_8^+(2)$
V_2	$\cong V_1 2$	$G_2(2)$	U_{OT}	$\cong U_O 2^6$	$L_4(2)$
V_3	$\cong 2^2$	$(S_3 \times U_3(3))2$	U_{OS}	$\cong U_O 2^{1+8}$	$S_3 \times S_3 \times S_3$
U_T	$\cong 2^{2+12}$	$S_3 \times L_4(2)$	U_{OC}	$\cong U_O 2^6$	$L_4(2)$
U_{TS}	$\cong U_T 2^4$	$S_3 \times S_3 \times S_3$	$U_{O\square}$	$\cong U_O 2^6$	$L_4(2)$
U_{TC}	$\cong U_T 2^3$	$S_3 \times L_3(2)$	U_{OTS}	$\cong U_O 2^6 2^4$	$S_3 \times S_3$
$U_{T\square}$	$\cong U_T 2^3$	$S_3 \times L_3(2)$	U_{OTC}	$\cong U_O 2^6 2^3$	$L_3(2)$
U_{TSC}	$\cong U_T 2^4 2$	$S_3 \times S_3$	$U_{OT\square}$	$\cong U_O 2^6 2^3$	$L_3(2)$
$U_{TS\square}$	$\cong U_T 2^4 2$	$S_3 \times S_3$	U_{OSC}	$\cong U_O 2^{1+8} 2$	$S_3 \times S_3$
$U_{TC\square}$	$\cong U_T 2^3 2^2$	$S_3 \times S_3$	$U_{OS\square}$	$\cong U_O 2^{1+8} 2$	$S_3 \times S_3$
$U_{TSC\square}$	$\cong U_T 2^4 2.2$	S_3	$U_{OC\square}$	$\cong U_O 2^6 2^3$	$L_3(2)$
U_S	$\cong 2^{4+12}$	$3S_6 \times S_3$	U_{OTSC}	$\cong U_O 2^6 2^4 2$	S_3
U_{SC}	$\cong U_S 2$	$3S_6$	$U_{OTS\square}$	$\cong U_O 2^6 2^4 2$	S_3
U_C	$\cong 2^{11}$	M_{24}	$U_{OTC\square}$	$\cong U_O 2^6 2^3 2^2$	S_3
			$U_{OSC\square}$	$\cong U_O 2^{1+8} 2.2$	S_3
			$U_{OTSC\square}$	$\cong [2^{21}]$	1

subgroups of the largest Conway simple group Co_1 of order $2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$ as example. $\mathcal{B}_2(Co_1)$ has been determined by the author in [8]. There are 30 conjugacy classes of 2-radical subgroups of Co_1 of which 27 are 2-centric, namely all those except for V_1, V_2, V_3 described in Table 1. (Note that the classification of 2-radical subgroups of the 26 sporadic simple groups has been completed, and a list of their references can be found in [15, Table 3].)

As presented above, each subgroup in $\mathcal{B}_2^{cen}(Co_1)$ is of the form U_F where F is a subset of $\{O, T, S, \mathcal{C}, \square\}$. It is shown in [9, Section 6.3] that we can remove, from $\mathcal{B}_2^{cen}(Co_1)$, all elements U_F with $\square \in F$ without changing Co_1-homotopy type. In other words, $\mathcal{B}_2^{cen}(Co_1)$ is Co_1-homotopy equivalent to the subposet $\mathcal{B}_2^{cen}(Co_1)^*$ consisting of those U_F's such that F does not contain $\square$. (Note that the original proof in [9, Section 6.3] shows that $\mathcal{B}_2^{cen}(Co_1)$ and $\mathcal{B}_2^{cen}(Co_1)^*$ are only homotopy equivalent, but it is extended to Co_1-homotopy equivalence using their observations of Thévenaz-Webb [13].) The inclusion-relation among elements in $\mathcal{B}_2^{cen}(Co_1)^*$ is described in Figure 1. Note that the index $|Co_1 : N_{Co_1}(U_F)|$ is attached for each U_F where $A = 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23$. Now it is shown in [9, Section 6.3] that each U_F is weakly closed in a Sylow 2-subgroup $P := U_{OTSC\square}$ with respect to Co_1; namely if $(U_F)^g$ for $g \in Co_1$ is contained in P then $(U_F)^g = U_F$. Thus it is easy to count n-simplices of the complex $\mathcal{B}_2^{cen}(Co_1)^*$ as follows:

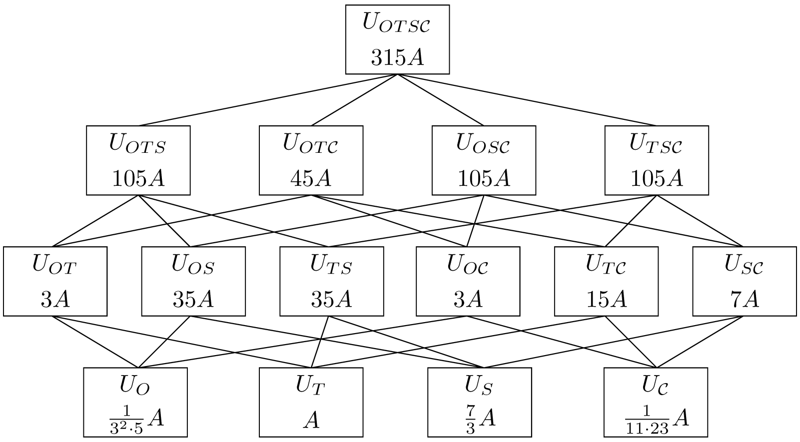

FIGURE 1. Inclusions in $\mathcal{B}_2^{cen}(Co_1)^*$.

$$\begin{array}{ll}
\text{0-simplices}: & [315+105+45+105+105+3+35+35+3+15+7 \\
& \qquad +\frac{1}{3^2\cdot 5}+1+\frac{7}{3}+\frac{1}{11\cdot 23}]A \\
\text{1-simplices}: & [(315\cdot 14)+(105\cdot 6)+(45\cdot 6)+(105\cdot 6)+(105\cdot 6) \\
& \qquad +(3\cdot 2)+(35\cdot 2)+(35\cdot 2)+(3\cdot 2)+(15\cdot 2)+(7\cdot 2)]A \\
\text{2-simplices}: & [(315\cdot 36)+(105\cdot 6)+(45\cdot 6)+(105\cdot 6)+(105\cdot 6)]A \\
\text{3-simplices}: & (315\cdot 24)A
\end{array}$$

From the above numbers of simplices, we have the Euler characteristic as follows:

$$\begin{aligned}
\chi(\mathcal{B}_2^{cen}(Co_1)) = \chi(\mathcal{B}_2^{cen}(Co_1)^*) &= \sum_{q=0}^{3}(-1)^q(\sharp \text{ of } q\text{-simplices in } \mathcal{B}_2^{cen}(Co_1)^*) \\
&= \Big(-52+\frac{1}{3^2\cdot 5}+\frac{7}{3}+\frac{1}{11\cdot 23}\Big)A \\
&= -104,144,306,175\,.
\end{aligned}$$

Thus we have that

$$\tilde{\chi}(\mathcal{B}_2^{cen}(Co_1)) = \chi(\mathcal{B}_2^{cen}(Co_1)) - 1 = -104,144,306,176 = 2^{18}\cdot(-397,279)\,.$$

This shows in particular that $\mathcal{B}_2(Co_1)$ is not homotopy equivalent to $\mathcal{B}_2^{cen}(Co_1)$ since $\tilde{\chi}(\mathcal{B}_2(Co_1))$ is divisible by the 2-part $2^{21} = |Co_1|_2$ of the order of the original group Co_1 (cf. [1, Theorem 6.7.2]), and since the reduced Euler characteristic is the homotopy invariant.

REMARK 5. Recall that the Ronan-Smith 2-local geometry for Co_1 is realized as the poset $\mathcal{B}_2^{cen}(Co_1)^*$ (cf. [9, Section 6.3]). Thus from the above observation, we can say that up to homotopy the smaller $\mathcal{B}_2^{cen}(Co_1)$, rather than the whole $\mathcal{B}_2(Co_1)$, gives a natural geometry to Co_1. The similar observation for other sporadic simple groups G can be found in [9, Section 6]; that is, some of the Ronan-Smith 2-local geometries for G can be obtained as the subgroup complex $\mathcal{B}_2^{cen}(G)$.

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