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Generalized Hyperfunctions and Algebra of Megafunctions

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Abstract. Sheaves of spaces of generalized hyperfunctions $\mathcal{BG}$ and algebras of megafunctions $\mathcal{MG}$ are introduced. The first one is a flabby sheaf. Moreover, there exist injective sheaf homomorphisms $\mathcal{G} \to \mathcal{BG}$ and $\mathcal{BG} \to \mathcal{MG}$, where $\mathcal{G}$ is the algebra of Colombeau generalized functions.

1. Introduction

Algebras of generalized functions are the framework for the study of linear problems with singularities and, more important, for non-linear problems where the distribution theory can not be used. We refer to several monographs and recent papers where Colombeau-type algebras are used and developed [3]– [5], [8]–[10], [15]. The aim of this note is to give a (natural) extension of Sato's hyperfunctions to generalized hyperfunctions, elements of $\mathcal{BG}$, containing Colombeau generalized function algebra $\mathcal{G}$ as a subspace. Since $\mathcal{BG}$ is not an algebra, we introduce the algebra of megafunctions $\mathcal{MG}$, containing $\mathcal{BG}$ as a subspace as well as algebra $\mathcal{G}$ as a subalgebra. The sheaf $\mathbf{R} \supset \omega \to \mathcal{BG}(\omega)$ is flabby and $\omega \to \mathcal{MG}(\omega)$ is supple. This gives possibilities for new microlocalizations involving all microlocalizations of embedded Schwartz distributions, ultradistributions and hyperfunctions as elements of generalized function algebras.

Sheaves of algebras of generalized and holomorphic generalized functions $\mathcal{G}_H(\Omega)$, where Ω is open in $\mathbf{C}$, is introduced by Colombeau [4], [5]. We refer to [7], [6], [1] and [2] for the properties of generalized holomorphic functions. In order to illustrate the specific properties related to holomorphic generalized functions, we note that it is known that for an open Ω an $f \in \mathcal{G}_H(\Omega)$ is equal to zero if it is equal to zero in an open set of Ω. Moreover, it is equal to zero if its value, in the sense of generalized complex numbers, at any point of Ω is equal to zero. This does not hold for generalized functions of $\mathcal{G}$ which are equal zero if and only if it holds in every generalized point. Also, if all the derivatives of $f \in \mathcal{G}_H(\Omega)$ are equal to zero, at a point $x \in \Omega$, it does not follow that $f = 0$ in $f \in \mathcal{G}(\Omega)$. (Being equal to zero means belonging to an ideal.)

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Let Ω be open in $\mathbf{C}$ and contain an open set $\omega \in \mathbf{R}$ as a closed subset. The space of generalized hyperfunctions is defined as $\mathcal{BG}(\omega) = \mathcal{G}_H(\Omega \setminus \omega)/\mathcal{G}_H(\Omega)$ while the algebra of megafunctions is defined as $\mathcal{MG}(\omega) = \mathcal{G}(\Omega \setminus \omega)/\mathcal{G}(\Omega)$. The analysis of corresponding sheaves is the subject of the paper.

1.1. Colombeau type algebras. Recall, if E is a vector space on $\mathbf{C}$ (or $\mathbf{R}$) with an increasing sequence of seminorms $\mu_n,\ n \in \mathbf{N}$, then the set of moderate nets of $\mathcal{E}_M(E)$, respectively of null nets of $\mathcal{N}(E)$, consists of nets $(R_\varepsilon)_{\varepsilon\in(0,1)} \in E^{(0,1)}$ with the properties

$$(\forall n \in \mathbf{N})\ (\exists a \in \mathbf{R})\ (\mu_n(R_\varepsilon) = O(\varepsilon^a))\,,$$

$$\text{respectively,}\ \ (\forall n \in \mathbf{N})\ (\forall b \in \mathbf{R})\ (\mu_n(R_\varepsilon) = O(\varepsilon^b))$$

(O is the Landau symbol). If $E = \mathbf{C}$ (or $E = \mathbf{R}$) and the seminorms are equal to the absolute value, then the corresponding spaces are $\mathcal{E}_M$ and $\mathcal{N}$. Moreover, they are algebras, $\mathcal{N}$ is an ideal in $\mathcal{E}$ and, as a quotient, one obtains Colombeau algebra of generalized complex numbers $\bar{\mathbf{C}} = \mathcal{E}_M/\mathcal{N}$ (or $\bar{\mathbf{R}}$). It is a ring, not a field.

Let ω be an open set in $\mathbf{R}^n$. If $E = \mathcal{E}(\omega)$ is the Schwartz space with the (usual) sequence of seminorms $\sup\{|\phi^{(\alpha)}(x)|; \alpha \le \nu,\ x \in K_\nu\},\ \nu \in \mathbf{N}_0$, where $(K_\nu)_\nu$ is an increasing sequence of compact sets exhausting ω, then the above definition gives algebras $\mathcal{E}_M(\omega), \mathcal{N}(\omega)$ (the latter is an ideal) and as a quotient, the simplified Colombeau algebra $\mathcal{G}(\omega)$. The embedding of Schwartz distribution space $\mathcal{E}'(\omega)$ is realized through the sheaf homomorphism

$$\mathcal{E}'(\omega) \ni f \mapsto [f * \phi_\varepsilon|_\omega] \in \mathcal{G}(\omega)\,,$$

where a fixed net of mollifiers $(\phi_\varepsilon)_\varepsilon$ is defined by $\phi_\varepsilon = \varepsilon^{-n}\phi(\cdot/\varepsilon),\ \ \varepsilon < 1,\ \phi \in \mathcal{S}(\mathbf{R}^n),\ \ \int \phi(t)dt = 1,\ \ \int t^m\phi(t)dt = 0, m \in \mathbf{N}_0^n, |m| > 0$. ($t^m = t_1^{m_1}\cdots t_n^{m_n}$ and $|m| = m_1 + \cdots + m_n$.) The extended sheaf homomorphism gives the embedding of $\mathcal{D}'(\omega)$ into $\mathcal{G}(\omega)$.

We will consider the one-dimensional case. The n-dimensional version of results is more complicated as in the classical hyperfunction theory.

2. Colombeau holomorphic generalized functions

Let Ω be an open set of $\mathbf{R}^2 = \mathbf{C}$. We will use notation (x, y) and $x + \sqrt{-1}y$ for the points of $\mathbf{R}^2$. Following [7], it is said that $G \in \mathcal{G}(\Omega)$ is a holomorphic generalized function if it satisfies $\bar{\partial}G = 0$ i.e. there is a representative $(G_\varepsilon)_\varepsilon$ of G such that $(\bar{\partial}G_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega)$:

$$(\forall K \subset\subset \Omega)\ (\forall a > 0)\ (\sup_{z\in K} |\bar{\partial}G_\varepsilon(z)| = o(\varepsilon^a))\,.$$

Note, by [9], it is not needed to assume the above estimates for the derivatives $\partial^\alpha\bar{\partial}G_\varepsilon,\ \varepsilon < 1$ ($|\alpha| > 0$).

Holomorphic generalized functions constitute a subalgebra of $\mathcal{G}(\Omega)$. It is denoted by $\mathcal{G}_H(\Omega)$. Every holomorphic function H defines a generalized holomorphic function with the constant net $(H)_\varepsilon$ as a representative.

EXAMPLE 1. *The next example will illustrate the boundary value representation as well as the comments quoted in the introduction. Let* $f_\varepsilon(x) = \frac{t}{\cosh(t/\varepsilon)}$, $t \in \boldsymbol{R}$, $\varepsilon \in (0,1)$. *This net determines a generalized function* $f \in \mathcal{G}(\boldsymbol{R})$ *such that for every* $t \in \boldsymbol{R}$, $f(t) = 0$ *in the sense of generalized real numbers* $\bar{\boldsymbol{R}}$ *but* $f \neq 0$ *in* $\mathcal{G}(\boldsymbol{R})$. *Moreover,* $f^{(\alpha)}(t) = 0$ *in* $\overline{\boldsymbol{R}}$, *for every* $t \neq 0$.

Now define

$$f_\varepsilon(z) = \frac{1}{2\pi\sqrt{-1}} \int_{-\infty}^{\infty} \frac{t}{(t-z)\cosh(t/\varepsilon)} dt\,, \quad z \in \boldsymbol{C}_+\,, \quad \varepsilon < 1\,.$$

This net defines an element of $\mathcal{G}_H(\boldsymbol{C}_+)$ *and it is different from zero in any point of* $\boldsymbol{C}_+$. *Note that for every* $\varepsilon \in (0,1)$,

$$\lim_{y\to 0+} F_\varepsilon(x+iy) = \frac{1}{2} f_\varepsilon(x) - \frac{1}{2\pi\sqrt{-1}} \mathcal{H}(f_\varepsilon)(x)\,, \quad x \in \boldsymbol{R}\,,$$

where $\mathcal{H}$ *is the Hilbert transform, and*

$$\lim_{y\to 0+} F_\varepsilon(x+\sqrt{-1}y) - F_\varepsilon(x-\sqrt{-1}y) = f_\varepsilon(x)\,, \quad x \in \boldsymbol{R}\,.$$

By $f_\varepsilon(z) = \frac{z}{\cosh(z/\varepsilon)}$, $\varepsilon \in (0,1)$ *is defined a moderate net in* $\boldsymbol{C} \setminus \{z; \Re z = 0\}$ *and this net determines the zero generalized function, there. For every fixed* ε, f_ε *is real analytic, but there does not exist a common open set* V *around* $\boldsymbol{R}$ *and* ε_0 *such that* f_ε *are analytic in* V *for* $\varepsilon < \varepsilon_0$.

We collect some results for holomorphic generalized functions. Only part v) is cruicual for our paper.

THEOREM 1. *Let* $G \in \mathcal{G}_H(\Omega)$.

i) *There exists a representative* $(G_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega)$ *with* $G_\varepsilon \in \mathcal{O}(\Omega)$ *(the space of holomorphic functions on* Ω*) for every* $\varepsilon < 1$.

ii) *If* $G = 0$ *in an open subset of* Ω, *then it is equal to zero in* Ω $((G_\varepsilon)_\varepsilon \in \mathcal{N}(\Omega))$.

iii) *If for some* $z_0 \in \Omega$, *there exists* $\eta > 0$ *such that for every* $a > 0$ *there exist* $C > 0$ *and* $\varepsilon_0 \in (0,1)$ *such that*

$$|G_\varepsilon^{(\alpha)}(z_0)| \leq C\eta^\alpha \alpha! \varepsilon^a\,, \quad \varepsilon < \varepsilon_0\,, \quad \alpha \in \boldsymbol{N}_0\,,$$

then $G = 0$.

iv) *If* $G(x) = 0$ *for every* $x \in \Omega$, *then* $F = 0$.

v) *Fundamental lemma.*

Let Ω_1 and Ω_2 be open sets in $\mathbf{C}$ such that $\Omega = \Omega_1 \cap \Omega_2 \neq \emptyset$ and let $F \in \mathcal{G}_H(\Omega)$. Then there exist $F_1 \in \mathcal{G}_H(\Omega_1)$ and $F_2 \in \mathcal{G}_H(\Omega_2)$ such that

$$F = F_1|_\Omega - F_2|_\Omega \, .$$

PROOF. Parts i), ii) and iii) are proved in [18]. Note that i) is also proved by Oberguggenberger (private communication) and ii) is proved by Colombeau and Galé [7] (the proof in [18] is a simplification). Part iv) is proved recently by Khelif and Scarpalezos [13].

v) The proof goes in the same way as the corresponding proof for holomorphic functions [12], [14]. First consider the case when F is extendable out of Ω i.e. there exist $\tilde{\Omega} \supset\supset \Omega$ and $\tilde{F} \in \mathcal{G}_H(\tilde{\Omega})$ such that $\tilde{F}_{|\Omega} = F$. Also, assume that the boundaries $\gamma_1 = \Omega_2 \cap \partial\Omega$ and $\gamma_2 = \Omega_1 \cap \partial\Omega$ are piecewise smooth, oriented in opposite directions and that they form disjoint closed curves $\gamma^s,\ s = 1, \cdots, r$ forming the boundaries of simply connected domains $\Omega^s,\ s = 1, \cdots, r$ so that

$$\Omega = \bigcup_{s=1}^{r} \Omega^s \, , \quad \gamma_1 = \bigcup_{s=1}^{r} \gamma_1^s \, , \quad \gamma_2 = \bigcup_{s=1}^{r} \gamma_2^s$$

where $\gamma_1^s = \gamma^s \cap \Omega_2,\ \gamma_2^s = \gamma^s \cap \Omega_1,\ s = 1, \cdots, r$. Then, F_ε, $\varepsilon \in (0, 1)$, are holomorphic on γ^s and applying the Cauchy formula

$$F_{j,\varepsilon}(z) = \frac{1}{2\pi\sqrt{-1}} \sum_{s=1}^{r} \int_{\gamma_j^s} \frac{F_\varepsilon(\zeta)}{\zeta - z} d\zeta \, , \quad z \in \mathbf{C} \setminus \gamma_j^s \, , \quad j = 1, 2 \, , \quad s = 1, \cdots, r \, , \quad \varepsilon < 1 \, ,$$

we obtain holomorphic functions:

$$F_{i,\varepsilon} \text{ in } \Omega_i \, , \quad i = 1, 2, \text{ such that } \quad F_\varepsilon = F_{1,\varepsilon}|_\Omega - F_{2,\varepsilon}|_\Omega \, , \quad \varepsilon < 1 \, .$$

The given integral representation and the simple estimates over compact sets K of Ω ($K \subset\subset \Omega$) imply that $(F_{i,\varepsilon})_\varepsilon$ in $\mathcal{E}_M(\Omega_i),\ i = 1, 2$. In the general case, we take sequences of domains $(\Omega_{j,k})_k,\ \Omega_{j,k} \subset\subset \Omega_{j,k+1},\ k \in \mathbf{N}$, which exhaust $\Omega_j,\ j = 1, 2$, such that for every $k \in \mathbf{N}$ $\Omega_{1,k} \cap \Omega_{2,k} = \Omega_k$ has a piecewise smooth boundary and that the pair $\Omega_1 \cup \Omega_2$ and $\Omega_{1,k} \cup \Omega_{2,k}$ forms a Runge pair.

Applying the first part of the proof, we have

$$F_\varepsilon|_{\Omega_k} = F_{1,k,\varepsilon}|_{\Omega_k} - F_{2,k,\varepsilon}|_{\Omega_k} \, , \quad \varepsilon < 1 \, , \quad k \in \mathbf{N} \, .$$

This implies

$$F_{1,k+1,\varepsilon} - F_{1,k,\varepsilon} = F_{2,k+1,\varepsilon} - F_{2,k,\varepsilon} \quad \text{on} \quad \Omega_{1,k} \cap \Omega_{2,k} \, , \quad \varepsilon < 1 \, , \ k \in \mathbf{N} \, ,$$

and we obtain, for every $k \in \mathbf{N}$, a net of holomorphic functions $(G_{k,\varepsilon})_\varepsilon$ on $\Omega_{1,k} \cup \Omega_{2,k}$ with the property $(G_{k,\varepsilon})_\varepsilon \in \mathcal{E}_M(\Omega_{1,k} \cup \Omega_{2,k})$. By Runge's theorem, there exists $(H_{k,\varepsilon})_\varepsilon \in (\mathcal{O}(\Omega_1 \cup \Omega_2))^{(0,1)}$

$$|G_{k,\varepsilon}^{(\alpha)}(z) - H_{k,\varepsilon}^{(\alpha)}(z)| < \frac{1}{2^k} \, , \quad z \in \Omega_{1,k-1} \cup \Omega_{2,k-1} \, , \quad \varepsilon < 1 \, . \tag{1}$$

This condition implies that for every ε, α and $z \in \Omega_{1,k-1} \cup \Omega_{2,k-1}$ the series $\sum_{j=k}^{\infty} |G^{(\alpha)}_{j,\varepsilon}(z) - H^{(\alpha)}_{j,\varepsilon}(z)|$ is convergent. Moreover, $(H_{k,\varepsilon})_\varepsilon \in \mathcal{E}_M(\Omega_1 \cup \Omega_2)$. We will show this.

Let us prove (1). Since the closure of $\Omega_{1,k-1} \cup \Omega_{2,k-1}$ in $\Omega_{1,k} \cup \Omega_{2,k}$ is compact, there exist a finite number of open balls $L_\nu,\ \nu = 1, \cdots, s$ so that $\bar{L}_\nu \subset \Omega_{1,k} \cup \Omega_{2,k}$ and $\Omega_{1,k-1} \cup \Omega_{2,k-1} \subset \bigcup_{\nu=1}^{s} L_\nu$. Taking connected components of this finite covering and their piecewise smooth boundaries $\Gamma_\mu,\ \mu = 1, \cdots, p$, (parts of circles), one obtains p simply connected domains which cover $\Omega_{1,k-1} \cup \Omega_{2,k-1}$. Put

$$h_{k,\varepsilon}(z) = \sum_{\mu=1}^{p} \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_\mu} \frac{G_{k,\varepsilon}(\zeta)d\zeta}{\zeta - z}, \quad z \in \Omega_{1,k-1} \cup \Omega_{2,k-1},$$

$$S_{k,\varepsilon}(z) = \sum_{\mu=1}^{p} \frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^{N_\mu(\varepsilon)} \frac{G_{k,\varepsilon}(\zeta_{\mu,i})\Delta\zeta_{\mu,i}}{\zeta_{\mu,i} - z}, \quad z \in \Omega_{1,k-1} \cup \Omega_{2,k-1}$$

where $S_{k,\varepsilon}$ are the corresponding Riemann sums. Consider $(\zeta_{\mu,i} - z)^{-1}$, $z \in \Omega_{1,k-1} \cup \Omega_{2,k-1}$. The Runge theorm (cf. the proof given in [11], p. 37, for example) implies that these functions can be approximated by polynomials with the prescibed precision. Noting this and that

$$\lim_{N_\mu(\varepsilon)\to\infty} \sum_{i=1}^{N_\mu(\varepsilon)} |\Delta\zeta_{\mu,i}| = |\Gamma_\mu| \text{ (the measure of } \Gamma_\mu),$$

we can prove that $H_{k,\varepsilon}$, of the form

$$H_{k,\varepsilon}(z) = \sum_{\mu=1}^{p} \sum_{i=1}^{N_\mu(\varepsilon)} G_{k,\varepsilon}(\zeta_{\mu,i})\Delta\zeta_{\mu,i} \sum_{j=1}^{N_{\mu,i}} a^i_j (z - \theta_{\mu,i})^j,$$

$$z \in \Omega_1 \cup \Omega_2, \quad a^i_j \in \mathbf{C}, \mu \le p, \quad i \le N_\mu, \quad j \le N_{\mu,i}, \quad \varepsilon < 1, \quad k \in \mathbf{N},$$

are elements of $\mathcal{E}_M(\Omega_1 \cup \Omega_2)$ such that (1) holds.

Moreover, we can choose the Riemann sums so that (1) holds for $\alpha \le k$.

Now one can prove that

$$\begin{aligned} F_{j,\varepsilon} &= F_{j,1,\varepsilon} + \sum_{k=1}^{\infty}(G_{k,\varepsilon} - H_{k,\varepsilon}) \\ &= F_{j,N,\varepsilon} + \sum_{n=N}^{\infty}(G_{n,\varepsilon} - H_{n,\varepsilon}) - \sum_{n=1}^{N-1} H_{n,\varepsilon}, \quad j = 1, 2,\ \varepsilon \in (0, 1), \end{aligned}$$

are nets of holomorphic functions in $\Omega_j,\ j = 1, 2$, determining generalized holomorphic functions F_1 and F_2 such that the assertion of the theorem holds.

3. Space $\mathcal{BG}(\omega)$

We follow the definition of classical Sato's hyperfunctions. We refer to [12] and [14] for an elementary introduction which is enough for our presentation and to [19] for the extension of the theory and the analysis of pseudo-differential operators.

Let ω be an open set in $\mathbf{R}$ and $\Omega \subset \mathbf{C}$ be an open set containing ω as a closed subset. The space of generalized hyperfunctions on ω is defined by

$$\mathcal{BG}(\omega) = \bigcup_{\Omega \supset \omega} \frac{\mathcal{G}_H(\Omega \setminus \omega)}{\mathcal{G}_H(\Omega)} .$$

Fundamental lemma implies that for any fixed Ω (containing ω as a closed set),

$$\mathcal{BG}(\omega) = \frac{\mathcal{G}_H(\Omega \setminus \omega)}{\mathcal{G}_H(\Omega)} .$$

Let $f = [(F_\varepsilon)_\varepsilon] \in \mathcal{GB}(\omega)$, where the representative $(F_\varepsilon)_\varepsilon$ constitutes of holomorphic functions in $\Omega \setminus \omega$. We will use the notation

$$f = [(F_{\varepsilon+})_\varepsilon] - [(F_{\varepsilon-})_\varepsilon] , \quad f = F_+(x + \sqrt{-1}\,0) - F_-(x - \sqrt{-1}\,0) = F_+ - F_- ,$$

where $F_+ = [(F_{\varepsilon+})_\varepsilon]$ and $F_- = [(F_{\varepsilon-})_\varepsilon]$ are holomorphic generalized functions in Ω_+ and Ω_-, respectively.

The next theorem is also a consequence of Fundamental lemma.

THEOREM 2. *$\boldsymbol{R} \supset \omega \to \mathcal{BG}(\omega)$, is a flabby sheaf.*

PROOF. The flabbiness is a direct consequence of Fundamental lemma.

The sheaf property concerning the support is clear. We have to prove that for an open covering ω_λ and given $f_\lambda \in \mathcal{BG}(\omega_\lambda)$, $\lambda \in \Lambda$, with the property $f_\lambda = f_\mu$ on $\omega_\lambda \cap \omega_\mu$, there exists an $f \in \mathcal{BG}(\omega)$ so that $f = f_\lambda$ on ω_λ, $\lambda \in \Lambda$. In fact, with the corresponding representatives in $\mathcal{O}(\Omega_\lambda \setminus \omega_\lambda)$, $\lambda \in \Lambda$, if $F_{\varepsilon,\mu} - F_{\varepsilon,\lambda} = F_{\varepsilon,\lambda,\mu} + r_{\varepsilon,\lambda,\mu}$, $\varepsilon < 1$, where $(F_{\varepsilon,\lambda,\mu}) \in (\mathcal{O}(\Omega_\lambda \cap \Omega_\mu))^{(0,1)} \cap \mathcal{E}_M(\Omega_\lambda \cap \Omega_\mu)$, $(r_{\varepsilon,\lambda,\mu})_\varepsilon \in \mathcal{N}(\Omega_\lambda \cap \Omega_\mu)$, $\lambda, \mu \in \Lambda$, then there exists an $(F_\varepsilon)_\varepsilon \in (\mathcal{O}(\Omega \setminus \omega))^{(0,1)} \cap \mathcal{E}_M(\Omega \setminus \omega)$ such that $F_\varepsilon - F_{\varepsilon,\lambda} \in \mathcal{O}(\Omega_\lambda)$, $\varepsilon < 1$, for every $\lambda \in \Lambda$. We can assume that $\omega = \bigcup_{\lambda=1}^\infty \omega_\lambda$, $\Omega = \bigcup_{\lambda=1}^\infty \Omega_\lambda$ and that the coverings are locally finite.

In the case of a finite covering, the assertion is a consequence of Fundamental lemma. (In fact, for $n > 2$ it is not simple but we skip this.) In the general case, let $\Omega = \bigcup_{n=1}^\infty \Theta_n$, $\Theta_n \subset\subset \Theta_{n+1}$ so that (Ω, Θ_n) makes a Runge pair for every n. Since for every fixed n we can apply the result for a finite covering, we construct the corresponding $F_{\varepsilon,n}$, $\varepsilon < 1$, $(F_{\varepsilon,n})_\varepsilon \in (\mathcal{O}(\Omega_n \setminus \omega))^{(0,1)} \cap \mathcal{E}_M(\Omega_n \setminus \omega)$, so that $F_{\varepsilon,\lambda,n} = F_{\varepsilon,n} - F_{\varepsilon,\lambda} \in \mathcal{O}(\Omega_\lambda \cap \Theta_n)$, $\varepsilon < 1$, $n \in \mathbf{N}$ ($F_{\varepsilon,\lambda,n} = 0$ if $\Theta_n \cap \Omega_\lambda = \emptyset$). Consider the family

$$F_{\varepsilon,\lambda,n+1} - F_{\varepsilon,\lambda,n} = F_{\varepsilon,n+1} - F_{\varepsilon,n} \quad \text{on} \quad \Theta_n \cap \Omega_\lambda , \quad \varepsilon < 1 , \quad \lambda \in \mathbf{N} .$$

Net of holomorphic functions $G_{\varepsilon,n}$, $\varepsilon < 1$ on Θ_n, defined by $G_{\varepsilon,n} = F_{\varepsilon,\lambda,n+1} - F_{\varepsilon,\lambda,n}$, $\lambda \in \Lambda$ on $\Theta_n \cap \Omega_\lambda$, $\varepsilon < 1$ is in $\mathcal{E}_M(\Theta_n)$. By the same explanation as in the proof of Fundamental lemma, there exists $(H_{\varepsilon,n})_\varepsilon \in (\mathcal{O}(\Omega))^{(0,1)} \cap \mathcal{E}_M(\Omega)$ so that for every $n \geq 2$ and $\alpha \leq n$,

$$|G^{(\alpha)}_{\varepsilon,n}(z) - H^{(\alpha)}_{\varepsilon,n}(z)| \leq \frac{1}{2^n}, \quad \varepsilon < 1, \quad z \in \Theta_{n-1}.$$

Define on $\Omega_\lambda \setminus \omega_\lambda$:

$$R_{\varepsilon,\lambda} = F_{\varepsilon,\lambda,1} + \sum_{n=1}^{\infty}(F_{\varepsilon,\lambda,n+1} - F_{\varepsilon,\lambda,n} - H_{\varepsilon,n}), \quad \varepsilon < 1.$$

Since $R_{\varepsilon,\lambda} = F_{\varepsilon,\lambda,N} + \sum_{n=N}^{\infty}(G_{\varepsilon,n} - H_{\varepsilon,n}) - \sum_{n=1}^{N-1} H_{\varepsilon,n}$, $\varepsilon < 1$, it follows that $(R_{\varepsilon,\lambda})_\varepsilon \in (\mathcal{O}(\Omega_\lambda \setminus \omega_\lambda))^{(0,1)} \cap \mathcal{E}_M(\Omega_\lambda \setminus \omega_\lambda)$ and that, on $\Omega_\lambda \cap \Omega_\mu$,

$$R_{\varepsilon,\lambda} - R_{\varepsilon,\mu} = F_{\varepsilon,\lambda,\mu}, \quad \varepsilon < 1.$$

Now, this property and $F_{\varepsilon,\lambda} - F_{\varepsilon,\mu} = F_{\varepsilon,\lambda,\mu}$, $\varepsilon < 1$ on $\Omega_\lambda \cap \Omega_\mu$, imply

$$F_{\varepsilon,\lambda} - R_{\varepsilon,\lambda} = F_{\varepsilon,\mu} - R_{\varepsilon,\mu}, \quad \varepsilon < 1, \text{ on } \Omega_\lambda \cap \Omega_\mu,$$

for every $\lambda, \mu \in \mathbf{N}$ and in this way we construct $(F_\varepsilon)_\varepsilon \in \mathcal{E}_M(\Omega \setminus \omega)$ so that $F_\varepsilon = F_{\varepsilon,\lambda} - R_{\varepsilon,\lambda}$ on $(\Omega_\lambda \setminus \omega_\lambda)$. This completes the proof.

3.1. Multiplication in $\mathcal{BG}$. As in the classical theory of hyperfunctions [12], we define the micro-analyticity at $x - \sqrt{-1}dx\infty$, $x \in \omega$.

Let $f \in \mathcal{BG}(\omega)$ have a representative determined by holomorphic functions F_ε, $\varepsilon \in (0,1)$ in $\Omega \setminus \omega$, such that the restrictions on Ω_+ and Ω_-, denoted by $F_{+,\varepsilon}$ and $F_{-,\varepsilon}$, $\varepsilon \in (0,1)$ respectively, have the property that $F_{+,\varepsilon}$ and $F_{-,\varepsilon}$ can be extended as analytic functions, to Ω_+, for every $\varepsilon \in (0,1)$. Then we say that f is micro-analytic at $x - \sqrt{-1}dx\infty$, $x \in \omega$. We define the micro-analyticity of f at $x + \sqrt{-1}dx\infty$, $x \in \omega$ in the similar way.

Then it is said that $x - \sqrt{-1}dx\infty$ (resp. $x + \sqrt{-1}dx\infty$) belongs to the singular spectrum of f, SSf, if f is not micro-analytic at $x - \sqrt{-1}dx\infty$ (resp. $x + \sqrt{-1}dx\infty$).

As in the hyperfunction theory, we define the product of generalized hyperfunctions f and g only in the cases when their singular spectrums are in a "good position" ($\mathcal{BG}(\omega)$ is not an algebra). So

$$\text{if} \quad f = F_+ - 0, \quad g = G_+ - 0, \quad \text{then} \quad fg = F_+(x + \sqrt{-1}0)G_+(x + \sqrt{-1}0) - 0,$$

$$\text{if} \quad f = 0 - F_-, \quad g = 0 - G_-, \quad \text{then} \quad fg = 0 - F_-(x - \sqrt{-1}0)G_-(x - \sqrt{-1}0).$$

A real analytic function ϕ is embedded into the space of hyperfunctions $\mathcal{B}(\omega)$ as $\phi = \Phi(x + \sqrt{-1}0) - 0 = \Phi_+$, where Φ is its analytic extension in a neighborhood of ω. Clearly, $\phi(x) = \Phi_+(x) = \Phi_-(x)$, $x \in \omega$. With this notation (and embedded ϕ in $\mathcal{BG}$ as a constant net), for an arbitrary $f = F_+ - F_- \in \mathcal{BG}$, we have

$$f\phi = F_+\Phi_+ - F_-\Phi_+.$$

Note, if Ψ is analytic in a neighborhood of ω, then, in the space of hyperfunctions $\mathcal{B}$,

$$\phi = (\Phi(x + \sqrt{-1}0) + \Psi(x + \sqrt{-1}0)) - \Psi(x - \sqrt{-1}0) .$$

For an arbitrary $f = F_+ - F_- \in \mathcal{BG}$ and an analytic ϕ the product in $\mathcal{BG}$ is:

$$\begin{aligned} f\phi &= (F_+ - F_-)(\Phi_+ + \Psi_+ - \Psi_-) \\ &= F_+(\Phi_+ + \Psi_+ - \Psi_-) - F_-(\Phi_+ + \Psi_+ - \Psi_-) = F_+\Phi_+ - F_-\Phi_+ , \end{aligned}$$

where the cancellations $F_+\Psi_+ - F_+\Psi_- = 0$, $F_-\Psi_+ - F_-\Psi_- = 0$, are done.

3.2. Embedding I. We will give a canonical embedding of the sheaf of Colombeau generalized functions into the sheaf of generalized hyperfunctions.

First, define: $f = [(F_\varepsilon)_\varepsilon] \in \mathcal{BG}(\omega)$ (F_ε are holomorphic functions in $\Omega \setminus \omega$) is an element of $\mathcal{G}(\omega)$ if for every $\omega_0 \subset\subset \omega$ there exists a net of smooth functions $(g_\varepsilon)_\varepsilon \in \mathcal{E}_M(\omega_0)$ such that for every $\varepsilon \in (0, 1)$ and every $\alpha \in \mathbf{N}_0$,

$$\lim_{y \to 0^+} (F_\varepsilon^{(\alpha)}(x + \sqrt{-1}y) - F_\varepsilon^{(\alpha)}(x - \sqrt{-1}y)) = g_\varepsilon^{(\alpha)}(x) \quad \text{uniformly for} \quad x \in \omega_0 .$$

Now, if $g = [(g_\varepsilon)_\varepsilon] \in \mathcal{G}(\omega)$ is compactly supported, $\operatorname{supp} g \subset\subset \omega$, then the corresponding generalized hyperfunction is defined by a net of holomorphic functions in $\Omega \setminus \omega$:

$$F_\varepsilon(x + \sqrt{-1}y) = \frac{1}{2\pi\sqrt{-1}} \int_\omega \frac{g_\varepsilon(t)}{t - (x + \sqrt{-1}y)} dt , \quad x + \sqrt{-1}y \in \Omega \setminus \omega,\ \varepsilon \in (0, 1) .$$

By Plemelj type theorems, we have that

$$\lim_{y \to 0+} F_\varepsilon(x + \sqrt{-1}y) - F_\varepsilon(x - \sqrt{-1}y) = g_\varepsilon(x) , \quad \text{for every} \quad \varepsilon < 1 .$$

In this way we define the sheaf homomorphism which can be extended in a usual way to the sheaf homomorphism $\mathcal{G}(\omega) \to \mathcal{BG}(\omega),\ \omega \in \mathbf{R}$.

The embedding of distributions and its subspaces is done in a similar way. For example, if $g \in \mathcal{E}'(\omega)$, let $g_\varepsilon = g * \phi_\varepsilon|_\varepsilon$, $\varepsilon \in (0, 1)$, where ϕ_ε, $\varepsilon \in (0, 1)$ is a net of mollifiers. This is a net in $\mathcal{E}_M(\omega)$ and as above we determine the corresponding representative in $\mathcal{BG}(\omega)$. Similarly, but using appropriate classes of mollifiers, we can embed some classes of algebras of generalized ultradistributions [17] and periodic hyperfunctions [20].

4. Algebra $\mathcal{MG}(\omega)$

As earlier, let ω be an open set in $\mathbf{R}$ and $\Omega \subset \mathbf{C}$ be an open set containing ω as a closed subset. The space of generalized megafunctions on ω is defined by

$$\mathcal{MG}(\omega) = \bigcup_{\Omega \supset \omega} \frac{\mathcal{G}(\Omega \setminus \omega)}{\mathcal{G}(\Omega)} .$$

Let $f \in \mathcal{MG}(\omega)$. We continue to use the notation

$$f = F_+(x + \sqrt{-1}\,0) - F_-(x - \sqrt{-1}\,0)\,, \quad f = F_+ - F_-\,,$$

$$F_+ \in \mathcal{G}(\Omega_+)\,, \quad F_- \in \mathcal{G}(\Omega_-)\,.$$

By the partition of unity, as in the case of Colombeau generalized functions, we have:

THEOREM 3. $\boldsymbol{R} \supset \omega \to \mathcal{MG}(\omega)$, *is a sheaf. Moreover, this sheaf is supple.*

PROOF. We will only prove the second part of the theorem (cf. [16] for the suppleness in $\mathcal{G}$).

Let $\tilde{Z}_1$ and $\tilde{Z}_2$ be closed sets in ω and $\tilde{Z}_1 \cup \tilde{Z}_2 = \tilde{Z}$. Let Z_1 and Z_2 be closed sets in Ω such that $Z_1 \cap \omega = \tilde{Z}_1$, $Z_2 \cap \omega = \tilde{Z}_2$, and $Z_1 \cup Z_2 = Z = \tilde{Z} \cap \omega$.

Let $f \in \mathcal{M}(\omega)$, $\operatorname{supp} f = \tilde{Z}$. We have to show that there exist $f_i \in \mathcal{M}(\omega)$, $i = 1, 2$, such that $f = f_1 + f_2$ and $\operatorname{supp} f_i \subset \tilde{Z}_i$, $i = 1, 2$.

Let f be represented by $F = [(F_\varepsilon)_\varepsilon] \in \mathcal{G}(\Omega \setminus \omega)$. Let $\eta_\varepsilon \in C^\infty(\Omega)$, $\varepsilon \in (0, 1)$ such that

$$\eta_\varepsilon(x, y) = \begin{cases} 1 & (x, y) \in Z_1 \\ 0 & (x, y) \in \Omega \setminus (Z_1)_\varepsilon\,, \quad \varepsilon \in (0, 1) \end{cases}$$

$$\text{and} \quad |\eta_\varepsilon^{(\alpha)}(x, y)| \le C_\alpha \varepsilon^{-p_\alpha}\,, \quad (x, y) \in \Omega\,, \quad \varepsilon \in (0, 1)\,,$$

where positive constants C_α and p_α may depend on α.
$((Z_1)_\varepsilon = \{(x, y) \in \Omega;\ \text{dist.}((x, y), Z_1) \le \varepsilon\}.)$

Let F_1 be represented by $(F_\varepsilon \eta_\varepsilon)_\varepsilon$ and F_2 be represented by $(F_\varepsilon(1 - \eta_\varepsilon))_\varepsilon$.

One can easily prove that $f = f_1 + f_2$, $\operatorname{supp} f_i \subset \tilde{Z}_i$, $i = 1, 2$, where, F_i represents f_i, $i = 1, 2$.

4.1. Multiplication in $\mathcal{MG}$. We will multiply elements of $\mathcal{MG}$ as germs: if $f \in \mathcal{G}(\Omega_1 \setminus \omega)/\mathcal{G}(\Omega_1)$ and $g \in \mathcal{G}(\Omega_2 \setminus \omega)/\mathcal{G}(\Omega_2)$, where open sets Ω_1 and Ω_2 contain ω as a closed set, then we will consider their product in $\mathcal{G}(\Omega \setminus \omega)/\mathcal{G}(\Omega)$, where $\Omega = \Omega_1 \cap \Omega_2$.

Let

$$k : \mathbf{C} \to \mathbf{C}\,, \quad x + \sqrt{-1}y \mapsto k(x + \sqrt{-1}y) = x - \sqrt{-1}y\,.$$

Assume $k(\Omega) = \Omega$, $k(\Omega_j) = \Omega_j$, $j = 1, 2$.

Let $F_+ \in \mathcal{G}(\mathbf{C}_+)$. Then

$$k^*(F_+)(x - \sqrt{-1}y) = F_+(k(x - \sqrt{-1}y))\,, \quad x - \sqrt{-1}y \in \mathbf{C}_-\,, \quad (y \text{ is positive})$$

is an element of $\mathcal{G}(\mathbf{C}_-)$ (k^* makes holomorphic antiholomorphic). With the same notation we have that

$$k^*(F_-)(x + \sqrt{-1}y) = F_-(k(x + \sqrt{-1}y))\,, \quad x + \sqrt{-1}y \in \mathbf{C}_+\,,$$

is an element of $\mathcal{G}(\mathbf{C}_+)$ if $F_- \in \mathcal{G}(\mathbf{C}_-)$. The corresponding boundary value notation is

$$k^*(F_+)(x-\sqrt{-1}0) = F_+(x+\sqrt{-1}0) \quad \text{and} \quad k^*(F_-)(x+\sqrt{-1}0) = F_-(x-\sqrt{-1}0)\,.$$

Let $f = F_+ - F_-,\ g = G_+ - G_- \in \mathcal{MG}(\omega)$. Then we define the product $fg \in \mathcal{MG}(\omega)$ by

$$fg = H_+ - H_-\,,$$

where

$$\begin{aligned} &H_+(x+\sqrt{-1}0) \\ &\quad = F_+(x+\sqrt{-1}0)G_+(x+\sqrt{-1}0) + k^*(F_-)(x+\sqrt{-1}0)k^*(G_-)(x+\sqrt{-1}0)\,, \end{aligned}$$

$$\begin{aligned} &H_-(x-\sqrt{-1}0) \\ &\quad = k^*(F_+)(x-\sqrt{-1}0)G_-(x-\sqrt{-1}0) + F_-(x-\sqrt{-1}0)k^*(G_+)(x-\sqrt{-1}0)\,. \end{aligned}$$

Let $g = 0$ in $\mathcal{MG}(\omega)$ be represented as $g = G_+ - G_-$, where G_+ and G_- have the same boundary value $G \in \mathcal{G}(\omega)$, $G(x) = G(x+\sqrt{-1}0) = G(x-\sqrt{-1}0)$. Then $fg = 0$ for any $f = F_+ - F_- \in \mathcal{MG}(\omega)$, because

$$\begin{aligned} &F_+(x+\sqrt{-1}0)G_+(x+\sqrt{-1}0) - k^*(F_+)(x-\sqrt{-1}0)G_-(x-\sqrt{-1}0) \\ &\quad = F_+(x+\sqrt{-1}0)(G_+(x+\sqrt{-1}0) - G_-(x-\sqrt{-1}0)) = 0\,, \end{aligned}$$

$$\begin{aligned} &k^*(F_-)(x+\sqrt{-1}0)k^*(G_-)(x+\sqrt{-1}0) - F_-(x-\sqrt{-1}0)k^*(G_+)(x-\sqrt{-1}0) \\ &\quad = F_-(x-\sqrt{-1}0)(G_-(x-\sqrt{-1}0) - G_+(x+\sqrt{-1}0)) = 0\,. \end{aligned}$$

One can easily prove the next theorem.

THEOREM 4. *$\mathcal{MG}(\omega)$ is a commutative and associative algebra.*

Let $f \in \mathcal{G}(\omega)$. Without loosing the generality assume that f is compactly supported. We determine its boundary value representation as in Section 3.2:

$$f = \lim_{y\to 0+} F_+(x+\sqrt{-1}y) - F_-(x+\sqrt{-1}y) = F_+(x+\sqrt{-1}0) - F_-(x+\sqrt{-1}0)\,,$$

where $F_+ = F|_{\Omega_+}$, $F_- = F|_{\Omega_-}$. Let

$$\tilde{F}_+(x+\sqrt{-1}y) = F_+(x+\sqrt{-1}y) - k^*(F_-)(x+\sqrt{-1}y)\,, \quad y>0\,.$$

It is an element of $\mathcal{G}(\Omega_+)$. With the given notation, we have

$$f = \lim_{y\to 0_+} \tilde{F}_+(x+\sqrt{-1}y) = \tilde{F}_+(x+\sqrt{-1}0) - 0\,.$$

This implies that any $f \in \mathcal{G}(\omega)$ can be written in the form $f = F_+ - 0$, where $F_+ \in \mathcal{G}(\Omega_+)$. The image of f in $\mathcal{MG}(\omega)$ is defined by

$$\mathcal{I}f = F_+ - 0\,.$$

Let $g \in \mathcal{G}(\omega)$ and $\mathcal{I}g = G_+ - 0$. We have in $\mathcal{MG}(\omega)$:

$$\mathcal{I}(fg) = F_+G_+ - 0 = (\mathcal{I}f)(\mathcal{I}g)\,.$$

This implies that $\mathcal{G}(\omega)$ is a subalgebra of $\mathcal{MG}(\omega)$.

4.2. Embedding II. We already defined the embedding of generalized functions algebra $\mathcal{G}$ into $\mathcal{MG}$. Note, if $f \in \mathcal{G}(\Omega)$ satisfies $f|_{\Omega_+} \in \mathcal{G}_H(\Omega_+)$ and $f|_{\Omega_-} \in \mathcal{G}_H(\Omega_-)$, then the corresponding element in $\mathcal{BG}(\omega)$ is equal to zero. This fact postulates the definition of a sheaf homomorphism $\mathcal{BG}(\omega) \to \mathcal{MG}(\omega)$ (of vector spaces) as follows.

Let $f \in \mathcal{BG}(\omega)$, $f = F_+ - F_-$. Then F_+ and F_- are elements of $\mathcal{G}(\Omega_+)$ and $\mathcal{G}(\Omega_-)$, respectively. They determine an element in $\mathcal{MG}(\omega)$ equals zero only in the case when they are the parts of the same Colombeau generalized function but in this case f is equal to zero in $\mathcal{BG}(\omega)$.

With this embedding, the multiplication of embedded generalized hyperfunctions coincides with the embedded product, if the product of generalized hyperfunctions exists (it is explained in [18]).

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