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On a Higher Class Number Formula of $\mathbf{Z}_p$-Extensions

Satoshi FUJII

Waseda University

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1. Introduction

Let p be a prime number and k a number field of finite degree over $\mathbf{Q}$, the rational number field. Let $\mathbf{Z}_p$ be the additive group of p-adic integers, and K/k a $\mathbf{Z}_p$-extension over k. For an integer $n \geq 0$, we denote by k_n the n-th layer of the extension K/k, namely k_n is the unique intermediate field of K/k such that $[k_n : k] = p^n$. Recently, Ozaki studied the maximal unramified pro-p extensions $\tilde{L}$ of K and $\tilde{L}_n$ of k_n as in what follows. Let $\tilde{G} = \mathrm{Gal}(\tilde{L}/K)$ and $\tilde{G}_n = \mathrm{Gal}(\tilde{L}_n/k_n)$ for all non-negative n. We define the subgroups $C_i(\tilde{G})$ of $\tilde{G}$ by the descending central series

$$\tilde{G} = C_1(\tilde{G}) \supseteq C_2(\tilde{G}) \supseteq \cdots \supseteq C_i(\tilde{G}) \supseteq \cdots, \quad C_{i+1}(\tilde{G}) = \overline{[C_i(\tilde{G}), \tilde{G}]}\,.$$

Then we consider the modules $X^{(i)} = C_i(\tilde{G})/C_{i+1}(\tilde{G})$, and call $X^{(i)}$ the i-th Iwasawa module. We define the subgroups $C_i(\tilde{G}_n) \subseteq \tilde{G}_n$ and the modules $X_n^{(i)}$ similar to $C_i(\tilde{G})$ and $X^{(i)}$, respectively. Note that $X_n^{(1)}$ is isomorphic to the Sylow p-subgroup of the ideal class group A_{k_n} of k_n and that $X^{(1)}$ is the Iwasawa module X_K of K/k which is defined as the projective limit $\varprojlim A_{k_n}$ with respect to the norm maps. By definition, the complete group ring $\Lambda_{K/k} = \mathbf{Z}_p[[\mathrm{Gal}(K/k)]]$ acts on $X^{(i)}$ in the natural way, namely $\mathrm{Gal}(K/k)$ acts via the inner automorphism. For $i = 1$, Iwasawa studied the $\Lambda_{K/k}$-module structure of X_K and deduced the following celebrated formula:

THEOREM A. *There exist non-negative integers $\lambda(K/k)$, $\mu(K/k)$ and an integer $\nu(K/k)$ such that*

$$\#A_{k_n} = p^{\lambda(K/k)n+\mu(K/k)p^n+\nu(K/k)}$$

for all sufficiently large n.

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These integers $\lambda(K/k)$, $\mu(K/k)$ and $\nu(K/k)$ are called the Iwasawa invariants of K/k. We remark that $\lambda(K/k)$ and $\mu(K/k)$ are the invariants of the $\Lambda_{K/k}$-module X_K. If $\mu(K/k) = 0$, then X_K is a finitely generated $\mathbf{Z}_p$-module with $\mathrm{rank}_{\mathbf{Z}_p} X_K = \lambda(K/k)$, and $X^{(i)}$ is also a finitely generated $\mathbf{Z}_p$-module. When $\mu(K/k) = 0$, we define the i-th λ-invariant of K/k by $\lambda^{(i)}(K/k) = \mathrm{rank}_{\mathbf{Z}_p} X^{(i)}$. Now we raise the following question on the higher Iwasawa modules $X^{(i)}$:

QUESTION. *Suppose $\mu(K/k) = 0$. Then, for each $i \geq 2$, does there exist an integer $\nu^{(i)}(K/k)$ such that $\# X_n^{(i)} = p^{\lambda^{(i)}(K/k)n + \nu^{(i)}(K/k)}$ for all sufficiently large n ?*

Ozaki found infinitely many fields where the above question is affirmatively answered for $i = 2, 3$.

THEOREM B. *Let p be an odd prime number and K/k the cyclotomic $\mathbf{Z}_p$-extension over a CM-field k with the maximal real subfield k^+. Assume that the following conditions are satisfied*:

(1) *the Iwasawa μ-invariant of K/k is* 0,
(2) *the class number of k^+ is prime to p*,
(3) *there is a unique prime of k^+ lying over p.*

Then there is an integer $\nu^{(i)}(K/k)(i = 2, 3)$ such that $\# X_n^{(i)} = p^{\lambda^{(i)}(K/k)n + \nu^{(i)}(K/k)}$ for all sufficiently large n.

For example, all imaginary quadratic fields satisfy the assumptions of Theorem B. In this paper, we will prove an asymptotic formula of $\# X_n^{(2)}$ in terms of $p^{\lambda^{(2)}(K/k)n}$, namely;

THEOREM. *Suppose that $\mu(K/k) = 0$ and p does not split in $K/\mathbf{Q}$. Then $\# X_n^{(2)} = p^{\lambda^{(2)}(K/k)n + O(1)}$.*

2. Lemmas

To prove this Theorem, we use the following lemmas. For a number field F, let E_F be the unit group of F. Denote by A_F the Sylow p-subgroup of the ideal class group of F. Let L be a finite Galois extension of F and $\mathrm{Gal}(L/F)$ its Galois group. Then we denote by $H_i(L/F, M)$ the i-th homology group of a $\mathrm{Gal}(L/F)$-module M for a non-negative integer i. We regard the additive group of p-adic integers $\mathbf{Z}_p$ as a $\mathrm{Gal}(L/F)$-module with trivial action. We denote by $M^{\mathrm{Gal}(L/F)}$ and $M_{\mathrm{Gal}(L/F)}$ the $\mathrm{Gal}(L/F)$-invariant submodule and the $\mathrm{Gal}(L/F)$-co-invariant module of M, respectively. For a $\mathbf{Z}_p$-module N, let $\mathrm{Tor}_{\mathbf{Z}_p} N$ be the maximal $\mathbf{Z}_p$-torsion submodule of N and put $N[p] = \{x \in N \mid px = 0\}$.

LEMMA 2.1. *Let F be a number field of finite degree and L/F an unramified finite p-extension of finite degree such that L contains the Hilbert p-class field of F and let $G = \mathrm{Gal}(L/F)$ (p-extension means a Galois extension with p-power degree). Put*

$\mathcal{H}_{L/F} = E_F/E_F \cap N_{L/F}L^\times$. *Then we have the exact sequence*

$$0 \longrightarrow \mathcal{H}_{L/F} \longrightarrow H_2(L/F, \mathbf{Z}_p) \longrightarrow (A_L)_G \longrightarrow 0\,.$$

Furthermore, for any subfield k of F such that L/k and F/k are Galois extensions, the above sequence is exact as $\mathrm{Gal}(F/k)$*-modules.*

PROOF. This lemma is well known as the central class field theory. For example, see Fröhlich [1]. □

Let $L_n = L_n^{(1)} = \tilde{L}_n^{C_2(\tilde{G}_n)}$ and $L_n^{(2)} = \tilde{L}_n^{C_3(\tilde{G}_n)}$. Then L_n is the Hilbert p-class field of k_n and $L_n^{(2)}$ is the central p-class field of L_n/k_n, respectively. It follows from the definition of $X_n^{(2)}$ that $X_n^{(2)} = \mathrm{Gal}(L_n^{(2)}/L_n) \simeq (A_{L_n})_{\mathrm{Gal}(L_n/k_n)}$. Since L_n/k_n is an abelian extension, we have $H_2(L_n/k_n, \mathbf{Z}_p) \simeq A_{k_n} \wedge A_{k_n}$, where $\wedge$ means the exterior product. For $m \geq n \geq 0$, let $N'_{m,n} : A_{k_m} \wedge A_{k_m} \to A_{k_n} \wedge A_{k_n}$ be the homomorphisms induced by the norm maps. Note that the diagram

$$\begin{array}{ccc} H_2(L_m/k_m, \mathbf{Z}_p) & \xrightarrow{\sim} & A_{k_m} \wedge A_{k_m} \\ {\scriptstyle d_{m,n}}\downarrow & & {\scriptstyle N'_{m,n}}\downarrow \\ H_2(L_n/k_n, \mathbf{Z}_p) & \xrightarrow{\sim} & A_{k_n} \wedge A_{k_n} \end{array}$$

is commutative for $m \geq n \geq 0$. Here, we denote by $d_{m,n}$ the map induced by the restriction map $\mathrm{Gal}(L_m/k_m) \to \mathrm{Gal}(L_n/k_n)$ $(\sigma \mapsto \sigma|_{L_n})$. By Lemma 2.1, we have the following:

LEMMA 2.2. *The following diagram is exact and commutative as Γ-modules for $m \geq n \geq 0$:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{H}_{L_m/k_m} & \longrightarrow & A_{k_m} \wedge A_{k_m} & \longrightarrow & X_m^{(2)} & \longrightarrow & 0 \\ & & {\scriptstyle \text{norm}}\downarrow & & \downarrow{\scriptstyle N'_{m,n}} & & \downarrow{\scriptstyle \text{restriction}} & & \\ 0 & \longrightarrow & \mathcal{H}_{L_n/k_n} & \longrightarrow & A_{k_n} \wedge A_{k_n} & \longrightarrow & X_n^{(2)} & \longrightarrow & 0\,. \end{array}$$

The action of $\sigma \in \Gamma$ on $A_{k_n} \wedge A_{k_n}$ is given by $\sigma(x \wedge y) = (\sigma x) \wedge (\sigma y)$ for $x, y \in A_{k_n}$.

The next lemma tells us that the knowledge of $A_{k_n} \wedge A_{k_n}$ gives information about $\mathcal{H}_{L_n/k_n}$ and $X_n^{(2)}$.

LEMMA 2.3. *Let $\{A_n\}$ and $\{B_n\}$ $(n \geq 0)$ denote projective systems of finite abelian p-groups with the following exact commutative diagram*:

$$\begin{array}{ccccc} 0 & \longrightarrow & A_{n+1} & \longrightarrow & B_{n+1} \\ & & {\scriptstyle \text{surjective}}\downarrow & & \downarrow{\scriptstyle \text{surjective}} \\ 0 & \longrightarrow & A_n & \longrightarrow & B_n\,, \end{array}$$

and let $A = \varprojlim A_n$, $B = \varprojlim B_n$.

(1) *Suppose that B is a finitely generated $\mathbf{Z}_p$-module. Then $\mathfrak{B} = \mathrm{Tor}_{\mathbf{Z}_p} B$ is isomorphic to a subgroup of B_n for all sufficiently large n.*

(2) *Suppose that B is a finitely generated $\mathbf{Z}_p$-module and*

$$\mathrm{Ker}(B_{n+1}/\mathfrak{B} \to B_n/\mathfrak{B}) = (B_{n+1}/\mathfrak{B})[p]$$

for all sufficiently large n. Then there exist integers a and b such that

$$\#A_n = p^{\lambda(A)n+a},$$
$$\#B_n = p^{\lambda(B)n+b}$$

for all sufficiently large n. Here we denote by $\lambda(M)$ the $\mathbf{Z}_p$-rank of a finitely generated $\mathbf{Z}_p$-module M.

PROOF. (1) Let $\mathfrak{b}_n = \mathrm{Ker}(B \to B_n)$. Then $\{\mathfrak{b}_n\}$ is a system of fundamental neighborhoods of B. Since $\mathfrak{B}$ is finite, there is n_0 such that $\mathfrak{b}_n \cap \mathfrak{B} = 0$ for $n \geq n_0$. It follows from the finiteness of B_n that $\mathfrak{b}_n$ is a free $\mathbf{Z}_p$-module of rank $\lambda(B)$ for all sufficiently large n. From the freeness of $\mathfrak{b}_n$, we see that $\mathfrak{B}$ maps to B_n injectively and B_n is a product of the image of $\mathfrak{B}$ and some subgroup of B_n.

(2) Let $B' = B/\mathfrak{B}$. By (1), we have

$$0 \longrightarrow \mathfrak{b}_n \longrightarrow B' \longrightarrow B_n/\mathfrak{B} \longrightarrow 0 \quad \text{(exact)},$$

and $\dim_{F_p}(B_n/\mathfrak{B})/p(B_n/\mathfrak{B}) = \lambda(B)$ for all sufficiently large n, where F_p is the finite field of p elements. By the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathfrak{b}_{n+1} & \longrightarrow & B' & \longrightarrow & B_{n+1}/\mathfrak{B} & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & \mathfrak{b}_n & \longrightarrow & B' & \longrightarrow & B_n/\mathfrak{B} & \longrightarrow & 0, \end{array}$$

and the snake lemma, we have $\mathrm{Ker}(B_{n+1}/\mathfrak{B} \to B_n/\mathfrak{B}) = (B_{n+1}/\mathfrak{B})[p] \simeq \mathfrak{b}_n/\mathfrak{b}_{n+1}$. It follows from $(B_{n+1}/\mathfrak{B})[p] \simeq (B_{n+1}/\mathfrak{B})/p(B_{n+1}/\mathfrak{B}) \simeq F_p^{\oplus\lambda(B)}$ that $\mathfrak{b}_{n+1} = p\mathfrak{b}_n$.

Fix an integer $n_0 \geq 0$ such that $\mathfrak{b}_n \cap \mathfrak{B} = 0$ and $\mathfrak{b}_{n+1} = p\mathfrak{b}_n$ for all $n \geq n_0$. Let $\#(B_{n_0}/\mathfrak{B}) = p^{\lambda(B)n_0+b'}$ for an integer b'. Since $\#\mathrm{Ker}(B_{n+1}/\mathfrak{B} \to B_n/\mathfrak{B}) = \#(B_{n+1}/\mathfrak{B})[p] = p^{\lambda(B)}$, we have $\#(B_n/\mathfrak{B}) = p^{\lambda(B)n+b'}$ if $n \geq n_0$. Then $\#B_n = \#(B_n/\mathfrak{B})\#\mathfrak{B} = p^{\lambda(B)n+b'}\#\mathfrak{B}$. Let $p^b = p^{b'}\#\mathfrak{B}$. Then $\#B_n = p^{\lambda(B)n+b}$ for all sufficiently large n.

Let $\mathfrak{a}_n = \mathrm{Ker}(A \to A_n)$ and $\mathfrak{A} = \mathrm{Tor}_{\mathbf{Z}_p} A$. Since $A \subseteq B$ we have $\mathfrak{A} = A \cap \mathfrak{B}$. It follows from the exact commutative diagram

$$\begin{array}{ccccc} 0 & \longrightarrow & A_{n+1}/\mathfrak{A} & \longrightarrow & B_{n+1}/\mathfrak{B} \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & A_n/\mathfrak{A} & \longrightarrow & B_n/\mathfrak{B} \end{array}$$

that $\mathrm{Ker}(A_{n+1}/\mathfrak{A} \to A_n/\mathfrak{A}) = (A_{n+1}/\mathfrak{A})[p]$. Then $\mathfrak{a}_{n+1} = p\mathfrak{a}_n$ for all sufficiently large n. The remaining part is proved as in the case of B_n. □

LEMMA 2.4. *Let K/k be a $\mathbf{Z}_p$-extension with a number field k. Assume that the Iwasawa μ-invariant of K/k is 0. Then we have the following commutative diagram for all sufficiently large n:*

$$\begin{array}{ccc} A_{k_{n+1}} & \xrightarrow{\sim} & \left(\bigoplus_{i=1}^{\lambda(K/k)} \mathbf{Z}/p^{a_i+n+1}\mathbf{Z}\right) \oplus \mathrm{Tor}_{\mathbf{Z}_p} X_K \\ {\scriptstyle\text{norm}}\downarrow & & \downarrow{\scriptstyle\text{natural surjection}} \\ A_{k_n} & \xrightarrow{\sim} & \left(\bigoplus_{i=1}^{\lambda(K/k)} \mathbf{Z}/p^{a_i+n}\mathbf{Z}\right) \oplus \mathrm{Tor}_{\mathbf{Z}_p} X_K \, , \end{array}$$

where $a_1, \cdots, a_{\lambda(K/k)}$ are integers independent of n and satisfy the inequalities $a_1 \le a_2 \le \cdots \le a_{\lambda(K/k)}$.

PROOF. For the proof of this lemma, see Grandet–Jaulent [2]. □

The following is keystone of the proof of main theorem.

LEMMA 2.5. *Let K/k be a $\mathbf{Z}_p$-extension, and let $\Gamma_n = \mathrm{Gal}(K/k_n)$. For $m \ge n \ge 0$, let $B_m^{(n)} = \{c \in A_{k_m} | {}^{\exists}\mathfrak{a} \in c \text{ s.t. } \sigma\mathfrak{a} = \mathfrak{a} \text{ for all } \sigma \in \Gamma_n\} \subseteq A_{k_m}^{\Gamma_n}$. If p is not decomposed in $K/\mathbf{Q}$, then $A_{k_m}^{\Gamma_n}/B_m^{(n)} \simeq \hat{H}^0(k_m/k_n, E_{k_m})$.*

PROOF. For the proof of this lemma, see Theorem 1 of Greenberg [3]. □

Here, we give a sketch of the proof of Theorem B for the case $i = 2$. By Lemma 2.4, we have

$$A_{k_n} \wedge A_{k_n} \simeq \left(\bigoplus_{1 \le i \le \lambda(K/k)} (\mathbf{Z}/p^{a_i+n}\mathbf{Z})^{\oplus(\lambda(K/k)-i)} \right) \oplus D$$

for some finite abelian p-group D independent of n. Now we prove the surjectivity of the norm map $N_{m,n} : E_{k_m} \to E_{k_n}$, which is equivalent to $\hat{H}^0(k_m/k_n, E_{k_m}) = E_{k_n}/N_{m,n}E_m = 0$. Since k is a CM-field and K/k is the cyclotomic $\mathbf{Z}_p$-extension, k_n is also a CM-field with the maximal real subfield k_n^+. Because the unit index of k_n/k_n^+ is 1 or 2 and p is odd, we have $\hat{H}^0(k_m/k_n, E_{k_m}) = \hat{H}^0(k_m^+/k_n^+, E_{k_m^+})$. By the assumptions (2), (3) and Iwasawa's theorem (Iwasawa [4]), the class number of k_n^+ is prime to p for $n \ge 0$. Therefore, $\hat{H}^0(k_m^+/k_n^+, E_{k_m^+}) = 0$ for $m \ge n \ge 0$ by Lemma 2.5. Applying Lemma 2.3 to $\mathcal{H}_{L_n/k_n}$ and $A_{k_n} \wedge A_{k_n}$, we see that there is an integer $\nu^{(2)}(K/k)$ such that $\#X_n^{(2)} = p^{\lambda^{(2)}(K/k)n+\nu^{(2)}(K/k)}$ for all sufficiently large n. The proof of the case $i = 3$ is much more difficult.

Let D_n be the subgroup of A_{k_n} generated by the classes each of which contains a prime above p. We put $A'_{k_n} = A_{k_n}/D_n$ and define $A'_{k_n} \to A'_{k_m}$ as the homomorphism induced by the natural inclusion $k_n \to k_m$.

LEMMA 2.6. *The order of the kernel of the homomorphism* $A'_{k_n} \to A'_{k_m}$ *is bounded for* $m \geq n \geq 0$.

PROOF. For the proof of this lemma, see Iwasawa [5]. □

3. Proof of Theorem

Let $\mathcal{H} = \varprojlim \mathcal{H}_{L_n/k_n}$, where the projective limit is taken with respect to the norm maps, and $\mathcal{I}_n = \mathrm{Im}(\mathcal{H} \to \mathcal{H}_{L_n/k_n})$ the image of the projection map. Applying Lemma 2.3 to $\mathcal{I}_n$ and $A_{k_n} \wedge A_{k_n}$, we see that there exist integers a and b such that $\#A_{k_n} \wedge A_{k_n} = p^{\lambda(X_K \wedge X_K)n+a}$ and $\#\mathcal{I}_n = p^{\lambda(\mathcal{H})n+b}$ for all sufficiently large n. By Lemma 2.2, we have

$$\begin{aligned}\#X_n^{(2)} &= \#A_{k_n} \wedge A_{k_n}/\#\mathcal{H}_{L_n/k_n} \\ &= p^{\lambda^{(2)}(K/k)n+(a-b)}/[\mathcal{H}_{L_n/k_n} : \mathcal{I}_n]\,.\end{aligned}$$

Hence, we have to prove that $[\mathcal{H}_{L_n/k_n} : \mathcal{I}_n]$ is bounded for $n \geq 0$. We can easily see that $\mathcal{I}_n = \bigcap_{m \geq n} N_{k_m/k_n}\mathcal{H}_{L_m/k_m}$. Therefore, if $\#\hat{H}^0(k_m/k_n, E_{k_m})$ is bounded for $m \geq n \geq 0$, then $[\mathcal{H}_{L_n/k_n} : \mathcal{I}_n]$ is bounded for n according to the following commutative diagram:

$$\begin{array}{ccc} E_{k_m} & \longrightarrow & \mathcal{H}_{L_m/k_m} \\ {\scriptstyle\text{norm}}\downarrow & & \downarrow{\scriptstyle\text{norm}} \\ E_{k_n} & \longrightarrow & \mathcal{H}_{L_n/k_n}\,. \end{array}$$

Therefore, we have only to prove that $\#\hat{H}^0(k_m/k_n, E_{k_m})$ is bounded for $m \geq n \geq 0$. Let $n_0 \geq 0$ be the integer such that k_{n_0} is the maximal unramified subextension of K/k. We deal with two cases separately.

Case 1. $n < n_0 \leq m$.
By our assumption that p is not decomposed in $K/\mathbf{Q}$ and Lemma 2.5, we have

$$\begin{aligned}\#\hat{H}^0(k_m/k_n, E_{k_m}) &= \#A_{k_m}^{\Gamma_n}/\#B_m^{(n)} \\ &\leq \#A_{k_m}^{\Gamma_n} \\ &\leq \#A_{k_m}^{\Gamma_{n_0}} \\ &= \#A_{k_{n_0}}\,.\end{aligned}$$

Hence the boundedness holds.

Case 2. $n_0 \leq n$.

Let $\mathfrak{p}_n$ be the unique prime of k_n above p. Since $\mathfrak{p}_n$ is Γ-invariant and $\#A_{k_n}^{\Gamma} \leq \#A_{k_n}^{\Gamma_{n_0}} = \#A_{k_{n_0}}$, we see that $D_n \subseteq A_{k_n}^{\Gamma}$ and the order of D_n is bounded. Then there is a constant $C_1 > 0$ such that $\#A_{k_n}/\#A'_{k_n} = \#D_n \leq C_1$ for all $n \geq 0$. Now we consider the homomorphism $A'_{k_n} \to A'_{k_m}$ induced by the natural inclusion $k_n \to k_m$. Clearly the image of the above map is contained in $B_m^{(n)}/D_m$. Conversely, let c mod D_m be an element of $B_m^{(n)}/D_m$. Then there is an ideal $\mathfrak{a} \in c$ of k_m such that $\sigma\mathfrak{a} = \mathfrak{a}$ for all $\sigma \in \Gamma_n$. Since every Γ_n-invariant ideal of k_m is a product of a power of the prime above p and an ideal of k_n, we may assume that the class c contains an ideal of k_n. Therefore $\mathrm{Im}(A'_{k_n} \to A'_{k_m}) = B_m^{(n)}/D_m$. By Lemma 2.6, there is a constant $C_2 > 0$ such that $\#A'_{k_n}\#D_m/\#B_m^{(n)} \le C_2$. Hence we have

$$\begin{aligned}\#\hat{H}^0(k_m/k_n, E_{k_m}) &= \#A_{k_m}^{\Gamma_n}/\#B_m^{(n)} \\ &\le \frac{\#A_{k_n}C_2}{\#A'_{k_n}\#D_m} \\ &\le C_1C_2/\#D_m \\ &\le C_1C_2\,,\end{aligned}$$

because $n \ge n_0$. This completes the proof of Theorem. □

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Note added in proof. After the author submitted, M. Ozaki showed that the Question of this paper is affirmatively answered for each $i \ge 1$. See his preprint: Non-abelian Iwasawa Theory of $\mathbf{Z}_p$-extensions. His article will be to appear to a journal.

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Present Address:
DEPARTMENT OF MATHEMATICAL SCIENCE, SCHOOL OF SCIENCE AND ENGINEERING,
WASEDA UNIVERSITY, OKUBO, SHINJUKU-KU, TOKYO, 169–8555 JAPAN.
e-mail: fujii@ruri.waseda.jp