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Another Natural Lift of a Kähler Submanifold of a Quaternionic Kähler Manifold to the Twistor Space

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Abstract. We study a Kähler submanifold M of a quaternionic Kähler manifold $\tilde{M}$. For such submanifold M we construct a totally real and minimal submanifold $\mathcal{Z}$ in the twistor space $\tilde{\mathcal{Z}}$ of $\tilde{M}$.

1. Introduction

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ be a quaternionic Kähler manifold with the quaternionic Kähler structure $(\tilde{g}, \tilde{Q})$, that is, $\tilde{g}$ is the Riemannian metric on $\tilde{M}$ and $\tilde{Q}$ is a rank 3 subbundle of End $T\tilde{M}$ which satisfies the following conditions:

(a) For each $p \in \tilde{M}$, there is a neighborhood U of p over which there exists a local frame field $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ of $\tilde{Q}$ satisfying

$$\tilde{I}^2 = \tilde{J}^2 = \tilde{K}^2 = -\mathrm{id}\,, \quad \tilde{I}\tilde{J} = -\tilde{J}\tilde{I} = \tilde{K}\,,$$
$$\tilde{J}\tilde{K} = -\tilde{K}\tilde{J} = \tilde{I}\,, \quad \tilde{K}\tilde{I} = -\tilde{I}\tilde{K} = \tilde{J}\,.$$

(b) For any element $L \in \tilde{Q}_p$, $\tilde{g}_p$ is invariant by L, i.e., $\tilde{g}_p(Lu, v) + \tilde{g}_p(u, Lv) = 0$ for $u, v \in T_p\tilde{M}, \quad p \in \tilde{M}$.

(c) The vector bundle $\tilde{Q}$ is parallel in End $T\tilde{M}$ with respect to the Riemannian connection $\tilde{\nabla}$ associated with $\tilde{g}$.

We call $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ in (a) *a local canonical basis* of $\tilde{Q}$. In this paper we assume that the dimension of $\tilde{M}^{4n}$ is not less than 8 and that $\tilde{M}^{4n}$ has nonvanishing scalar curvature. A submanifold M^{2m} of $\tilde{M}$ is said to be *almost Hermitian* if there exists a section $\tilde{I}$ of the bundle $\tilde{Q}|_M$ such that (1) $\tilde{I}^2 = -\mathrm{id}$, (2) $\tilde{I}TM = TM$ (cf. D. V. Alekseevsky and S. Marchiafava [1]). We denote by I the almost complex structure on M induced from $\tilde{I}$. Evidently (M, I) with the induced metric g is an almost Hermitian manifold. If (M, g, I) is Kähler, we call it a

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Kähler submanifold of a quaternionic Kähler manifold $\tilde{M}$. An almost Hermitian submanifold M together with a section $\tilde{I}$ of $\tilde{Q}|_M$ is said to be *totally complex* if at each point $p \in M$ we have $LT_pM \perp T_pM$, for each $L \in \tilde{Q}_p$ with $\tilde{g}(L, \tilde{I}_p) = 0$ (cf. S. Funabashi [4]). Alekseevsky and Marchiafava [1] proved that in a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ with nonvanishing scalar curvature, a $2m(m \geq 2)$-dimensional almost Hermitian submanifold is Kähler if and only if it is totally complex.

We recall the theory of twistor spaces of quaternionic Kähler manifolds, which is an important ingredient for the study of quaternionic Kähler manifolds. The *twistor space* $\tilde{\mathcal{Z}}$ of a quaternionic Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ is defined by $\tilde{\mathcal{Z}} = \{\tilde{I} \in \tilde{Q} | \tilde{I}^2 = -\mathrm{id}\}$. Then the natural projection $\tilde{\pi} : \tilde{\mathcal{Z}} \to \tilde{M}$ is an S^2-bundle over $\tilde{M}$. The twistor space $\tilde{\mathcal{Z}}$ has a natural complex structure and a holomorphic contact structure $\mathcal{H}$ defined by S. Salamon [7]. Moreover $\tilde{\mathcal{Z}}$ of a quaternionic Kähler manifold $\tilde{M}$ of positive scalar curvature admits a Kähler-Einstein metric. Let M^{2m} be an almost Hermitian submanifold of $\tilde{M}$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. Then the map $M \ni p \mapsto \tilde{I}_p \in \tilde{\mathcal{Z}}_p$ is a section of the bundle $\tilde{\mathcal{Z}}|_M$ over M. The submanifold $\tilde{I}(M)$ of $\tilde{\mathcal{Z}}$ is called the *natural lift* of an almost Hermitian submanifold (D. V. Alekseevsky and S. Marchiafava [2]). Alekseevsky and Marchiafava have shown that a $2m(m \geq 2)$-dimensional almost Hermitian submanifold M is Kähler if and only if its natural lift $\tilde{I}(M)$ is a complex submanifold of $\tilde{\mathcal{Z}}$ which is an integral submanifold of the holomorphic contact structure $\mathcal{H}$. In particular the natural lift $\tilde{I}(M^{2n})$ of a half dimensional Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ is a Legendrian submanifold of the twistor space $\tilde{\mathcal{Z}}$. Conversely, any Legendrian submanifold N of $\tilde{\mathcal{Z}}$ defines a half dimensional Kähler submanifold $M = \tilde{\pi}(N)$ of $\tilde{M}$.

We consider another natural lift. Let M be an almost Hermitian submanifold of $\tilde{M}$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. Then the bundle $\tilde{Q}|_M$ has the following decomposition:

$$\tilde{Q}|_M = \mathbf{R}\tilde{I} + Q',$$

where Q' is defined by $Q'_p = \{L \in \tilde{Q}_p | \tilde{g}(L, \tilde{I}_p) = 0\}$ at each point $p \in M$. We put $\mathcal{Z} = Q' \cap \tilde{\mathcal{Z}}|_M$. Then the natural projection $\pi : \mathcal{Z} \to M$ is an S^1-bundle over M. It may be viewed as a kind of tube along the natural lift $\tilde{I}(M)$. Our observation is the following.

THEOREM 1.1. *Let M^{2m} be a $2m(m \geq 2)$-dimensional Kähler submanifold of a quaternionic Kähler manifold $\tilde{M}$ of positive scalar curvature. Then $\mathcal{Z}$ is a totally real and minimal submanifold of the twistor space $\tilde{\mathcal{Z}}$. In particular the space $\mathcal{Z}$ of a half dimensional Kähler submanifold M^{2n} of $\tilde{M}^{4n}$ is a minimal Lagrangian submanifold of $\tilde{\mathcal{Z}}$.*

2. Proof of Theorem

First we recall a complex structure J and a Kähler metric $\tilde{k}$ on the twistor space $\tilde{\mathcal{Z}}$ of a quaternioninc Kähler manifold $(\tilde{M}^{4n}, \tilde{g}, \tilde{Q})$ defined by Salamon [7] (see also Besse Chapter 14 [3]). We normalize the fibre metric $\langle , \rangle$ of the bundle $\tilde{Q}$ such that a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is an orthonormal basis, putting $\langle , \rangle = \frac{1}{4n}\tilde{g}$. Then the fibre $\tilde{\mathcal{Z}}_p$ of $\tilde{\mathcal{Z}}$ at $p \in \tilde{M}$ is given by

$$\tilde{\mathcal{Z}}_p = \{\tilde{I} \in \tilde{Q}_p | \tilde{I}^2 = -\mathrm{id}\} = \{\tilde{I} \in \tilde{Q}_p | \langle \tilde{I}, \tilde{I} \rangle = 1\} .$$

Since $\tilde{\mathcal{Z}}$ is a parallel fibre subbundle in $\tilde{Q}$ with respect to the Riemannian connection $\tilde{\nabla}$, the tangent bundle $T\tilde{\mathcal{Z}}$ is decomposed to the direct sum

$$T\tilde{\mathcal{Z}} = \mathcal{V} + \mathcal{H} , \tag{2.1}$$

where $\mathcal{V}$ is the vertical distribution tangent to the fibres of $\tilde{\pi}$ and $\mathcal{H}$ is the supplementary horizontal distribution defined by the Riemannian connection. For each $z \in \tilde{\mathcal{Z}}$ we define a natural complex structure J on $T_z\tilde{\mathcal{Z}} = \mathcal{V}_z + \mathcal{H}_z$ as follows:

(i) J leaves $\mathcal{V}_z$ and $\mathcal{H}_z$ invariant;

(ii) $J = (\tilde{\pi}_*|_{\mathcal{H}_z})^{-1} \circ z \circ \tilde{\pi}_*|_{\mathcal{H}_z}$ on $\mathcal{H}_z$;

(iii) J on $\mathcal{V}_z$ is induced by the canonical complex structure on S^2, under the identification of $\mathcal{V}_z$ with $T_z(\tilde{\mathcal{Z}}_{\tilde{\pi}(z)}) = T_zS^2$.

In (ii) we note that $\tilde{\pi}_*|_{\mathcal{H}_z} : \mathcal{H}_z \to T_{\tilde{\pi}(z)}\tilde{M}$ is a linear isomorphism and that $z \in \tilde{\mathcal{Z}}_{\tilde{\pi}(z)} \subset \tilde{Q}_{\tilde{\pi}(z)}$ is a complex structure on $T_{\tilde{\pi}(z)}\tilde{M}$. Next we explain (iii) more explicitly. Since the fibre $\tilde{\mathcal{Z}}_{\tilde{\pi}(z)}$ is a unit sphere in $\tilde{Q}_{\tilde{\pi}(z)}$, the tangent space $T_z(\tilde{\mathcal{Z}}_{\tilde{\pi}(z)})$ at z is identified with the orthogonal complement $z^\perp = \{L \in \tilde{Q}_{\tilde{\pi}(z)} | \langle z, L\rangle = 0\}$. Under this identification, the complex structure J on $\mathcal{V}_z$ is given by $JL = zL$ for $L \in \mathcal{V}_z \cong z^\perp$, where zL means the composition in End $T_{\tilde{\pi}(z)}\tilde{M}$. It is known that the almost complex structure J on $\tilde{\mathcal{Z}}$ thus defined is integrable and that $\mathcal{H}$ is a holomorphic contact structure.

From now on we assume that the scalar curvature $\tilde{\tau}$ of $(\tilde{M}, \tilde{g})$ is positive. $(\tilde{M}, \tilde{g})$ is Einstein so, up to a homothety, we may choose $Ric = (n + 2)\tilde{g}$. We define a Hermitian metric $\tilde{k}$ on $(\tilde{\mathcal{Z}}, J)$ as follows:

(i) $\tilde{k}(\mathcal{V}_z, \mathcal{H}_z) = \{0\}$;

(ii) $\tilde{k}(X, Y) = \tilde{g}(\tilde{\pi}_*X, \tilde{\pi}_*Y)$ for $X, Y \in \mathcal{H}_z$;

(iii) $\tilde{k}$ on $\mathcal{V}_z$ is induced by the inner product $\langle , \rangle$ in $\tilde{Q}_{\tilde{\pi}(z)}$ under the identification $\mathcal{V}_z = T_z(\tilde{\mathcal{Z}}_{\tilde{\pi}(z)})$.

Then the projection $\tilde{\pi} : (\tilde{\mathcal{Z}}, \tilde{k}) \to (\tilde{M}, \tilde{g})$ is a Riemannian submersion with totally geodesic fibres. It is known that $\tilde{k}$ is a Kähler-Einstein metric (cf. [7]).

For later convenience, we describe the integrability tensor A of the Riemannian submersion $\tilde{\pi} : (\tilde{\mathcal{Z}}, \tilde{k}) \to (\tilde{M}, \tilde{g})$ introduced by O'Neill [6]. We denote by $\bar{\nabla}$ the Riemannian connection of $(\tilde{\mathcal{Z}}, \tilde{k})$. For horizontal vector fields $X, Y \in \Gamma(\mathcal{H})$ and a vertical vector field U, we define $A_X Y$ and $A_X U$ by

$$A_X Y = \mathcal{V}(\bar{\nabla}_X Y) = \frac{1}{2}\mathcal{V}([X, Y])\,,$$

$$A_X U = \mathcal{H}(\bar{\nabla}_X U)\,,$$

where $\mathcal{V}(*)$ and $\mathcal{H}(*)$ denote the vertical component and horizontal component of $*$ with respect to the decomposition (2.1), respectively. Then at $z = S \in \tilde{\mathcal{Z}}$ we have

$$(A_X Y)_z = -\frac{1}{2}\tilde{R}(\tilde{\pi}_* X, \tilde{\pi}_* Y) \cdot S \quad \text{for } X, Y \in \mathcal{H}\,. \tag{2.2}$$

Here we identify $\mathcal{V}_z \cong z^{\perp} = S^{\perp} \subset \tilde{Q}_{\tilde{\pi}(z)}$ and denote by $\tilde{R}$ the curvature tensor of the base manifold $\tilde{M}$.

We recall the formula of the curvature tensor $\tilde{R}$ for a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ (S. Ishihara [5]):

$$\tilde{R}(X, Y) \cdot \tilde{I} = G(X, Y)\tilde{J} - F(X, Y)\tilde{K}\,, \tag{2.3}$$

$$\tilde{R}(X, Y) \cdot \tilde{J} = -G(X, Y)\tilde{I} + E(X, Y)\tilde{K}\,, \tag{2.4}$$

$$\tilde{R}(X, Y) \cdot \tilde{K} = F(X, Y)\tilde{I} - E(X, Y)\tilde{J}\,, \tag{2.5}$$

where these forms E, F, and G are given as follows;

$$E(X, Y) = -\frac{\tilde{\tau}}{4n(n+2)}\tilde{g}(\tilde{I}X, Y)\,,$$

$$F(X, Y) = -\frac{\tilde{\tau}}{4n(n+2)}\tilde{g}(\tilde{J}X, Y)\,,$$

$$G(X, Y) = -\frac{\tilde{\tau}}{4n(n+2)}\tilde{g}(\tilde{K}X, Y)\,.$$

For $z \in \tilde{\mathcal{Z}}$, we choose a local canonical basis $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ around $\tilde{\pi}(z)$ such that $\tilde{I}_{\tilde{\pi}(z)} = z$. Then the vertical subspace $\mathcal{V}_z$ is spanned by $\tilde{J}_{\tilde{\pi}(z)}$ and $\tilde{K}_{\tilde{\pi}(z)}$. Applying (2.2) and (2.3), we have for $X, Y \in \mathcal{H}_z$

$$(A_X Y)_z = \frac{\tilde{\tau}}{8n(n+2)}\{\tilde{g}(\tilde{K}\tilde{\pi}_* X, \tilde{\pi}_* Y)\tilde{J}_{\tilde{\pi}(z)} - \tilde{g}(\tilde{J}\tilde{\pi}_* X, \tilde{\pi}_* Y)\tilde{K}_{\tilde{\pi}(z)}\}\,.$$

By this we obtain

$$(A_X \tilde{J}_{\tilde{\pi}(z)})_z = -\frac{\tilde{\tau}}{8n(n+2)}(\tilde{\pi}_*|_{\mathcal{H}})^{-1}(\tilde{K}\tilde{\pi}_* X)\,,$$

$$(A_X \tilde{K}_{\tilde{\pi}(z)})_z = \frac{\tilde{\tau}}{8n(n+2)}(\tilde{\pi}_*|_{\mathcal{H}})^{-1}(\tilde{J}\tilde{\pi}_* X)\,.$$

Now we will prove our Theorem. Let M^{2m} be a $2m(m \geq 2)$-dimensional Kähler submanifold of $\tilde{M}$ together with a section $\tilde{I}$ of $\tilde{Q}|_M$. Then by Theorem 1.12 in [1] M is totally complex. The bundle $\tilde{Q}|_M$ has the orthogonal decomposition $\tilde{Q}|_M = \mathbf{R}\tilde{I} + Q'$. We put $\mathcal{Z} = Q' \cap \tilde{\mathcal{Z}}|_M$. The natural projection $\pi : \mathcal{Z} \to M$ is an S^1-bundle over M. Let $\hat{f} : \mathcal{Z} \to \tilde{\mathcal{Z}}$ and $f : M \to \tilde{M}$ be inclusion maps.

LEMMA 2.1. *We have a commutative diagram:*

$$\begin{array}{ccc} (\mathcal{Z}, k) & \xrightarrow{\hat{f}} & (\tilde{\mathcal{Z}}, \tilde{k}) \\ \pi \big\downarrow & & \big\downarrow \tilde{\pi} \\ (M, g) & \xrightarrow{f} & (\tilde{M}, \tilde{g}) \end{array} .$$

The natural projection $\pi : (\mathcal{Z}, k) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres with respect to the induced metrics k and g.

PROOF OF LEMMA 2.1. Since M is totally complex, by Lemma 2.10 in [9] the section $\tilde{I}$ of $\tilde{Q}|_M$ and the vector subbundle Q' are parallel with respect to the induced connection $\tilde{\nabla}$ on $\tilde{Q}|_M$. Therefore $\mathcal{Z}$ is a parallel fibre subbundle in $\tilde{Q}|_M$ with respect to $\tilde{\nabla}$. Then the tangent bundle $T\mathcal{Z}$ is decomposed to the direct sum

$$T\mathcal{Z} = \mathcal{V}' + \mathcal{H}' , \tag{2.6}$$

where $\mathcal{V}'$ is the vertical distribution tangent to the fibres of π and $\mathcal{H}'$ is the supplementary horizontal distribution defined by the induced connection. From this, it follows that $\mathcal{H}'_z$ is a subspace of $\mathcal{H}_z$ at each point $z \in \mathcal{Z}$. This is a crucial fact for our argument. We denote by k the induced metric on $\mathcal{Z}$ from $\tilde{k}$. Since $\mathcal{V}'_z \subset \mathcal{V}_z$ and $\mathcal{H}'_z \subset \mathcal{H}_z$, we have $k(\mathcal{V}'_z, \mathcal{H}'_z) = \{0\}$ and a commutative diagram:

$$\begin{array}{ccc} \mathcal{H}'_z & \xrightarrow{\hat{f}_*} & \mathcal{H}_z \\ \pi_* \big\downarrow & & \big\downarrow \tilde{\pi}_* \\ T_{\pi(z)}M & \xrightarrow{f_*} & T_{\tilde{\pi}(z)}\tilde{M} \end{array} . \tag{2.7}$$

Therefore $\pi_*|_{\mathcal{H}'_z} : (\mathcal{H}'_z, k_z) \to (T_{\pi(z)}M, g_{\pi(z)})$ is a linear isometry. Consequently we see that $\pi : (\mathcal{Z}, k) \to (M, g)$ is a Riemannian submersion. For each point $p \in M$, the fibre $\mathcal{Z}_p$ is a geodesic circle in the unit sphere $\tilde{\mathcal{Z}}_p$. Since $\tilde{\mathcal{Z}}_p$ is totally geodesic in $\tilde{\mathcal{Z}}$, $\mathcal{Z}_p$ is a geodesic in $\tilde{\mathcal{Z}}$. From this it follows that each fibre $\mathcal{Z}_p$ is totally geodesic in $\mathcal{Z}$ □

LEMMA 2.2. *$\mathcal{Z}$ is a totally real submanifold of $\tilde{\mathcal{Z}}$.*

PROOF OF LEMMA 2.2. For an arbitrary point $z \in \mathcal{Z}$, we will prove that $J(T_z\mathcal{Z})$ is orthogonal to $T_z\mathcal{Z}$, where J is the complex structure on $T_z\tilde{\mathcal{Z}}$. For $z \in \mathcal{Z}$ we choose a local section $\tilde{J}$ of $\mathcal{Z}$ defined on a neighborhood of $\pi(z)$ such that $\tilde{J}_{\pi(z)} = z$. We put $\tilde{K} = \tilde{I}\tilde{J}$. Then $\{\tilde{I}, \tilde{J}, \tilde{K}\}$ is a local canonical basis of $\tilde{Q}|_M$ and $\{\tilde{J}, \tilde{K}\}$ is a local basis of Q'. We recall the following orthogonal decompositions of $T_z\mathcal{Z}$ and $T_z\tilde{\mathcal{Z}}$: $T_z\mathcal{Z} = \mathcal{V}'_z + \mathcal{H}'_z$, $T_z\tilde{\mathcal{Z}} = \mathcal{V}_z + \mathcal{H}_z$, $\mathcal{V}'_z \subset \mathcal{V}_z$, $\mathcal{H}'_z \subset \mathcal{H}_z$. The vertical subspace $\mathcal{V}'_z = T_z(\mathcal{Z}_{\pi(z)})$ is spanned by $\tilde{K}_{\pi(z)}$ and $\mathcal{V}_z = T_z(\tilde{\mathcal{Z}}_{\pi(z)})$ is spanned by $\tilde{I}_{\pi(z)}$ and $\tilde{K}_{\pi(z)}$. Since $J\tilde{K}_{\pi(z)} = z\tilde{K}_{\pi(z)} = \tilde{J}_{\pi(z)}\tilde{K}_{\pi(z)} = \tilde{I}_{\pi(z)}$, $J\mathcal{V}'_z$ is orthogonal to $T_z\mathcal{Z}$. By (2.7), we have $\mathcal{H}'_z = (\tilde{\pi}_*|_{\mathcal{H}_z})^{-1}T_{\pi(z)}M$. For any $X \in \mathcal{H}'_z$ we have

$$JX = (\tilde{\pi}_*|_{\mathcal{H}_z})^{-1}z(\tilde{\pi}_*X) = (\tilde{\pi}_*|_{\mathcal{H}_z})^{-1}\tilde{J}_{\pi(z)}(\tilde{\pi}_*X)\,.$$

Since M is totally complex, $\tilde{J}_{\pi(z)}(\tilde{\pi}_*X)$ is orthogonal to $T_{\pi(z)}M$ and hence JX is orthogonal to $\mathcal{H}'_z$. Therefore JX is orthogonal to $T_z\mathcal{Z}$. □

We denote by σ and $\hat{\sigma}$ the second fundamental forms of the submanifolds M in $\tilde{M}$ and $\mathcal{Z}$ in $\tilde{\mathcal{Z}}$, respectively. Then the following holds.

LEMMA 2.3. *For each $z \in \mathcal{Z}$, the image of $\hat{\sigma}$ is contained in the horizontal subspace $\mathcal{H}_z$. Moreover we have*

$$\tilde{\pi}_*\hat{\sigma}(X, Y) = \sigma(\pi_*X, \pi_*Y) \qquad \textit{for} \quad X, Y \in T_z\mathcal{Z}\,.$$

PROOF OF LEMMA 2.3. Since the fibre $\mathcal{Z}_{\pi(z)}$ is a geodesic circle in the unit sphere $\tilde{\mathcal{Z}}_{\pi(z)}$ and $\tilde{\mathcal{Z}}_{\pi(z)}$ is a totally geodesic sphere of $\tilde{\mathcal{Z}}$, we have $\hat{\sigma}(\mathcal{V}'_z, \mathcal{V}'_z) = \{0\}$. As in the proof of Lemma 2.2, we choose a local basis $\{\tilde{J}, \tilde{K}\}$ of Q' around $\pi(z)$ such that $\tilde{J}_{\pi(z)} = z$. Let $\bar{X}$ and $\bar{Y}$ be basic vector fields on $\mathcal{Z}$, that is, $\bar{X}$ and $\bar{Y}$ are horizontal vector fields which are π-related to vector fields X and Y on M, respectively. By Lemma 1 in [6], $\tilde{\pi}_*(\mathcal{H}(\bar{\nabla}_{\bar{X}}\bar{Y})) = \tilde{\pi}_*(\bar{\nabla}_{\bar{X}}\bar{Y}) = \tilde{\nabla}_X Y$. Futhermore the vertical component $\mathcal{V}(\bar{\nabla}_{\bar{X}}\bar{Y})_z$ is given by

$$\begin{aligned}\mathcal{V}(\bar{\nabla}_{\bar{X}}\bar{Y})_z &= (A_{\bar{X}}\bar{Y})_z = -\frac{1}{2}\tilde{R}(X, Y)\cdot\tilde{J}_{\pi(z)}\\ &= \frac{\tilde{\tau}}{8n(n+2)}\{-\tilde{g}(\tilde{K}X, Y)\tilde{I}_{\pi(z)} + \tilde{g}(\tilde{I}X, Y)\tilde{K}_{\pi(z)}\}\\ &= \frac{\tilde{\tau}}{8n(n+2)}g(IX, Y)\tilde{K}_{\pi(z)}\,.\end{aligned}$$

Here we remark that $\tilde{g}(\tilde{K}X, Y) = 0$. Since $\mathcal{V}'_z$ is spanned by $\tilde{K}_{\pi(z)}$, $\mathcal{V}(\bar{\nabla}_{\bar{X}}\bar{Y})_z \in T_z\mathcal{Z}$. From these, it follows that $\hat{\sigma}(\bar{X}, \bar{Y}) \in \mathcal{H}_z$ and that $\tilde{\pi}_*\hat{\sigma}(\bar{X}, \bar{Y})_z = \sigma(X_{\pi(z)}, Y_{\pi(z)})$.

Let U be a vertical vector field of $\mathcal{Z}$ around z which satisfies $U_z = \tilde{K}_{\pi(z)}$. We will prove that $\hat{\sigma}(\bar{X}_z, U_z) = 0$. We have

$$\mathcal{H}(\bar{\nabla}_{\bar{X}}U)_z = (A_{\bar{X}}\tilde{K}_{\pi(z)})_z = -\frac{\tilde{\tau}}{8n(n+2)}(\tilde{\pi}_*|_{\mathcal{H}_z})^{-1}(\tilde{I}\pi_*\bar{X})\,.$$

Here $\tilde{I}(\pi_*\bar{X}) = \tilde{I}X$ is a tangential vector of M and hence $\mathcal{H}(\bar{\nabla}_{\bar{X}}U)_z \in \mathcal{H}'_z \subset T_z\mathcal{Z}$. Secondly we will compute the vertical component of $(\bar{\nabla}_{\bar{X}}U)_z$. Since $\bar{\nabla}_{\bar{X}}U = \bar{\nabla}_U\bar{X} + [\bar{X}, U]$,

$\mathcal{V}(\bar{\nabla}_{\bar{X}} U)_z = \mathcal{V}(\bar{\nabla}_U \bar{X})_z + \mathcal{V}([\bar{X}, U]_z)$. Each fibre of $\tilde{\mathcal{Z}}$ is a totally geodesic submanifold. This implies that $(\bar{\nabla}_U \bar{X})_z$ is horizontal and hence $\mathcal{V}(\bar{\nabla}_U \bar{X})_z = 0$. Since $\bar{X}$ and U are tangential vector fields of $\mathcal{Z}$, $[\bar{X}, U]$ is also a tangential vector field of $\mathcal{Z}$. On the other hand $\pi_*[\bar{X}, U] = [X, 0] = 0$ and hence $[\bar{X}, U]$ is a vertical vector field. Therefore $\mathcal{V}([\bar{X}, U]_z) = [\bar{X}, U]_z \in T_z\mathcal{Z}$. Consequently we have $\mathcal{V}(\bar{\nabla}_{\bar{X}} U)_z \in T_z\mathcal{Z}$. From these, it follows that $\hat{\sigma}(\bar{X}_z, U_z) = 0$. Thus Lemma 2.3 has been proved. □

By Lemma 2.3 we obtain the following.

COROLLARY 2.4. *If M^{2m} is a $2m(m \geq 2)$-dimensional totally geodesic Kähler submanifold of $\tilde{M}$, then $\mathcal{Z}$ is a totally geodesic submanifold of $\tilde{\mathcal{Z}}$.*

PROOF OF THEOREM. Finally we will prove that $\mathcal{Z}$ is a minimal submanifold of $\tilde{\mathcal{Z}}$. It is known that a totally complex submanifold M is minimal in $\tilde{M}$ (cf. [4]). This, together with Lemma 2.3, implies that $\mathcal{Z}$ is minimal in $\tilde{\mathcal{Z}}$. □

3. Examples

We will explain examples of half dimensional totally complex submanifolds of a quaternionic Kähler manifold.

(1) M. Takeuchi [8] studied a complete totally complex totally geodesic submanifold M of a quaternionic symmetric space $\tilde{M}$ of compact type or non-compact type with $2 \dim M = \dim \tilde{M}$. He called such a pair a *TCG-pair* and classified TCG-pairs. He also studied the twistor space $\tilde{\mathcal{Z}}$ of $\tilde{M}$ and constructed the diagram as in Lemma 2.1 for a TCG-pair $(\tilde{M}, M)$. In this case, he showed that a natural lift $\mathcal{Z}$ of M is given by the set of fixed points of an anti-holomorphic involution of $\tilde{\mathcal{Z}}$.

(2) K. Tsukada [9] studied totally complex submanifolds with parallel second fundamental form in a quaternion projective space $\mathbf{H}P^n$ and classified them.

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