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Cyclic Lagrangian Submanifolds and Lagrangian Fibrations

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Abstract. Let (M, ω) be a symplectic manifold and $L \subset M$ be a Lagrangian submanifold. In [Oh2], the cyclic condition of L was defined. Y.-G. Oh proved that, in [Oh2], if (M, ω) is Kähler-Einstein with non-zero scalar curvature and L is minimal, then L is cyclic. In this article, first, we prove that L is cyclic if and only if the "mean cuvature cohomology class" of L is rational, when (M, ω) is Kähler-Einstein with non-zero scalar curvature. Secondly, we see that there are non-cyclic minimal Lagrangian submanifolds when (M, ω) is a prequantizable Ricci-flat Kähler manifold. Thirdly, if (M, ω) is Kähler-Einstein with non-zero scalar curvature, there are not minimal Lagrangian fibration structures on M by a result of [Oh2]. Nevertheless we construct Hamiltonian minimal Lagrangian fibration.

1. Introduction

Let (M, ω) be a symplectic manifold. We define the period group Γ_ω of (M, ω) as

$$\Gamma_\omega = \{\langle [\omega], A\rangle \mid A \in H_2(M; \mathbf{Z})\} .$$

Following A. Weinstein [W], we call (M, ω) prequantizable if Γ_ω is trivial or discrete. Moreover, when $H_1(M; \mathbf{Z}) = 0$, we call a Lagrangian submanifold L in M cyclic if

$$\Gamma_{\omega,L} = \{\langle [\omega], B\rangle \mid B \in H_2(M, L; \mathbf{Z})\}$$

is a discrete subgroup of $\mathbf{R}$. Note that Γ_ω is a subgroup of $\Gamma_{\omega,L}$ for any L.

One of the main theorems in [Oh2] is the following.

THEOREM 1.1 ([Oh2]). *Let (M, J, ω) be a Kähler-Einstein manifold with non-zero scalar curvature, $H_1(M; \mathbf{Z}) = 0$, and $i : L \hookrightarrow M$ be a Lagrangian submanifold with its mean curvature vector H. Suppose that the one form $\alpha_H = i^*(H \lrcorner \omega)$ on L is exact. Then L is cyclic. Moreover, the following holds:*

1. *When L is orientable, then $n_L | \gamma_{c_1}$.*
2. *When L is not orientable, then $n_L | 2\gamma_{c_1}$,*

where $n_L = \gamma_\omega / \gamma_{\omega,L}$ for $\Gamma_\omega = \gamma_\omega \mathbf{Z}$, $\Gamma_{\omega,L} = \gamma_{\omega,L}\mathbf{Z}$ and $\Gamma_{c_1} = \gamma_{c_1}\mathbf{Z}$ (c_1 is the first Chern class of (M, J)).

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For example, if (M, J, ω) is Kähler-Einstein with non-zero scalar curvature and $H_1(M; \mathbf{Z}) = 0$, then minimal Lagrangian submanifolds are cyclic. In Section 2, as the generalization of Theorem 1.1, we prove the following theorem.

THEOREM 1.2. *Let (M, J, ω) be a Kähler-Einstein manifold with non-zero scalar curvature, $H_1(M; \mathbf{Z}) = 0$, and $i : L \hookrightarrow M$ be a Lagrangian submanifold with its mean curvature vector H. Then L is cyclic if and only if there is a positive integer $\gamma_{\alpha,L}$ such that $\gamma_{\alpha,L}[\alpha_H] \in \mathrm{Image}(H^1(L; 2\pi\mathbf{Z}) \to H^1(L; \mathbf{R}))$, where we choose $\gamma_{\alpha,L}$ as the minimal integer which satisfies such condition. Moreover the following holds*:

1. *When L is orientable, then $n_L = l\gamma_{\alpha,L}$, where l is a divisor of γ_{c_1}.*
2. *When L is not orientable, then $2n_L = l\gamma_{\alpha,L}$, where l is a divisor of $4\gamma_{c_1}$.*

In contrast with Theorem 1.1, if (M, J, ω) is prequantizable Ricci-flat Kähler, there are non-cyclic minimal Lagrangian submanifolds.

THEOREM 1.3. *Let (M, J, ω) be a prequantizable Ricci-flat K3 surface. Then one of the following holds.*

1. *There are not embedded Lagrangian tori L with $0 \neq [L] \in H_2(M; \mathbf{Z})$.*
2. *There are non-cyclic minimal Lagrangian tori.*

We can obtain such examples by considering special Lagrangian fibrations and their action-angle coordinates.

On the other hand, if (M, J, ω) is Kähler-Einstein with non-zero scalar curvature, then M cannot admit "(local) minimal Lagrangian fibration", by Theorem 1.1 and an action-angle coordinate of a (local) Lagrangian fibration. However we can see that "Hamiltonian minimal Lagrangian fibration" is admissible. Here, a Lagrangian submanifold L in a Kähler manifold (M, J, ω) is called Hamiltonian minimal if the first variation of volume is zero for all Hamiltonian deformations of L.

THEOREM 1.4. *Let (M, J, ω) be a compact Kähler manifold with its real dimension $2n$. Suppose that we have an effective isometric Hamiltonian T^n-action on M. Then any connected regular fiber of the moment map is Hamiltonian minimal.*

Note here that, E. Goldstein proved in [G] that there is a minimal regular fiber of the moment map, which maximizes volume among the fibers. Therefore, these fibers are examples of non-minimal Hamiltonian minimal Lagrangian submanifolds.

2. Cyclic condition of Lagrangian submanifolds

In this section, we review the notion of the cyclic condition of Lagrangian submanifolds (see [Oh2] or [W]). Let (M, ω) be a prequantizable symplectic manifold, namely, there exists a non-negative number γ_ω such that the period group $\Gamma_\omega = \{\langle[\omega], A\rangle | A \in H_2(M; \mathbf{Z})\} \subset \mathbf{R}$ of (M, ω) satisfies $\Gamma_\omega = \gamma_\omega\mathbf{Z}$. It is well-known that if (M, ω) is prequantizable, then there is a complex line bundle $E \to M$ with a connection ∇ such that its curvature R_∇ satisfies

$$R_\nabla = \frac{2\pi i}{\gamma_\omega}\omega .$$

We call (E, ∇) a prequantization bundle. In general, there are many equivalence classes of such connections (see e.g [K]). However, if we suppose $H_1(M; \mathbf{Z}) = 0$, then there is an unique equivalence class of connections which satisfies $R_\nabla = \frac{2\pi i}{\gamma_\omega}\omega$ and the holonomy of ∇ around a loop $l \subset M$ is equal to

$$\exp\left(\frac{2\pi i}{\gamma_\omega}\int_S \omega\right) , \tag{2.1}$$

where $S \subset M$ is a surface with $\partial S = l$ (see e.g. [K]).

Let $i : L \hookrightarrow M$ be a Lagrangian submanifold. Then $(i^*E, i^*\nabla)$ is flat and we have its holonomy homomorphism $Hol_L : \pi_1(L) \to S^1$. We call L cyclic if the image of Hol_L is cyclic in S^1. Since L is Lagrangian, L is cyclic if and only if the subgroup $\Gamma_{\omega,L} = \{\langle[\omega], B\rangle | B \in H_2(M, L; \mathbf{Z})\}$ of $\mathbf{R}$ is discrete.

Let (M, J, ω) be Kähler-Einstein with its Ricci form $\rho = c\omega$, $c \neq 0$. In this case, (M, ω) is prequantizable. Moreover if $H_1(M; \mathbf{Z}) = 0$, then the prequantization bundle (E, ∇) satisfies

$$(K, \nabla_{LC}) = \begin{cases} (E^{\otimes m}, \nabla^{\otimes m}) & (\text{when } c > 0) , \\ ((E^*)^{\otimes m}, (\nabla^*)^{\otimes m}) & (\text{when } c < 0) , \end{cases} \tag{2.2}$$

where K is the canonical bundle of M, ∇_{LC} is the connection on K induced from the Levi-Civita connection and m satisfies $\{\langle c_1(M), A\rangle | A \in H_2(M; \mathbf{Z})\} = m\mathbf{Z}$.

LEMMA 2.1. *Let (M, ω) be a Kähler manifold and $\iota : L \hookrightarrow M$ be a Lagrangian submanifold.*

1. *If L is orientable, then the holonomy of $(\iota^*K, \iota^*\nabla_{LC})$ along a loop $\gamma \subset L$ is $\exp(i\int_\gamma \alpha_H)$, where H is the mean curvature vector of ι and $\alpha_H = \iota^*(H \lrcorner \omega)$.*
2. *If L is not orientable, then the holonomy of $(\iota^*K^{\otimes 2}, \iota^*\nabla_{LC}^{\otimes 2})$ along a loop $\gamma \subset L$ is $\exp(2i\int_\gamma \alpha_H)$.*

PROOF. If L is orientable, then there is the non-vanishing section Ω of ι^*K, which is the complex extension of the volume form of L. By Proposition 2.2 in [Oh2], the connection form with respect to this trivialization is $i\alpha_H$. Hence 1. is proved. The proof of non-orientable case is similar. □

PROOF OF THEOREM 1.2. Since the proof for the case $c < 0$ is similar, we prove the theorem only for the case $c > 0$. Let L be a cyclic Lagrangian submanifold in M. Then we have, for any loop $\gamma \subset L$,

$$(Hol(\gamma))^{n_L} = 1 , \tag{2.3}$$

where Hol is the holonomy of $(\iota^* E, \iota^* \nabla)$ and $n_L = \frac{\gamma_\omega}{\gamma_{\omega,L}}$.

On the other hand, by (2.2) and Lemma 2.1, we have the following.

$$\begin{cases} \text{If } L \text{ is orientable, then } (Hol(\gamma))^{\gamma_{c_1}} = \exp(i \int_\gamma \alpha_H)\,. \\ \text{If } L \text{ is non-orientable, then } (Hol(\gamma))^{2\gamma_{c_1}} = \exp(2i \int_\gamma \alpha_H)\,. \end{cases} \tag{2.4}$$

Hence, by (2.3) and (2.4), we have the following.

$$\begin{cases} \text{If } L \text{ is orientable, then } n_L[\alpha_H] \in \mathrm{Image}(H^1(L; 2\pi\mathbf{Z}) \to H^1(L; \mathbf{R}))\,. \\ \text{If } L \text{ is non-orientable, then } 2n_L[\alpha_H] \in \mathrm{Image}(H^1(L; 2\pi\mathbf{Z}) \to H^1(L; \mathbf{R}))\,. \end{cases} \tag{2.5}$$

Conversely, suppose that there is the positive integer $\gamma_{\alpha,L}$ such that $\gamma_{\alpha,L}[\alpha_H] \in \mathrm{Image}(H^1(L; 2\pi\mathbf{Z}) \to H^1(L; \mathbf{R}))$ for a Lagrangian submanifold L. Here, we choose $\gamma_{\alpha,L}$ as the minimal integer which satisfies the condition above. Then the following holds;

$$\exp\left(i\gamma_{\alpha,L}\int_\gamma \alpha_H\right) = 1 \tag{2.6}$$

for any loop $\gamma \subset L$. Hence, by (2.2), (2.6) and Lemma 2.1, we have the following.

$$\begin{cases} \text{If } L \text{ is orientable, then } (Hol(\gamma))^{\gamma_{\alpha,L}\gamma_{c_1}} = 1\,. \\ \text{If } L \text{ is non-orientable, then } (Hol(\gamma))^{2\gamma_{\alpha,L}\gamma_{c_1}} = 1\,. \end{cases} \tag{2.7}$$

Thus L is cyclic.

Moreover, when L is orientable, there are non-zero integers l and l' such that $l\gamma_{\alpha,L} = n_L$ and $\gamma_{\alpha,L}\gamma_{c_1} = l'n_L$, by (2.5) and (2.7). Hence $\gamma_{c_1} = ll'$ and $l\gamma_{\alpha,L} = n_L$. The proof for the case L is non-orientable is similar. □

If (M, J, ω) is Ricci-flat, there is not the relationship between the prequantization bundle of (M, ω) and the canonical bundle of (M, J). Hence it is an interesting question that whether minimal Lagrangian submanifolds in Ricci-flat (M, J, ω) are cyclic. In the next section, we see that there are non-cyclic minimal Lagrangian submanifolds in K3-surfaces.

3. Lagrangian fibrations

To construct examples of non-cyclic minimal Lagrangian submanifolds in a Ricci-flat Kähler manifold, we use Lagrangian fibrations and their action-angle coordinates. Since we only use a local fibration structure, if we say "$\pi : M \to B$ is a Lagrangian fibration", then it may admit singular fibers.

Let (M^{2n}, ω) be a symplectic manifold which has a Lagrangian fibration $\pi : M \to B^n$. It is well-known that a compact connected regular fiber $\pi^{-1}(b_0)$ is a Lagrangian torus and that there is an action-angle coordinate of an open neighborhood of $\pi^{-1}(b_0)$ as follows (see e.g. [A], [D]);

there is an open neigborhood U of $\pi^{-1}(b_0)$ in M and a diffeomorphism

$$(a, \alpha) : U \to V \times T^n$$

with V open in $\mathbf{R}^n$, such that $a = \chi \circ \pi$ for some diffeomorphism $\chi : \pi(U) \to V$ and

$$\omega = \sum_{j=1}^{n} d\alpha_j \wedge da_j \quad \text{on } U .$$

We can construct the map $\chi = (\chi_1, \cdots, \chi_n) : \pi(U) \to V$ as follows. Let $\gamma_i(b)$, $i = 1, \cdots, n$, be loops in $\pi^{-1}(b)$, depending smoothly on $b \in \pi(U)$, such that their homology classes form a basis in $H_1(\pi^{-1}(b); \mathbf{Z})$. Then we define the maps

$$\chi_i(b) = \int_{\gamma_i(b)} \theta ,$$

where θ is an 1-form on U such that $\omega = d\theta$ on U.

PROPOSITION 3.1. *Let (M, ω) be a prequantizable symplectic manifold with $H_1(M; \mathbf{Z}) = 0$. Suppose that (M, ω) has a Lagrangian torus fibration $\pi : M \to B$. If $\pi^{-1}(b_0)$ is a cyclic regular fiber, then $\pi^{-1}(b)$ is not cyclic for almost all $b \in B$ near b_0.*

PROOF. Let $\pi^{-1}(b_0)$ be a cyclic regular fiber and $(a, \alpha) : U \to V \times T^n$ be an action-angle coordinate around $\pi^{-1}(b_0)$. There are tubes $\Sigma_i(b) \subset U$ connecting the loops $\gamma_i(b_0)$ with $\gamma_i(b)$ such that the action coordinate is

$$a(b) = a(b_0) + \left(\int_{\Sigma_1(b)} \omega, \cdots, \int_{\Sigma_n(b)} \omega \right) .$$

By the definition of the cyclic condition, we see that $\pi^{-1}(b)$, if it is regular orbit, is cyclic if and only if $a(b) - a(b_0) \in (\gamma_\omega \mathbf{Q})^n$. □

By Proposition 3.1, there are not cyclic Lagrangian fibrations, even locally. Hence, by Theorem 1.1, we have the following corollary.

COROLLARY 3.2. *Let (M, J, ω) be Kähler-Einstein with non-zero scalar curvature and $H_1(M; \mathbf{Z}) = 0$. Then there are not minimal Lagrangian torus fibration structures of M, even locally.*

On the other hand, it is well-known that there are Ricci-flat Kähler manifolds which have "special Lagrangian fibrations". Here we recall special Lagrangian submanifolds (see [HL]).

Let (X, J) be an n-dimensional Calabi-Yau manifold with a Ricci-flat Kähler metric g, Kähler form ω, and holomorphic n-form Ω. An n-dimensional real submanifold $\iota : L \hookrightarrow X$ is special Lagrangian if $\iota^*\omega = 0$ and $\iota^*(Im\Omega) = 0$. Any special Lagrangian submanifold is homologically volume minimizing in X, i.e., $\mathrm{Vol}(L) \leq \mathrm{Vol}(L')$ for any submanifold $L' \subset X$ such that $[L] = [L'] \in H_n(X; \mathbf{Z})$. Hence special Lagrangian submanifolds are minimal.

Next, we see examples of prequantizable Ricci-flat Kähler manifolds with special Lagrangian fibrations. By Proposition 3.1, generic fibers of such fibrations are non-cyclic minimal Lagrangian submanifolds.

Example(K3 surface) Let (M, J) be a K3-surface with a Ricci-flat Kähler metric g, Kähler form ω, and holomorphic n-form Ω. In [Gr], it was proved that (M, J, ω) has special Lagrangian fibration if and only if there is a nonzero cohomology class $E \in H^2(M; \mathbf{Z})$ such that $E \cdot E = 0$ and $[\omega] \cdot E = 0$ by using hyperKähler trick. Therefore, by using Proposition 3.1, Theorem 1.3 holds.

PROOF OF THEOREM 1.3. Suppose that there is an embedded Lagrangian torus L with $0 \neq [L] \in H_2(M; \mathbf{Z})$. Note that, since L is a Lagrangian embedded torus, the normal bundle NL is trivial. Hence $[L] \cdot [L] = 0$. Then the Poincaré dual $E = P.D.([L]) \in H^2(M; \mathbf{Z})$ of $[L]$ satisfies the conditions $E \cdot E = 0$ and $[\omega] \cdot E = 0$. Hence (M, J, ω) has a special Lagrangian fibration. □

4. Hamiltonian minimal Lagrangian fibrations

In the previous section, we saw that, if (M, J, ω) is a prequantizable Kähler-Einstein manifold with non-zero scalar curvature, then (M, J, ω) admits no minimal Lagrangian fibration, even locally. Nevertheless, in this section, we will see that there are Hamiltonian minimal Lagrangian fibrations.

First, we recall the Hamiltonian minimality of Lagrangian submanifolds in a Kähler manifold (M, J, ω) (see [Oh1]). Let (M, J, ω) be a Kähler manifold, $L \subset M$ be a Lagrangian submanifold and V be a normal variation vector along L. Since L is Lagrangian, we can regard $(V \rfloor \omega)_{|L}$ as an 1-form on L. If $(V \rfloor \omega)_{|L}$ is exact, V is called a Hamiltonian variation vector. A smooth family $\{\iota_t\}$ of embeddings of L into M is called a Hamiltonian deformation, if its derivative is Hamiltonian. Note that Hamiltonian deformations leave Lagrangian submanifolds Lagrangian. We say that a Lagrangian submanifold is Hamiltonian minimal, if the first variation of volume is zero for all Hamiltonian deformations of L. In [Oh1], Oh proved the following proposition.

PROPOSITION 4.1. *Let (M, J, ω) be a Kähler manifold. A Lagrangian submanifold $i : L \subset M$ is Hamiltonian minimal if and only if its mean curvature vector H satisfies*

$$\delta(i^*(H \lrcorner \omega)) = 0$$

on L where δ is the adjoint of d on L with respect to the induced metric from M.

PROOF OF THEOREM 1.4. Let (M, ω) be a real $2n$-dimensional compact Kähler manifold, which has an effective isometric Hamiltonian T^n-action and $i : L \hookrightarrow M$ be a regular connected fiber of the moment map. Note that L is an orbit of the torus action. This theorem is proved by the following lemmas. □

LEMMA 4.2. *The 1-form $\alpha_H = i^*(H \lrcorner \omega)$ on L is T^n-invariant.*

PROOF. Let $h \in T^n$. We write the action of h on M and L as $m_{M,h} : M \to M$ and $m_{L,h} : L \to L$ respectively. Note that these maps are isometries, here the metric on L is induced metric, and $m_{M,h} \circ i = i \circ m_{L,h}$ for any $h \in T^n$. Hence we have the invariance of the mean curvature vector H;

$$\begin{aligned}(m_{M,h})_* H &= \sum_{j=1}^{n} (m_{M,h})_* (\bar{\nabla}_{i_* e_j} i_* e_j - i_*(\nabla_{e_j} e_j)) \\ &= \sum_{j=1}^{n} (\bar{\nabla}_{(m_{M,h})_* i_* e_j} (m_{M,h})_* i_* e_j - (m_{M,h})_* i_*(\nabla_{e_j} e_j)) \\ &= \sum_{j=1}^{n} (\bar{\nabla}_{i_*(m_{L,h})_* e_j} i_*(m_{L,h})_* e_j - i_*(\nabla_{(m_{L,h})_* e_j} (m_{L,h})_* e_j)) \\ &= H\,,\end{aligned}$$

where $\{e_1, \cdots, e_n\}$ is an orthonormal basis of $T_x L$, $\bar{\nabla}$ and ∇ are the Levi-Civita connections on M and L respectively. By the definition of α_H, we have

$$\begin{aligned}(m^*_{L,h} \alpha_H)(X) &= \omega(H, i_*(m_{L,h})_* X) \\ &= \omega(H, (m_{M,h})_* i_* X) \\ &= \omega((m_{M,h^{-1}})_* H, i_* X) \\ &= \omega(H, i_* X) \\ &= \alpha_H(X)\,,\end{aligned}$$

where X is a vector field on L. □

Since $L \simeq T^n$ and the induced metric on L is T^n-invariant, the following lemma is well-known (see e.g [H]).

LEMMA 4.3. *Let g be the induced metric on L. A p-form β on L is harmonic with respect to g if and only if β is T^n-invariant.*

Therefore, by Proposition 4.1, Lemma 4.2 and Lemma 4.3, the proof of Theorem 1.4 has finished.

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