TOKYO J. MATH.
VOL. 28, NO. 1, 2005

Foliations Associated with Nambu-Jacobi Structures

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(Communicated by S. Miyoshi)

Dedicated to Professor Takushiro Ochiai on his 60th birthday

Abstract. We define a Nambu-Jacobi structure as a bracket of several functions which satisfies the Fundamental Identity. Then we express the Nambu-Jacobi structure in terms of two tensor fields and show the necessary and sufficient conditions that they should satisfy. We investigate the foliations associated with a Nambu-Jacobi structure. This allows us to give many examples of Nambu-Jacobi manifolds.

1. Introduction and definitions

It is well-known that a Poisson manifold has its associated foliation. It is a generalized foliation in the sense of Stefan and Sussmann whose leaves are immersed symplectic manifolds. In the case of a Jacobi manifold (in the sense of Lichnerowicz [5] or called a manifold with local Lie algebra structure by Kirillov [4]), we have also a generalized foliation whose leaves are either symplectic or a contact manifolds. In this paper, we consider the cases of Nambu-Poisson and Nambu-Jacobi manifolds. First, we recall briefly a Nambu-Poisson manifold to generalize it to a *Nambu-Jacobi* manifold.

1.1. Nambu-Poisson structures. Let $C^\infty(M)$ be the set of smooth functions on a manifold M.

DEFINITION 1. $\{\cdots\}$ is called a *Nambu-Poisson bracket* of degree q if it is a skew-symmetric q-linear map over R

$$\{\cdots\} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{q} \longrightarrow C^\infty(M)$$

which satisfies the following:

(1) (Fundamental Identity or Generalized Jacobi Identity)

$$\{f_1, \cdots, f_{q-1}, \{g_1, \cdots, g_q\}\} = \sum_{i=1}^{q} \{g_1, \cdots, \{f_1, \cdots, f_{q-1}, g_i\}, \cdots, g_q\}$$

Received September 9, 2003; revised February 9, 2004
2000 Mathematics Subject Classification. Primary 52C12; Secondary 70G45.

where $f_1, \cdots, f_{q-1}, g_1, \cdots, g_q \in C^\infty(M)$.

(2) (Leibniz rule) For each argument of the bracket, the usual derivation rule holds, that is, for $f_1, \cdots, f_{q+1} \in C^\infty(M)$

$$\{f_1, \cdots, f_{q-1}, f_q f_{q+1}\} = \{f_1, \cdots, f_{q-1}, f_q\} f_{q+1} + f_q \{f_1, \cdots, f_{q-1}, f_{q+1}\}$$

holds.

As in the case of usual Poisson bracket, this is also equivalent to the existence of a q-vector field η on M satisfying

$$\eta(df_1, \cdots, df_q) = \{f_1, \cdots, f_q\} \quad \text{for} \quad f_1, \cdots, f_q \in C^\infty(M)$$
$$L_{\eta(d\boldsymbol{f}_{q-1}, \cdot)}\eta = 0 \quad \text{for} \quad f_1, \cdots, f_{q-1} \in C^\infty(M)$$

where $d\boldsymbol{f}_{q-1}$ is the abbreviation of $df_1 \wedge \cdots \wedge df_{q-1}$, $\eta(d\boldsymbol{f}_{q-1}, \cdot)$ stands for the vector field $\eta(df_1, \cdots, df_{q-1}, \cdot)$ on M, which is the Hamiltonian vector field determined by several functions (cf. [1]), and $L_{\eta(d\boldsymbol{f}_{q-1})}\eta$ is a Lie derivative.

DEFINITION 2. η is called a *Nambu-Poisson tensor* of degree q.

A Nambu-Poisson tensor has the following striking property.

THEOREM 1.1 (P. Gautheron [3], K. Mikami [6], N. Nakanishi [8]). *If q is greater than 2, Nambu-Poisson tensor of degree q is locally decomposable. This means that if η is non-zero at a point, then on a neighborhood of the point there exist vector fields $Y_1, \cdots, Y_q$ so that η can be written as*

$$\eta = Y_1 \wedge \cdots \wedge Y_q .$$

1.2. Nambu-Jacobi structures. Now we define a Nambu-Jacobi manifold and see that the bracket of a Nambu-Jacobi manifold is described by a pair of two multi-vector fields. Let us begin with a general definition.

Let M be a C^∞ manifold and $C^\infty(M)$ the algebra of smooth functions on M. For an integer $q \geq 1$, we consider an alternating q-linear map

$$\mathcal{A} : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{q} \longrightarrow C^\infty(M)$$

and we always assume $\mathcal{A}$ satisfies the following conditions:

(a) The map $\mathcal{A}$ is continuous with respect to C^∞ topology and

(b) $\operatorname{supp}\mathcal{A}(f_1, \cdots, f_q) \subset \operatorname{supp} f_1 \cap \cdots \cap \operatorname{supp} f_q$.

We call the map $\mathcal{A}$ a *support-non-increasing bracket of degree q* and often write $\{f_1, \cdots, f_q\}$ for $\mathcal{A}(f_1, \cdots, f_q)$.

When $q = 1$, we understand $\mathcal{A}$ is just a linear map. By a theorem of Peetre [9], these conditions assure that the support-non-increasing bracket is a differential operator and the resulting function is written in terms of the derivatives of the argument functions of $\mathcal{A}$.

DEFINITION 3. If $\mathcal{A}$ is a support-non-increasing bracket of degree q on M which satisfies the Fundamental Identity in Definition 1, we call $\mathcal{A}$ a *Nambu-Jacobi bracket*. A smooth manifold with a Nambu-Jacobi bracket is called a *Nambu-Jacobi manifold*.

REMARK 1.1. We remark that when $q = 2$, a Nambu-Jacobi manifold reduces to a usual Jacobi manifold (cf. [4], [5]). The Leibniz rule in the definition of a Nambu-Poisson bracket clearly implies our condition (a) and (b) on bracket. Thus a Nambu-Poisson manifold is a special case of a Nambu-Jacobi manifold.

Along the line of the proof of Kirillov ([4]) for the usual Jacobi bracket, we have the following:

THEOREM 1.2 (cf. [2]). *A Nambu-Jacobi bracket is a differential operator of order at most* 1.

This theorem allows us to describe a Nambu-Jacobi bracket in terms of a pair of two multi-vector fields on M. Let P be a p-vector field on M. P naturally defines a bracket of degree p by $(f_1, \cdots, f_p) \mapsto P(df_1, \cdots, df_p)$. We denote this bracket by $\{f_1, \cdots, f_p\}^P$ or sometimes by $P(f_1, \cdots, f_p)$ when there is no danger of confusion. This notation is analogous to the notation $X(f)$ to denote the derivative of a function f by a vector field X. We define a new bracket $1 \wedge P$ of degree $(p+1)$ by the formula

$$(1 \wedge P)(f_1, \cdots, f_{p+1}) = \sum_{i=1}^{p+1} (-1)^{i-1} f_i P(df_1, \cdots, \hat{df_i}, \cdots, df_{p+1})\,. \tag{1}$$

Then we have the following observation on a Nambu-Jacobi bracket.

LEMMA 1.1 (cf. [2]). *Let $\mathcal{A}$ be a Nambu-Jacobi bracket of degree q $(q \geq 2)$. Then there uniquely exist a q-vector field Q and a $(q-1)$-vector field P which are Nambu-Poisson tensors such that $\mathcal{A} = Q + 1 \wedge P$ holds.*

PROOF. Let $p = q - 1$ and put $\mathcal{B}(f_1, \cdots, f_p) = \mathcal{A}(1, f_1, \cdots, f_p)$. Then it is easily seen that $\mathcal{B}$ is a bracket of degree p and satisfies the Fundamental Identity. From the skewness of $\mathcal{A}$, $\mathcal{B}(f_1, \cdots, f_p) = 0$ if one of the arguments is a constant function. Thus, the order of $\mathcal{B}$ as a differential operator is exactly equal to 1. This means $\mathcal{B}$ is defined by a p-vector field P. Now put $\mathcal{C} = \mathcal{A} - 1 \wedge P$. Then by the same reason $\mathcal{C}$ is also a bracket defined by a q-vector field. Denoting it by Q, we obtain Lemma 1.1. Uniqueness is verified easily. □

DEFINITION 4. We call the pair (Q, P) a *Nambu-Jacobi pair* if $Q + 1 \wedge P$ defines a Nambu-Jacobi bracket.

NOTATION. In the sequel, we frequently use the contraction of tensor fields. For example, let Q be a q-vector field and $\alpha = \alpha_1 \wedge \cdots \wedge \alpha_p$ a p-form $(p \leq q)$. We denote the contraction Q and α by the following various notations, interchangeably. $i_\alpha Q = Q(\alpha) = Q(\alpha, \cdot) = Q(\alpha_1 \wedge \cdots \wedge \alpha_p, \cdot)$.

1.3. Fundamental Identity. We consider the bracket $\mathcal{A}$ defined by $(p+1)$-vector field Q and p-vector field P, by the equality $\mathcal{A} = Q + 1 \wedge P$. We now look for the conditions on P and Q under which $\mathcal{A}$ satisfies the Fundamental Identity, namely the conditions that make (Q, P) a Nambu-Jacobi pair. When $\deg P = 1$ and $\deg Q = 2$, the Nambu-Jacobi pair $Q + 1 \wedge P$ is a usual Jacobi bracket and the conditions on P and Q are well known. Namely, they satisfy $[P, Q] = 0$ and $[Q, Q] = -2P \wedge Q$ if and only if the bracket satisfies the Jacobi identity, where $[\cdot, \cdot]$ is the Schouten-Nijenhuis bracket (cf. (15)). Therefore our interest is in the case $\deg P \geq 2$.

To proceed our calculations, we introduce the following notations.

DEFINITION 5. For a p-vector field P and a q-vector field Q $(p \geq 2, q \geq 1)$ which are both considered as brackets, we define a map

$$J^P Q : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{p+q-1} \longrightarrow C^\infty(M)$$

by

$$\begin{aligned}(J^P Q)(f_1, \cdots, f_{p-1}; g_1, \cdots, g_q) = P(f_1, \cdots, f_{p-1}, Q(g_1, \cdots, g_q)) \\ - (Q(P(f_1, \cdots, f_{p-1}, g_1), g_2, \cdots, g_q) \\ - Q(g_1, P(f_1, \cdots, f_{p-1}, g_2), g_3, \cdots, g_q) \\ - \cdots - Q(g_1, \cdots, g_{q-1}, P(f_1, \cdots, f_{p-1}, g_q)),\end{aligned} \tag{2}$$

for $f_1, \cdots, f_{p-1}, g_1, \cdots, g_q \in C^\infty(M)$.

We remark here that $J^P Q$ can be considered as a contravariant tensor field but *does not* define a bracket since it is not fully alternating with respect to the argument functions. Note that $J^P P = 0$ means that P satisfies the Fundamental Identity. The same formula as $J^P Q$ can be defined for any brackets (not necessarily given by multi-vector fields). In the present case, where the initial brackets are defined by multi-vector fields (P and Q), we have the following equivalent expression.

$$J^P Q(f_1, \cdots, f_{p-1}; g_1, \cdots, g_q) = [P(d\boldsymbol{f}_{p-1}, \cdot), Q](dg_1, \cdots, dg_q) \tag{3}$$

where $d\boldsymbol{f}_{p-1} = df_1 \wedge \cdots \wedge df_{p-1}$ as before, $P(d\boldsymbol{f}_{p-1}, \cdot) = P(df_1, \cdots, df_{p-1}, \cdot)$ is a vector field and $[P(d\boldsymbol{f}_{p-1}, \cdot), Q] = L_{P(d\boldsymbol{f}_{p-1}, \cdot)} Q$ is a Lie derivation. Thus $J^P Q = 0$ is equivalent to that the Hamiltonian vector fields preserve Q .

We also need the following map.

DEFINITION 6. For a p-vector field P and a q-vector field Q $(p \geq 2, q \geq 1)$, we define a map

$$P \vdash Q : \underbrace{C^\infty(M) \times \cdots \times C^\infty(M)}_{p+q} \longrightarrow C^\infty(M)$$

by

$$(P \vdash Q)(f_1, \cdots, f_{p-1}; g_0, \cdots, g_q) = (P(d\boldsymbol{f}_{p-1}, \cdot) \wedge Q)(dg_0, \cdots, dg_q) \tag{4}$$

where $d\boldsymbol{f}_{p-1} = df_1 \wedge \cdots \wedge df_{p-1}$.

Also, $P \vdash Q$ is not a bracket in general. Note that $P \vdash P = 0$ if and only if P is a locally decomposable. In order to get the relation between P and Q in $\mathcal{A} = Q + 1 \wedge P$, we need to calculate $J^{\mathcal{A}}\mathcal{A}$ since the condition that $\mathcal{A}$ is to be a Nambu-Jacobi bracket is

$$J^{\mathcal{A}}\mathcal{A}(f_1, \cdots, f_p; g_1, \cdots, g_{p+1}) = 0\,. \tag{5}$$

By a direct computation we can express the left hand side of this equation (5) so that a certain sum of multiples of functions consisting of $\{\cdots\}^P$, $\{\cdots\}^Q$, and f_i, g_j which are outside of the brackets $\{\cdots\}^P$ or $\{\cdots\}^Q$. Although it is straightforward, the computation is lengthy. We will do it in Appendix separately.

The relations of P and Q which we obtain are in the following:

PROPOSITION 1.3. *Let $\mathcal{A} = Q + 1 \wedge P$ be a bracket of degree $q = p + 1$ defined by p-vector field P and q-vector field Q. Then a necessary and sufficient condition for the bracket $\mathcal{A}$ to be a Nambu-Jacobi bracket is that P and Q satisfy the following four identities.*

(1) $J^P P = 0\,,$

(2) $J^P Q = 0\,,$

(3) $$J^Q P(d\boldsymbol{f}_p; \cdots) + (-1)^{p+1}Q(d(P(d\boldsymbol{f}_p)), \cdots) + \sum_{i=1}^{p}(-1)^i (P \vdash P)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0\,,$$

(4) $$(J^Q Q)(d\boldsymbol{f}_p; \cdots) + \sum_{i=1}^{p}(-1)^i (P \vdash Q)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0\,,$$

where $d\boldsymbol{f}_p$ stands for differential form $df_1 \wedge \cdots \wedge df_p$.

PROOF. As is stated above, this is done in Appendix by a direct but a long calculation. □

To simplify the above identities we need the following two lemmas.

LEMMA 1.2. *Let P and Q be multi-vector fields of degree p and q, respectively. If $P \vdash Q$ is also a multi-vector field (i.e. $P \vdash Q$ is a skew symmetric tensor field) and $p \geq 3$ and $q \geq 1$ then $P \vdash Q$ vanishes identically.*

PROOF. Considering the equation at each point of the manifold, we may assume that P and Q are alternating multi-linear maps on finite dimensional vector space. Put $B = P \vdash Q$. It suffices to show that $B_{|V} = 0$ for arbitrary $(p+q)$-dimensional subspace V. From the definition, if $Q_{|W} = 0$ for any q-dimensional subspace $W \subset V$, then clearly $B_{|V} = 0$. So assume $Q_{|W} \neq 0$ for a q-dimensional subspace W. Then there exist $y_1, \cdots, y_q \in W$

satisfying

$$Q(y_1, \cdots, y_q) \neq 0\,.$$

We can find an element $x \in V$ such that $x \wedge y_1 \wedge \cdots \wedge y_q \neq 0$ and

$$Q(y_1, \cdots, \underbrace{x}_{i}, \cdots, y_q) \neq 0 \quad \text{for some} \quad i\,. \qquad \square$$

PROOF [(Proof of Claim)]. Consider the following linear functional on V

$$\varphi : V \longrightarrow R : x \mapsto \sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q)\,.$$

We have

$$\varphi(y_1) = \cdots = \varphi(y_q) = Q(y_1, \cdots, y_q) \neq 0$$

and

$$y_2 - y_1, \cdots, y_q - y_1 \in \operatorname{Ker}\varphi$$

Since $\dim \operatorname{Ker}\varphi = p + q - 1$ and $p \geq 1$, we find an element $z \in \operatorname{Ker}\varphi$ so that

$$z,\, y_2 - y_1, \cdots, y_q - y_1 \in \operatorname{Ker}\varphi$$

are linearly independent. Then it can be seen that the set

$$z + y_1,\, y_1,\, y_2, \cdots, y_q$$

is linearly independent and $x = z + y_1$ is an element with desired property. Indeed, we have

$$\sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q) = \varphi(x) = \varphi(z + y_1) = \varphi(y_1) = Q(y_1, \cdots, y_q) \neq 0\,.$$

This means there exists some i such that

$$Q(y_1, \cdots, \underbrace{x}_{i}, \cdots, y_q) \neq 0\,.$$

(end of the proof of claim). $\square$

For x and $Y = (y_1, \cdots, y_q)$ which we found above and for any $(p-2)$-tuple T, we have

$$\begin{aligned}
0 =& B(T, x; x, Y) = (P(T, x, \cdot) \wedge Q)(x, Y) \\
=& P(T, x, x)Q(Y) - \sum_{j=1}^{q} P(T, x, y_j) Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q) \\
=& -P(T, x, \sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q) y_j)\,.
\end{aligned}$$

If we put $u = \sum_{j=1}^{q} Q(y_1, \cdots, \underbrace{x}_{j}, \cdots, y_q) y_j$, this shows

$$P(x, u, T) = 0 \quad \text{for all } (p-2)\text{-tuple} T .$$

For any $(p-3)$-tuple of vectors T' and q-tuple of vectors T'' in V, we have

$$B(x, u, T'; T'') = (P(x, u, T', \cdot) \wedge Q)(T'') = 0$$

($p \geq 3$ is necessary here).

Since we are assuming B is a multi-vector and $\{x, u\}$ are linearly independent, this clearly shows $B_{|V} = 0$.

The next lemma shows that in our case, $P \vdash P$, $P \vdash Q$ and $Q \vdash Q$ are proved to be multi-vector fields and we can apply Lemma 1.2 to the identities in Proposition 1.3.

LEMMA 1.3. *Let $\mathcal{A} = Q + 1 \wedge P$ be a Nambu-Jacobi bracket determined by q-vector field and p-vector field P $(q = p+1)$. Then $P \vdash P$, $P \vdash Q$ and $Q \vdash Q$ are fully alternating and hence they are multi-vector fields.*

PROOF. To prove $P \vdash Q$ is a multi-vector field, we have only to verify the skewness of $P \vdash Q$, namely the identity

$$\begin{aligned}&(P \vdash Q)(h_1, f_2, \cdots, f_{p-1}; h_2, g_1, \cdots, g_{p+1}) \\ &\quad + (P \vdash Q)(h_2, f_2, \cdots, f_{p-1}; h_1, g_1, \cdots, g_{p+1}) = 0 \end{aligned} \tag{6}$$

for arbitrary functions $h_1, h_2, f_2, \cdots, f_{p-1}, g_1, \cdots, g_{p+1}$.

For this, we calculate

$$\begin{aligned}&J^P Q(h_1 h_2, f_2, \cdots, f_{p-1}; g_1, \cdots, g_{p+1}) \\ &\quad = [P(d(h_1 h_2), df_2, \cdots, df_{p-1}, \cdot), Q](dg_1, \cdots, dg_{p+1})\end{aligned}$$

which is identically equal to 0 by Proposition 1.3 (2). Thus we have

$$\begin{aligned}0 =& [P(d(h_1 h_2), df_2, \cdots, df_{p-1}, \cdot), Q] = [P(h_1 dh_2 + h_2 dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] \\ =& [h_1 P(dh_2, df_2, \cdots, df_{p-1}, \cdot), Q] + [h_2 P(dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] \\ =& h_1[P(dh_2, df_2, \cdots, df_{p-1}, \cdot), Q] - (P(dh_2, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_1)) \\ &+ h_2[P(dh_1, df_2, \cdots, df_{p-1}, \cdot), Q] - (P(dh_1, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_2) \\ =& -(P(dh_2, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_1)) - (P(dh_1, df_2, \cdots, df_{p-1}, \cdot) \wedge Q(dh_2) .\end{aligned}$$

From this it is easy to see the identity (6) holds.

The cases of $P \vdash P$ and $Q \vdash Q$ are proved by similar calculations using (1) and (4) in Proposition 1.3. □

REMARK 1.2. In a similar way, what we obtain from (3) in Proposition 1.3 is the following. *The function*

$$(Q \vdash P)(f_1, \cdots, f_p; g_1, \cdots, g_{p+1}) + (-1)^{p+1} P(df_1, \cdots, df_p) Q(dg_1, \cdots, dg_{p+1}) \tag{7}$$

is skew symmetric with respect to the all arguments. In particular, if $f_1 = g_1$ *the above function vanishes.*

By the above two lemmas, we have most part of the following:

PROPOSITION 1.4. *If* (Q, P) *is a Nambu-Jacobi pair and* $\deg P = p \geq 2$, *then* $P \vdash P = 0$, $P \vdash Q = 0$ *and* $Q \vdash Q = 0$.

PROOF. For $p > 2$, the statement is obvious from Lemma 1.2 and Lemma 1.3. The case $p = 2$ is treated separately.

[Proof when $p = 2$] Assume $\mathcal{A}$ is a Nambu-Jacobi bracket. First, we prove P is decomposable 2-vector at a point where $Q \neq 0$. As before, we consider P and Q are 2-vector and 3-vector of a vector space V. Since $\deg Q = 3$, by Lemma 1.2 and Lemma 1.3, we have $Q \vdash Q = 0$ that is Q is decomposable. The condition $P \vdash Q$ is fully skew symmetric means that

$$P(x, \cdot) \wedge Q(x, \cdot, \cdot) = 0 \quad \text{for } x \in V^* . \tag{8}$$

Taking the value at (y, z, w) we have

$$P(x, y)Q(x, z, w) - P(x, z)Q(x, y, w) + P(x, w)Q(x, y, z) = 0 . \tag{9}$$

Regard Q as a linear map $V^* \to \bigwedge^2 V$ and fix a direct sum decomposition

$$V^* = K \oplus L \tag{10}$$

where $K = \ker Q$ and L is a complementary subspace which is isomorphic to $\operatorname{Im} Q$. If $w \in K$, from the above relation we have

$$P(x, w)Q(x, y, z) = 0 . \tag{11}$$

Given $0 \neq x \in L$, we can choose y, z so that $Q(x, y, z) \neq 0$. Thus, we have

$$P(x, w) = 0 \quad \text{for any} \quad x \in L, w \in K .$$

If we replace x by $x + v$, $(v \in K)$, in (11), we have

$$(P(x, w) + P(v, w))Q(x, y, z) = P(v, w)Q(x, y, z) = 0 , \quad w \in K . \tag{12}$$

From this we see that

$$P(x, y) = 0 \quad \text{if} \quad x \in K . \tag{13}$$

Since $\operatorname{rank} Q = \dim L = 3$, $\operatorname{rank} P$ must be 2 and P is a decomposable 2-vector hence $P \vdash P = 0$.

Next we prove $P \vdash Q = 0$. Regard Q and P as linear maps $\bigwedge^2 V^* \to V$, $V^* \to V$, respectively. Since we have

$$P(x, y)Q(x, z, \cdot) - P(x, z)Q(x, y, \cdot) + P(x, \cdot)Q(x, y, z) = 0 , \tag{14}$$

and as we saw above P maps K to 0. This means $\operatorname{Im} P \subset \operatorname{Im} Q$. From this we have $P(x, \cdot) \wedge Q = 0$ for any $x \in V^*$. This shows $P \vdash Q = 0$.

If at a point a, $Q = 0$ and a is in the closure of the set where $Q \neq 0$, the semi-continuity of the rank assures that rank $P \leq 2$ and we have $P \vdash P = 0$, $P \vdash Q = 0$ in this case too.

Finally, we consider the point where Q vanishes identically on some neighborhood of the point. In this case Proposition 1.3 (3) says P is a 2-vector satisfying

$$P(x, y)P - P(x, \cdot) \wedge P(y, \cdot) = 0 \quad \text{for} \quad x, y \in V^* .$$

This clearly shows that P is decomposable.

Thus we proved $P \vdash P = 0$ and $P \vdash Q = 0$ hold everywhere. This finishes the proof in the case where $p = 2$. □

By the above Proposition 1.4, the identities in Proposition 1.3 are simplified as in the following form for $p \geq 2$.

THEOREM 1.5. *Let $\mathcal{A} = Q + 1 \wedge P$ be a bracket on a manifold M, which is given by a $(p+1)$-vector field Q and a p-vector field P where $p \geq 2$ and assume* rank $P \leq 2$ *when $p = 2$. Then $\mathcal{A}$ is a Nambu-Jacobi bracket if and only if the following conditions are satisfied.*

For any functions $f_1, f_2, \cdots, f_p \in C^\infty(M)$,

$$\begin{aligned} &(1) \quad [P(d\boldsymbol{f}_{p-1}, \cdot), P] = 0 \,, \\ &(2) \quad [P(d\boldsymbol{f}_{p-1}, \cdot), Q] = 0 \,, \\ &(3) \quad [Q(d\boldsymbol{f}_p, \cdot), P] = (-1)^p Q(d(P(d\boldsymbol{f}_p)), \cdot) \,, \\ &(4) \quad [Q(d\boldsymbol{f}_p, \cdot), Q] = 0 \end{aligned}$$

hold, where $d\boldsymbol{f}_{p-1}$ stands for $df_1 \wedge \cdots \wedge df_{p-1}$.

PROOF. By Proposition 1.4, the conditions in Proposition 1.3 reduce to the above formulas. Conversely, assume that P and Q satisfy the above formulas. Then we have the same conclusion as those of Lemma 1.3. Thus if $p > 2$, we have $P \vdash P = 0$, $P \vdash Q = 0$, $Q \vdash Q = 0$ by Lemma 1.2. If we assume P is decomposable when $p = 2$, we can get the same conclusion by the argument of Lemma 1.4. Consequently, P and Q satisfy the conditions in Proposition 1.3. □

REMARK 1.3. Corollary 5.1 in [2] says if $Q + 1 \wedge P$ is a Namubu-Jacobi bracket, then conditions (2) and (1) in our theorem hold. Also, Corollary 5.6 in [2] says if $Q + 1 \wedge P$ is a Namubu-Jacobi bracket, then conditions (4) and (3) in our theorem hold. We emphasize here that our theorem states that those are necessary and sufficient conditions for $Q + 1 \wedge P$ to be a Nambu-Jacobi bracket.

Since $P \vdash Q = 0$ means $P(d\boldsymbol{f}_{p-1}, \cdot) \wedge Q = 0$, in the case when P is non-zero, Q is a multiple of P. Thus we have a vector field v satisfying $Q = v \wedge P$. It is desirable, in this case, to find the conditions on P and v which imply the Fundamental Identity. This will be done in the next section.

2. Associated Foliations

In this section, we investigate some geometric structure of Nambu-Jacobi manifold, namely the associated foliation which is given by the characteristic distributions of the structure. As is well-known, the Jacobi identity of a Poisson manifold implies the integrability of the characteristic distribution of the Poisson structure. This leads us to the foliation by symplectic leaves. This foliation is singular in general in the sense that the dimension of the leaves varies from point to point. Similarly on a Nambu-Poisson manifold we have a foliation and a contravariant *volume tensor* (multi-vector field of highest degree on a manifold) on each leaf. Theorem 2.1 below may be considered as a geometric characterization of a Nambu-Poisson manifold. We mean by the characteristic distribution of a p-vector field η the image of the bundle map $B_\eta : \bigwedge^{p-1} T^*M \to TM$ where $B_\eta(\alpha) = \eta(\alpha, \cdot)$.

Recall that *the generalized divergence* of η is defined as follows. Let ∇ be a torsion free affine connection on TM. ∇ gives a map $\nabla : \Gamma(\bigwedge^p TM) \to \Gamma(T^*M) \otimes \Gamma(\bigwedge^p TM)$. Let

$$c : \Gamma(T^*M) \otimes \Gamma(\bigwedge^p TM) \to \Gamma(\bigwedge^{p-1} TM)$$

be the map given by the contraction of 1-forms and p-vector fields. The generalized divergence $\mathrm{Div}\,\eta$ associated with ∇ is defined by

$$\mathrm{Div}\,\eta = c(\nabla(\eta)) .$$

One of the definition of the Schouten bracket of multi-vector fields is given by the formula

$$[P, Q] = \mathrm{Div}(P \wedge Q) - (\mathrm{Div}\,P) \wedge Q - (-1)^p P \wedge \mathrm{Div}\,Q \tag{15}$$

where p is the degree of P. It is independent of the choice of connections. In what follows, we choose once and for all a Riemannian connection on TM and the Div will be the one which is associated with this connection (See also [7]).

THEOREM 2.1. *Let η be a decomposable C^∞ p-vector field on a C^∞-manifold M. Then the following statements are equivalent.*

(1) *The bracket $\{f_1, \cdots, f_p\}^\eta = \eta(df_1, \cdots, df_p)$ satisfies the Fundamental Identity.*

(2) *The characteristic distribution of η is integrable* (*in the sense of Sussmann and Stefan*).

(3) *On the open set U where η is non-zero*, *there exists a smooth* 1*-form γ which satisfies the equality*

$$\mathrm{Div}\,\eta = i_\gamma \eta .$$

PROOF. Equivalence of (1) and (2) is known in Theorem 4.3 of [10].
(2) $\Rightarrow$ (3). Since η is decomposable p-vector field, on a neighborhood of each point $a \in U$, we can choose a set of vector fields $X_1, X_2, \cdots, X_p \quad (X_i \in \Gamma(\mathrm{Im} B_\eta))$ such that

$$\eta = X_1 \wedge X_2 \wedge \cdots \wedge X_p .$$

Then we have

$$\begin{aligned}\operatorname{Div}\eta &= \sum_{i=1}^{p}(-1)^{i-1}X_1 \wedge \cdots \wedge (\operatorname{Div} X_i) \wedge \cdots \wedge X_p \\ &+ \sum_{i<j}(-1)^{i+j-1}[X_i, X_j] \wedge \cdots \hat{X}_i \wedge \cdots \wedge \hat{X}_j \wedge \cdots \wedge X_p\end{aligned}$$

Since $\operatorname{Im} B_\eta$ is integrable, $[X_i, X_j]$ is a linear combination of $X_1, X_2, \cdots, X_p$ at each point of U. Thus $\operatorname{Div}\eta$ is a $(p-1)$-vector field which is generated by $X_1, X_2, \cdots, X_p$ and hence a cross-section of the bundle $\bigwedge^{p-1}(\operatorname{Im} B_\eta)$. Define

$$J_\eta : (\operatorname{Im} B_\eta)^* \to \bigwedge^{p-1}(\operatorname{Im} B_\eta)$$

to be the bundle map given by $J_\eta(\alpha) = \eta(\alpha, \cdot) = i_\alpha \eta$. Clearly it is a bundle isomorphism on U where η is non-zero. This assures that there exists a 1-form γ on U such that $i_\gamma \eta = \operatorname{Div}\eta$. □

(3)⇒(1). First we note the following formula.

LEMMA 2.1. *Let β be a $(p-1)$-form and η a decomposable p-vector field on M. Then we have the following equality.*

$$[\eta(\beta, \cdot), \eta] = (-1)^p \eta(d\beta)\eta + \eta(\beta, \cdot) \wedge \operatorname{Div}\eta + (-1)^p(\operatorname{Div}\eta)(\beta)\eta \tag{16}$$

PROOF [(Proof of Lemma)]. Taking the contraction on both sides of

$$\nabla(\eta(\beta, \cdot)) = (\nabla\eta)(\beta, \cdot) + \eta(\nabla\beta, \cdot)\,,$$

we have

$$\operatorname{Div}(\eta(\beta, \cdot)) = (-1)^{p-1}(\operatorname{Div}\eta)(\beta) + (-1)^{p-1}\eta(d\beta)\,. \tag{17}$$

Thus we have

$$\begin{aligned}[\eta(\beta, \cdot), \eta] &= \operatorname{Div}(\eta(\beta, \cdot) \wedge \eta) - (\operatorname{Div}\eta(\beta, \cdot))\eta + \eta(\beta, \cdot) \wedge \operatorname{Div}\eta \\ &= (-1)^p(\operatorname{Div}\eta)(\beta)\eta + (-1)^p\eta(d\beta)\eta + \eta(\beta, \cdot) \wedge \operatorname{Div}\eta\,.\end{aligned}$$

Note that $\eta(\beta, \cdot) \wedge \eta = 0$ holds by the decomposability. □

We continue the proof of Theorem. Since $\eta \wedge \eta(df_1, \cdots, df_{p-1}, \cdot) = 0$ on U, we have

$$\begin{aligned}0 &= i_\gamma(\eta \wedge \eta(df_1, \cdots, df_{p-1}, \cdot)) \\ &= i_\gamma(\eta) \wedge \eta(df_1, \cdots, df_{p-1}, \cdot) + (-1)^p\eta(df_1, \cdots, df_{p-1}, \gamma)\eta \\ &= (\operatorname{Div}\eta) \wedge \eta(df_1, \cdots, df_{p-1}, \cdot) - ((\operatorname{Div}\eta)(df_1, \cdots, df_{p-1}))\eta \\ &= (-1)^{p-1}[\eta(df_1, \cdots, df_{p-1}, \cdot), \eta]\,.\end{aligned}$$

We used Lemma above for $\beta = df_1 \wedge \cdots \wedge df_{p-1}$.

If $a \in M \setminus U$, $\eta_{|a} = 0$, the right hand side of the above lemma is equal to 0 and thus $[\eta(df_1, \cdots, df_{p-1}, \cdot), \eta]_{|a} = 0$.

Consequently, we have

$$[\eta(df_1, \cdots, df_{p-1}, \cdot), \eta] = 0$$

on the whole M and the bracket $\{\cdots\}^{\eta}$ satisfies the Fundamental Identity. □

Now we are going to investigate the foliation associated with a Nambu-Jacobi structure. Let $\mathcal{A} = Q + 1 \wedge P$ be a Nambu-Jacobi bracket on a manifold M, which is given by a $(p+1)$-vector field Q and a p-vector field P. Then by Theorem 1.5 (1) of preceding section, $[P(df_1, \cdots, df_{p-1}, \cdot), P] = 0$. Thus P is a Nambu-Poisson tensor and its characteristic distribution is integrable, giving a generalized foliation (Theorem 2.1). We denote this foliation by $\mathcal{F}_P$. Exactly the same thing holds for the $(p+1)$-vector field Q. Thus we have two foliations $\mathcal{F}_P$ and $\mathcal{F}_Q$ of M. First we restrict our attention to the case when P is non-zero or it may be said that we consider the foliations of the open set of M where P is non-zero. By Proposition 1.4, $P \vdash Q = 0$. This is equivalent to

$$P(df_1, \cdots, df_{p-1}, \cdot) \wedge Q = 0\,,$$

for any $(p-1)$ functions $f_1, \cdots, f_{p-1}$. On a neighborhood of a point where $P \neq 0$, we have functions $f_1, \cdots, f_p$ such that $P(df_1, \cdots, df_p) \neq 0$. Thus the set of vector fields $\{X_1, \cdots, X_p\}$ where $X_i = P(df_1, \cdots, \hat{df}_i, \cdots, df_p)$ is linearly independent at each point. From the above relation, Q is a multiple of X_i's and consequently, there is a vector field v such that $Q = v \wedge P$. A partition of unity argument assures that we may consider v a global one.

PROPOSITION 2.2. *The vector field v preserves the associated foliation $\mathcal{F}_P$. In fact, there exists a function φ such that $L_v P = \varphi P$ holds.*

PROOF. By Theorem 1.5, we have

$$\begin{aligned} 0 = [P(d\boldsymbol{f}_{p-1}, \cdot), Q] &= [P(d\boldsymbol{f}_{p-1}, \cdot), v \wedge P] \\ &= [P(d\boldsymbol{f}_{p-1}, \cdot), v] \wedge P + v \wedge [P(d\boldsymbol{f}_{p-1}, \cdot), P] \\ &= [P(d\boldsymbol{f}_{p-1}, \cdot), v] \wedge P \end{aligned}$$

because $[P(d\boldsymbol{f}_{p-1}), P] = 0$. But $[P(d\boldsymbol{f}_{p-1}, \cdot), v] \wedge P = 0$ is expressed as

$$-((L_v P)(d\boldsymbol{f}_{p-1}, \cdot)) \wedge P - (P(L_v d\boldsymbol{f}_{p-1}, \cdot)) \wedge P = 0$$

and $(P(L_v d\boldsymbol{f}_{p-1}, \cdot)) \wedge P = 0$ because of the decomposability of P. Thus we have

$$((L_v P)(d\boldsymbol{f}_{p-1}, \cdot)) \wedge P = 0$$

for arbitrary $(p-1)$-form $d\boldsymbol{f}_{p-1}$. Again by the decomposability of P, we see $L_v P$ is a multiple of P. Thus we have a function φ satisfying $L_v P = \varphi P$. The equation

$$[v, P(d\boldsymbol{f}_{p-1}, \cdot)] = \varphi P(d\boldsymbol{f}_{p-1}, \cdot) + P(L_v d\boldsymbol{f}_{p-1})$$

shows that v preserves foliation $\mathcal{F}_P$. □

We have the converse.

THEOREM 2.3. *Let P be a regular Nambu-Poisson tensor of degree $p \geq 2$, which we assume decomposable when $p = 2$. Suppose that there exists a vector field v which satisfies*

$$L_v P = \varphi P$$

for some smooth function φ. Define $(p+1)$-vector field Q by $Q = v \wedge P$. Then the pair (Q, P) is a Nambu-Jacobi pair, namely the bracket

$$\mathcal{A} = Q + 1 \wedge P$$

defines a Nambu-Jacobi structure.

PROOF. Since we are assuming the decomposability of P, it is sufficient to verify the conditions (1)–(4) of Theorem 1.5.
Condition (1) is our assumption. Condition (2) asserts that $[P(dg, \cdot), Q] = 0$ holds for any $dg := dg_1 \wedge \cdots \wedge dg_{p-1}$. This is easily verified as follows by using $[P(dg, \cdot), P] = 0$ and the decomposability of P;

$$\begin{aligned}[P(dg, \cdot), v \wedge P] &= [P(dg, \cdot), v] \wedge P + v \wedge [P(dg, \cdot), P] \\ &= -(\varphi P(dg, \cdot) + P(L_v(dg), \cdot)) \wedge P = 0\,.\end{aligned}$$

We verify Condition (4).

Since $Q = v \wedge P$ is decomposable, from the view point of Theorem 2.1, it is enough to see the integrability of the characteristic distribution $\mathcal{F}_Q$ of Q on open set where $Q \neq 0$. Locally, we can write P and Q as follows.

$$P = hX_1 \wedge \cdots \wedge X_p\,, \quad Q = hv \wedge X_1 \wedge \cdots \wedge X_p$$

where X_i is a local vector field of the form $P(df_1 \wedge \cdots \wedge \hat{df_i} \wedge \cdots \wedge df_p)$ and h is a function. The vector fields $X_1, \cdots, X_p, v$ generate the distribution $\mathcal{F}_Q$ and they form a involutive system since $\mathcal{F}_P$ is integrable by assumption and since we have the following:

$$[v, P(dg, \cdot)] = L_v((P(dg, \cdot)) = (L_v P)(dg, \cdot) + P(L_v dg, \cdot) = P(\varphi dg + L_v dg, \cdot)\,.$$

Thus $\mathcal{F}_Q$ is integrable and we have $[Q, Q] = 0$.

To verify Condition (3), we must prove the equality

$$[Q(d\boldsymbol{f}; \cdot), P](\cdots) = (-1)^p Q(d(P(d\boldsymbol{f})), \cdots) \tag{18}$$

for $d\boldsymbol{f} = df_1 \wedge \cdots \wedge df_p$.

First we calculate the left hand side of this equality. Using

$$Q(d\boldsymbol{f}, \cdot) = (v \wedge P)(d\boldsymbol{f}) = P(i_v(d\boldsymbol{f}), \cdot) + (-1)^p P(d\boldsymbol{f})v$$

and a general formula

$$\operatorname{Div}(P(\alpha)) = (-1)^{\deg \alpha}(\operatorname{Div} P)(\alpha) + (-1)^{\deg \alpha} P(d\alpha)\,, \tag{19}$$

we calculate as follows:

$$
\begin{aligned}
[Q(df,\cdot),P] &= [P(i_v(df),\cdot),P] + (-1)^p[P(df)v,P] \\
&= \mathrm{Div}(P(i_v(df,\cdot))\wedge P) - \mathrm{Div}(P(i_v(df)),\cdot)P \\
&\quad + P(i_v(df),\cdot)\wedge \mathrm{Div}\,P \\
&\quad + (-1)^p P(df)[v,P] + (-1)^{p+1} v\wedge P(d(P(df)),\cdots) \\
&= (-1)^p((\mathrm{Div}\,P)(i_v(df)))P + (-1)^p P(d(i_v(df)))P \\
&\quad + P(i_v(df),\cdot)\wedge \mathrm{Div}\,P \\
&\quad + (-1)^p P(df)[v,P] + (-1)^{p+1} v\wedge P(d(P(df)),\cdot)\,.
\end{aligned} \tag{20}
$$

Now we use the assumption that there exists a 1-form such that $\mathrm{Div}\,P = i_\gamma P$ (Theorem 2.1). Then the above (20) is equal to

$$
\begin{aligned}
&(-1)^p(i_\gamma P)(i_v(df))P + (-1)^p P(d(i_v(df)))P + P(i_v(df))\wedge(i_\gamma P) \\
&\quad + (-1)^p P(df)[v,P] + (-1)^{p+1} v\wedge P(d(P(df)),\cdot) \\
&= -i_\gamma(P(i_v(df)))P + (-1)^p P(d(i_v(df)))P + P(i_v df)\wedge(i_\gamma P) \\
&\quad + (-1)^p P(df)[v,P] + (-1)^{p+1} v\wedge P(d(P(df)),\cdot) \\
&= -i_\gamma(P(i_v df)\wedge P) + (-1)^p P(di_v(df))P \\
&\quad + (-1)^p P(df)\varphi P + (-1)^{p+1} v\wedge P(d(P(df)),\cdot) \\
&= (-1)^p P(L_v(df))P + (-1)^p P(df)\varphi P + (-1)^{p+1} v\wedge P(d(P(df)),\cdot) \\
&= (-1)^p L_v(P(df)P + (-1)^{p+1}(L_v P)(df)P \\
&\quad + (-1)^p P(df)(\varphi P) + (-1)^{p+1} v\wedge P(d(P(df)),\cdot) \\
&= (-1)^{p+1} v\wedge P(d(P(df)),\cdot) + (-1)^p L_v(P(df))P\,.
\end{aligned}
$$

This can be seen to be equal to the right hand side of (18), since we have

$$
\begin{aligned}
(-1)^p Q(d(P(df)),\cdot) &= (-1)^p(v\wedge P)(d(P(df)),\cdots) \\
&= (-1)^p v(d(P(df)))P + (-1)^{p+1} v\wedge P(d(P(df)),\cdot)\,.
\end{aligned}
$$

Thus, the pair $(Q = v\wedge P, P)$ satisfy the conditions (1)–(4) and the bracket is a Nambu-Jacobi pair. □

Next we consider a Nambu-Jacobi structure $Q + 1\wedge P$ where Q is regular, that is Q is nowhere zero. In this case we obtain the following:

THEOREM 2.4. *Let Q be a Nambu Poisson tensor of degree $q(\geq 2)$. We assume when $q = 2$, Q is decomposable. Let α be a 1-form which is closed on the leaves of $\mathcal{F}_Q$. That is $Q(d\alpha,\cdot) = 0$. Put $P = Q(\alpha,\cdot)$. Then (Q, P) makes a Nambu-Jacobi pair. Conversely,* *if (Q, P) is a Nambu-Jacobi pair and Q is regular, there exists a 1-form α which is closed along the leaves of Q such that $P = Q(\alpha, \cdot)$.*

PROOF. We first verify the condition $J^P Q = 0$. Namely, we prove

$$[P(d\boldsymbol{f}_{p-1}, \cdot), Q] = [Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot), Q] = 0\,.$$

Using the decomposability of Q and the formula (19) for $\mathrm{Div}(Q(\alpha))$, we calculate as follows:

$$\begin{aligned}[Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot), Q] &= \mathrm{Div}(Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q) - \mathrm{Div}(Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot)) \wedge Q \\ &\quad + Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q \\ &= (-1)^{p+1}(\mathrm{Div}\, Q)(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q + (-1)^{p+1} Q(d\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q \\ &\quad + Q(\alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge \mathrm{Div}\, Q\,.\end{aligned}$$

Clearly, this is equal to 0 where $Q = 0$. On the other hand, on the open set where $Q \neq 0$, we have a 1-form γ such that $\mathrm{Div}\, Q = Q(\gamma, \cdot)$ and the above is equal to

$$\begin{aligned}&(-1)^{p+1} Q(\gamma \wedge \alpha \wedge d\boldsymbol{f}_{p-1}, \cdot) \wedge Q + Q(\alpha \wedge d\boldsymbol{f}_{p-1}) \wedge Q(\gamma) \\ &\quad = -i_\gamma(Q(\alpha \wedge d\boldsymbol{f}_{p-1}) \wedge Q) = 0\,.\end{aligned}$$

Thus we proved $J^P Q = 0$.
Secondly, we prove

$$[P(d\boldsymbol{f}_{p-1}, \cdot), P] = 0$$

for any functions $f_1, \cdots, f_{p-1}$. We use the abbreviated notations that $p = q-1$ and $d\boldsymbol{f}_{p-1} = df_1 \wedge \cdots \wedge df_{p-1}$ as before. Then we calculate as follows;

$$\begin{aligned}[P(d\boldsymbol{f}_{p-1}, \cdot), P] &= [P(d\boldsymbol{f}_{p-1}, \cdot), Q(\alpha, \cdot)] \\ &= [P(d\boldsymbol{f}_{p-1}, \cdot), Q](\alpha) + Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha)\,. \end{aligned} \tag{21}$$

As we showed above, $[P(d\boldsymbol{f}_{p-1}, \cdot), Q](\alpha) = 0$ and $Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha) = 0$ is verified as follows.

$$\begin{aligned}Q(L_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha, \cdot) &= Q(di_{P(d\boldsymbol{f}_{p-1}, \cdot)}\alpha + i_{P(d\boldsymbol{f}_{p-1}, \cdot)}d\alpha, \cdot) \\ &= Q(i_{P(d\boldsymbol{f}_{p-1}, \cdot)}d\alpha) = Q(d\alpha(P(d\boldsymbol{f}_{p-1}, \cdot), \cdot)\,.\end{aligned}$$

The rightmost term vanishes since if write $Q = X_1 \wedge \cdots \wedge X_q$, this is equal to

$$\sum_{i=1}^{q} (-1)^{i-1} d\alpha(P(d\boldsymbol{f}_{p-1}, \cdot), X_i) X_1 \wedge \cdots, \wedge \hat{X}_i \wedge \cdots \wedge X_q$$

and since α is closed on $\mathrm{Im}\, Q$.

Next, we prove

$$[Q(d\mathbf{g}, \cdot), P] = (-1)^p Q(d(P(d\mathbf{g}), \cdot) \quad \text{for} \quad d\mathbf{g} = dg_1 \wedge \cdots \wedge dg_p\,.$$

This is shown as follows.

$$\begin{aligned}[][Q(d\mathbf{g},\cdot), Q(\alpha)] &= [Q(d\mathbf{g},\cdot), Q](\alpha) + Q(L_{Q(d\mathbf{g},\cdot)}\alpha,\cdot) = Q(di_{Q(d\mathbf{g},\cdot)}\alpha + i_{Q(d\mathbf{g},\cdot)}d\alpha) \\ &= Q(di_{Q(d\mathbf{g},\cdot)}\alpha) = Q(d(Q(d\mathbf{g},\alpha))) = (-1)^p Q(d(P(d\mathbf{g}))).\end{aligned}$$

$Q(i_{Q(d\mathbf{g},\cdot)}d\alpha) = 0$ follows from $d\alpha = 0$ on $\mathcal{F}_Q$.

Now we prove the converse. Namely, assuming (Q, P) is a Nambu-Jacobi pair on M and Q is non-singular, we prove that there exists a 1-form α such that $P = Q(\alpha,\cdot)$ and $Q(d\alpha) = 0$. By assumption, (Q, P) satisfies (1)–(4) in Theorem 1.5. If we consider Q as a bundle map $\bigwedge^p T^*M \to TM$, $\operatorname{Im} Q$ is a $(p+1)$-dimensional sub-bundle of TM. Q is also considered as a non-zero cross section of $\bigwedge^{p+1}\operatorname{Im} Q$ and gives a natural isomorphism $(\operatorname{Im} Q)^* \to \bigwedge^p \operatorname{Im} Q$. Let $B_Q : \bigwedge^p \operatorname{Im} Q \to (\operatorname{Im} Q)^*$ denote the inverse isomorphism. Since we have $P(df_1 \wedge \cdots \wedge df_{p-1},\cdot) \wedge Q = 0$, $\operatorname{Im} P \subset \operatorname{Im} Q$ (see Proposition 1.4). Thus P is a cross section of the bundle $\bigwedge^p \operatorname{Im} Q$. Put $\alpha' = B_Q(P)$ and choose a 1-form α so that α projects to α' under the natural surjection $T^*M \to (\operatorname{Im} Q)^*$. Then we can see that

$$Q(\alpha,\cdot) = Q(\alpha',\cdot) = P.$$

Now by a characterization of Nambu-Poisson tensor field, there exists a 1-form on M, satisfying $\operatorname{Div} Q = Q(\gamma,\cdot)$. Since $Q(\alpha,\cdot) = P$ is also a Nambu-Poisson tensor field, there exists a 1-form λ on the open set where $Q(\alpha,\cdot) \neq 0$, satisfying $\operatorname{Div}(Q(\alpha,\cdot)) = Q(\alpha,\lambda,\cdot)$. By the condition $[Q(\alpha, d\boldsymbol{f}_{p-1},\cdot), Q] = 0$, and the decomposability of Q, we have

$$0 = -(\operatorname{Div}(Q(\alpha, d\boldsymbol{f}_{p-1},\cdot))Q + Q(\alpha, d\boldsymbol{f}_{p-1},\cdot) \wedge \operatorname{Div} Q.$$

But we have the following

$$\begin{aligned}\operatorname{Div}(Q(\alpha, d\boldsymbol{f}_{p-1},\cdot)) &= (-1)^{p-1}\operatorname{Div}(Q(\alpha,\cdot))(d\boldsymbol{f}_{p-1}) + (-1)^{p-1}Q(\alpha, dd\boldsymbol{f}_{p-1}) \\ &= (-1)^{p-1}Q(\alpha,\lambda, d\boldsymbol{f}_{p-1}).\end{aligned}$$

Thus we have

$$\begin{aligned}(-1)^{p-1}Q(\alpha,\lambda,d\boldsymbol{f}_{p-1})Q &= Q(\alpha, d\boldsymbol{f}_{p-1},\cdot) \wedge \operatorname{Div} Q = Q(\alpha, d\boldsymbol{f}_{p-1},\cdot) \wedge Q(\gamma,\cdot) \\ &= -i_\gamma(Q(\alpha, d\boldsymbol{f}_{p-1},\cdot) \wedge Q) + Q(\alpha, d\boldsymbol{f}_{p-1},\gamma)Q.\end{aligned}$$

Since $Q(\alpha, d\boldsymbol{f}_{p-1},\cdot) \wedge Q = 0$ by the decomposability, this means $Q(\alpha, \lambda - \gamma,\cdot) = 0$. If we use the formula

$$\operatorname{Div}(Q(\alpha,\cdot)) = -(\operatorname{Div} Q)(\alpha,\cdot) - Q(d\alpha,\cdot),$$

we have $Q(d\alpha,\cdot) = -Q(\alpha,\gamma-\lambda,\cdot) = 0$. This is what we wanted to show. □

REMARK 2.1. This theorem has also been proved in [2] when α is an exact 1-form df for some function f on M.

3. Examples

By Theorem 2.3 and Theorem 2.4, we obtain concrete examples of Nambu-Jacobi manifolds. Here, we have a few of them.

1. We consider the Reeb foliation of S^3 as the underlying foliation. There exists a 2-vector field P which is non-singular on each leaf and tangent to it. We can assume every thing is invariant under the natural S^1-action on S^3. Let v be the vector field on S^3 which is obtained from this action. Then $L_v P = 0$ and $(Q = v \wedge P, P)$ is a Nambu-Jacobi pair by Theorem 2.3. Q vanishes exactly along the toral leaf.

2. Let $\mathfrak{F}$ be the Anosov foliation on the circle bundle over a closed surface of genus $g \geq 2$. The leaves are diffeomorphic to either R^2 or cylinder $S^1 \times R$. Since both types of leaves are dense, there is no non-trivial vector field transverse to $\mathfrak{F}$ which preserves the foliation. Therefore the only possible Jacobi pair is trivial one, namely it is $(0, P)$.

3. Let $A : T^n \to T^n$ be a hyperbolic toral automorphism. The mapping torus M_A of A has a foliation foliated by the weak unstable manifolds. Let Q denote a natural tensor field which gives a volume tensor field along each leaf. Let α be the 1-form on $M_A = T^n \times [0, 1]/ \sim \to S^1$, which is the pull-back of $d\theta$ by the projection $M_A \to S^1$. Then α is closed and $(Q, P = Q(\alpha, \cdot))$ is a Nambu-Jacobi pair. P defines a foliation foliated by strong unstable manifolds.

4. For any Nambu-Poisson structure Q on M, $(Q, \operatorname{Div} Q)$ is a Nambu-Jacobi pair. Here Div is a divergence associated with a connection which preserves a volume form of M. If $\operatorname{Div} Q = Q(\gamma, \cdot)$, we have $Q(d\gamma, \cdot) = -\operatorname{Div}^2 Q$ and $\operatorname{Div}^2 = 0$ since we assumed the connection preserves a volume form. Thus by Theorem 2.4, we have the result.
On a Nambu-Jacobi manifold for which the tensor fields P and Q are both non-singular, we have a regular foliation $\mathcal{F}_Q$ and its subfoliation $\mathcal{F}_P$. By our theorem, on each leaf of $\mathcal{F}_Q$ there exits a non-singular vector field and the subfoliation $\mathcal{F}_P$ is defined by a closed 1-form on the leaf. These impose a rather strong restriction on the foliated structure of such a Nambu-Jacobi structure. It seems an interesting topological question to find which manifold has such a foliated structure.

4. Appendix

In this appendix, we prove Proposition 1.3. We denote the bracket defined by a p-vector field P by $\{\cdots\}^P$. Namely, $\{f_1, \cdots, f_p\}^P = P(df_1, \cdots, df_p)$. The bracket $\{\cdots\} := \{\cdots\}^{Q+1\wedge P} = Q + 1 \wedge P$ determined by a $q(= p+1)$-vector field Q and a p-vector field P, is by definition is the following:

$$\{f_1, \cdots, f_q\} = \{f_1, \cdots, f_q\}^Q + \sum_{j=1}^{q} (-1)^{j-1} f_j \{f_1, \cdots, \hat{f}_j, \cdots, f_q\}^P$$

$$= Q(df_1, \cdots, df_q) + \sum_{j=1}^{q} (-1)^{j-1} f_j P(df_1, \cdots, \hat{df_j}, \cdots, df_q)\,.$$

We would like to write down the Fundamental Identity for this bracket in terms of the brackets of Q and P and find the relations which Q and P satisfy.

For the brackets $\{\cdots\}^P$ and $\{\cdots\}^Q$ of degree p and q, respectively, we defined $J^P Q$ and $P \vdash Q$ as follows.

$$\begin{aligned}
&J^P Q(f_1, \cdots, f_{p-1}; g_1, \cdots, g_q) \\
&\quad = \{f_1, \cdots, f_{p-1}, \{g_1, \cdots, g_q\}^Q\}^P - \{\{f_1, \cdots, f_{p-1}, g_1\}^P, g_2, \cdots, g_q\}^Q \\
&\qquad - \{g_1, \{f_1, \cdots, f_{p-1}, g_2\}^P, g_3, \cdots, g_q\}^Q - \cdots \\
&\qquad - \{g_1, \cdots, g_{q-1}, \{f_1, \cdots, f_{p-1}, g_q\}^P\}^Q \\
&\quad = [P(df_1 \wedge \cdots \wedge df_{p-1}, \cdot), Q](dg_1, \cdots, dg_q)\,,
\end{aligned}$$

$$\begin{aligned}
&(P \vdash Q)(f_1, \cdots, f_{p-1}; g_0, \cdots, g_q) \\
&\quad = \sum_{j=0}^{q} (-1)^j \{f_1, \cdots, f_{p-1}, g_j\}^P \{g_0, \cdots, \hat{g}_j, \cdots, g_q\}^Q \\
&\quad = (P(df_1, \cdots, df_{p-1}, \cdot) \wedge Q)(dg_0, \cdots, dg_q)\,.
\end{aligned}$$

The Fundamental Identity for $\{\cdots\} = \{\cdots\}^{Q+1\wedge P}$ is the following identity for any C^∞ functions $f_1, \cdots, f_{q-1}, g_1, \cdots, g_q$ on M.

$$\{f_1, \cdots, f_{q-1}, \{g_1, \cdots, g_q\}\} = \sum_{i=1}^{q} (-1)^{i-1} \{\{f_1, \cdots, f_{q-1}, g_i\}, g_1, \cdots, \hat{g}_i, \cdots, g_q\}\,.$$

In this appendix, however, for our notational convenience, we adopt the following equivalent equation as the Fundamental Identity.

$$\{\{f_1, \cdots, f_q\}, g_2, \cdots, g_q\} = \sum_{j=1}^{q} \{f_1, \cdots, f_{j-1}, \{f_j, g_2, \cdots, g_q\}, f_{j+1}, \cdots, f_q\}\,. \tag{22}$$

We now start the computation. Since by definition,

$$\{f_1, \cdots, f_q\} := \{f_1, \cdots, f_q\}^Q + \sum_{j=1}^{q} (-1)^{j-1} f_j \{f_1, \cdots, \hat{f}_j, \cdots, f_q\}^P\,,$$

the left hand side of (22) is calculated as follows.

$$\begin{aligned}
&\{\{f_1, \cdots, f_q\}, g_2, \cdots, g_q\} \\
&\quad = \{\{\mathcal{F}\}^Q, \mathcal{G}\}^Q + \{\mathcal{F}\}^Q \{\mathcal{G}\}^P + \sum_{i=1}^{q} (-1)^{i-1} \{\mathcal{F}_i\}^P \{f_i, \mathcal{G}\}^Q
\end{aligned}$$

$$+\sum_{i=1}^{q}(-1)^{i-1}f_i(\{\{\mathcal{F}_i\}^P,\mathcal{G}\}^Q+\{\mathcal{F}_i\}^P\{\mathcal{G}\}^P+\sum_{k=2}^{q}(-1)^{k-1}g_k\{\{\mathcal{F}_i\}^P,\mathcal{G}_k\}^P)$$

$$+\sum_{k=2}^{q}(-1)^{k-1}g_k(\{\{\mathcal{F}\}^Q,\mathcal{G}_k\}^P+\sum_{i=1}^{q}(-1)^{i-1}\{\mathcal{F}_i\}^P\{f_i,\mathcal{G}_k\}^P)$$

where $\mathcal{F},\mathcal{G}$ denote the sequences of $f_1,\cdots,f_q$ and $g_2,\cdots,g_q$ respectively. $\mathcal{F}_i$ denotes the sequence which obtained from $\mathcal{F}$ by deleting the i-th component, and $\mathcal{G}_j$ denotes the sequence which obtained from $\mathcal{G}$ by deleting the $(j-1)$-th component. This is the left hand side of (22) expressed in terms of f_i's, g_i's and their brackets with respect to $\{\cdots\}^P$ and $\{\cdots\}^Q$.

In a similar way, we calculate the right hand side of the Fundamental Identity (22), by applying the Leibniz rule several times.

$$\begin{aligned}
&\sum_{j=1}^{q}\{f_1,,,f_{j-1},\{f_j,g_2,,,g_q\},f_{j+1},,,f_q\}\\
&\quad=\sum_{j=1}^{q}(-1)^{j-1}\{\{f_j,\mathcal{G}\}^Q+f_j\{\mathcal{G}\}^P+\sum_{k=2}^{q}(-1)^{k-1}g_k\{f_j,\mathcal{G}_k\}^P,\mathcal{F}_j\}\\
&\quad=\sum_{j=1}^{q}(-1)^{j-1}\{\{f_j,\mathcal{G}\}^Q,\mathcal{F}_j\}^Q+\sum_{j=1}^{q}(-1)^{j-1}\{f_j,\mathcal{G}\}^Q\{\mathcal{F}_j\}^P\\
&\qquad+\sum_{j=1}^{q}\sum_{\ell\neq j}(-1)^{j-1+\ell}f_\ell\{\{f_j,\mathcal{G}\}^Q,\mathcal{F}_{\ell j}\}^P+\sum_{j=1}^{q}(-1)^{j-1}f_j\{\{\mathcal{G}\}^P,\mathcal{F}_j\}^Q\\
&\qquad+q\{\mathcal{G}\}^P(\{\mathcal{F}\}^Q+\sum_{j=1}^{q}(-1)^{j-1}f_i\{\mathcal{F}_i\}^P)+\sum_{j=1}^{q}\sum_{\ell\neq j}(-1)^{j-1+\ell}f_jf_\ell\{\{\mathcal{G}\}^P,\mathcal{F}_{\ell j}\}^P\\
&\qquad+\sum_{j=1}^{q}\sum_{\ell=1}^{q-1}(-1)^{j+\ell}g_\ell(\{\{f_j,\mathcal{G}_\ell\}^P,\mathcal{F}_j\}^P+\{g_\ell,\mathcal{F}_j\}^Q\{f_j,\mathcal{G}_\ell\}^P+\{f_j,\mathcal{G}_\ell\}^P\{\mathcal{F}_j\}^P)\\
&\qquad+\sum_{j=1}^{q}\sum_{\ell=1}^{q-1}\sum_{m\neq j}(-1)^{j+\ell+m}f_m(g_\ell\{\{f_j,\mathcal{G}_\ell\}^P,\mathcal{F}_{mj}\}^P+\{f_j,\mathcal{G}_\ell\}^P\{g_\ell,\mathcal{F}_{mj}\}^P)\,,
\end{aligned}$$

where we used the notation $\mathcal{F}_{ij}$ which denotes if $i<j$ then the sequence $(\cdots,\hat{f}_i,,,\hat{f}_j,,)$ and if $i>j$ then it denotes the sequence which is obtained by dropping two entries and then changing the sign of the first entry, namely $\mathcal{F}_{ij}:=(-f_1,f_2,,,\hat{f}_j,,,\hat{f}_i,,)$. We will not simplify these any further since from the computation we can obtain necessary conditions on P and Q for the bracket $\{\cdots\}$ to satisfy the Fundamental Identity.

First we note that the sums containing the product f_if_j cancel out because of the skewness of the bracket.

To get the conditions, we put $f_q = g_q \equiv 1 (q = p+1)$ and compare the left hand side and the right hand side of (22) which we computed above. Since $\{\cdots, 1\}^P$ and $\{\cdots, 1\}^Q$ are both constantly equal to 0, we obtain the following relations:

$$\{\{f_1, \cdots, f_p\}^P, g_2, \cdots, g_p\}^P = \sum_{j=1}^{p} \{f_1, \cdots, f_{j-1}, \{f_j, g_2, \cdots, g_p\}^P, f_{j+1}, \cdots, f_p\}^P .$$

This is nothing but (1) of Proposition 1.3 and the Fundamental Identity for $\{\cdots\}^P$. Namely we get the condition

$$J^P P = 0 . \tag{23}$$

Putting this condition in our computation, we see that the terms containing $f_i g_k$ all cancel out.

Next, we put $g_q \equiv 1$, and by the same reason as before, we get the relation

$$\{\{f_1, \cdots, f_q\}^Q, g_2, \cdots, g_p\}^P = \sum_{j=1}^{q} \{f_1, \cdots, f_{j-1}, \{f_j, g_2, \cdots, g_p\}^P, f_{j+1}, \cdots, f_q\}^Q .$$

This shows

$$(-1)^p J^P Q(g_2, \cdots, g_p; f_1, \cdots, f_q) = 0 ,$$

and we get

$$J^P Q = 0 . \tag{24}$$

By this relation, we see that the terms which are the multiple of the function g_k all cancel out. Similarly, knowing the relations (23) and (24) and by putting $f_q \equiv 1$, we obtain the following relation

$$\begin{aligned}
&\{\{f_1, , , f_p\}^P, g_2, , , g_q\}^Q \\
&\quad = \sum_{j=1}^{p} \{f_1, , , f_{j-1}, \{f_j, g_2, , , g_q\}^Q, f_{j+1}, , f_p\}^P \\
&\qquad + (-1)^p \{f_1, , , f_p, \{g_2, , , g_q\}^P\}^Q + p\{f_1, , , f_p\}^P \{g_2, , , g_q\}^P \\
&\qquad + \sum_{j=1}^{p} \sum_{k=2}^{p+1} (-1)^{p+k+j+1} \{g_2, , , \hat{g}_k, , , g_q, f_j\}^P \{g_k, f_1, , , f_{j-1}, f_{j+1}, , , f_p\}^P .
\end{aligned}$$

A little computation shows that this is equivalent to the following:

$$\begin{aligned}
J^Q P(g_2, \cdots, g_q; f_1, \cdots, f_p) &= (-1)^p \{\{g_2, \cdots, g_q\}^P f_1, \cdots, f_p\}^Q \\
&\quad + \sum_{k=2}^{q} (-1)^k (P \vdash P)(g_2, \cdots, \hat{g}_k, \cdots, g_q; g_k, f_1, \cdots, f_p) .
\end{aligned} \tag{25}$$

This is the relation equivalent to (3) of Proposition 1.3. Note that

$$
\begin{aligned}
&(P \vdash P)(g_2, \cdots, \hat{g}_k, \cdots, g_q; g_k, f_1, \cdots, f_p) \\
&\quad = \sum_{j=1}^{p} (-1)^j \{g_2, \cdots, \hat{g}_k, \cdots, g_q, f_j\}^P \{g_k, f_1 \cdots, \hat{f}_j, \cdots, f_p\}^P \\
&\qquad + (-1)^{q-k} \{g_2, \cdots, g_q\}^P \{f_1, \cdots, f_p\}^P .
\end{aligned}
$$

If P and Q satisfy the above condition (25), the terms of the form $f_j\{\cdots\}$ cancel out.

Finally, in the same way, we obtain the following condition on Q.

$$
\begin{aligned}
&\{\{f_1, , , f_q\}^Q, g_2, , , g_q\}^Q \\
&\quad = \sum_{j=1}^{q} \{f_1, , , f_{j-1}, \{f_j, g_2, , , g_q\}^Q, f_{j+1}, , , f_q\}^Q \\
&\qquad + \sum_{j=1}^{q} \sum_{k=2}^{q} (-1)^{k-1} \{f_j, g_2, , , \hat{g}_k, , , g_q\}^P \{f_1, , , f_{j-1}, g_k, f_{j+1}, , , f_q\}^Q. \\
&\qquad + p\{f_1, , , f_q\}^Q \{g_2, , , g_q\}^P .
\end{aligned}
$$

This is expressed as

$$
J^Q Q(g_2, \cdots, g_q; f_1, \cdots, f_q) = \sum_{k=2}^{q} (-1)^k (P \vdash Q)(g_2, \cdots, \hat{g}_k, \cdots, g_q; g_k, f_1, \cdots, f_q) . \tag{26}
$$

This is nothing but (4) of Proposition 1.3.

We have shown that (1)–(4) of Proposition 1.3 are necessary conditions for $\{\cdots\}^{Q+1\wedge P}$ satisfying the Fundamental Identity. Conversely, from our computation, we can easily see that if the relations (23),(24),(25) and (26) on the brackets $\{\cdots\}^P$ and $\{\cdots\}^Q$ hold, the Fundamental Identity of the bracket $\{\cdots\} = \{\cdots\}^{Q+1\wedge P}$ is true. Thus the relations (23),(24),(25) and (26) together are equivalent to the Fundamental Identity for $\{\cdots\}^{Q+1\wedge P}$. In this way, we obtained

PROPOSITION 4.1. *Let $\mathcal{A} = Q + 1 \wedge P$ be a Nambu-Jacobi bracket degree $Q = q = p + 1 \geq 3$. Then we have the following identities*

(1) $J^P P = 0$,

(2) $J^P Q = 0$,

(3) $J^Q P(d\boldsymbol{f}_p; \cdots) + (-1)^{p+1} Q(dP(d\boldsymbol{f}_p), \cdots)$
$\quad + \sum_{i=1}^{p} (-1)^i (P \vdash P)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0$,

(4) $J^Q Q(d\boldsymbol{f}_p; \cdots) + \sum_{i=1}^{p} (-1)^i (P \vdash Q)(df_1 \cdots \hat{df_i} \cdots df_p; df_i, \cdots) = 0$.

These together are also sufficient for the bracket $\mathcal{A} = Q + 1 \wedge P$ to satisfy the Fundamental Identity.

REMARK 4.1. When $p = 1$, if we interpret the formulas properly, the relation obtained from the above is expressed as

$$[P, Q] = 0, \quad [Q, Q] = -2P \wedge Q, \tag{27}$$

which is the usual definition of Jacobi structure.

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