TOKYO J. MATH.
VOL. 28, NO. 2, 2005

On Generalized Circuit of the Collatz Conjecture

Tomoaki MIMURO

Hosei University

(Communicated by K. Shinoda)

Abstract. The Collatz conjecture is that there exists a positive integer n which satisfies $f^n(m) = 1$ for any integer $m \geq 3$, where f is the function on the rational number field defined by $f(m) = m/2$ if the numerator of m is even and $f(m) = (3m+1)/2$ if the numerator of m is odd. Let m be a rational number such that $f^n(m) = m > 1$. Then we show that, if m has some simple sequences, then the total number of positive integer m is finite, by estimating $f(m) - m$.

1. Introduction

We define a function f on the set of the rational numbers by

$$f(m) = \begin{cases} \dfrac{m}{2} & \text{if } m \text{ is even}, \\ \dfrac{3m+1}{2} & \text{if } m \text{ is odd}, \end{cases}$$

where m is a positeive integer. We denote by $f^n = f \circ f^{n-1}$ the n-fold iterate of f, for each positive integer n. The Collatz conjecture is that there exists a positive integer n which satisfies $f^n(m) = 1$ for any integer $m \geq 2$. We call m the "starting-number" and the smallest n the "total-sequence".

This conjecture is equivalent to the next two conditions for every odd integer $m > 1$:

(1) $f^n(m) \neq m$ for any $n \geq 1$. (If $f^n(m) = m$ holds, then we call m "cycle-number".)

(2) m has total-sequence. ($f^n(m)$ dose not diverge.)

We consider the condition, (1) and assume that m is odd, since even number is mapped to an odd number by iterating f. We know only one cycle-number: $m = 1$. We call it the "trivial-cycle".

Let m be a cycle-number. We define the numbers l_i $(i \geq 0)$ and m_i $(i \geq 1)$ by the following rules:

(i) We put $l_0 = -1$ and $m_1 = m$.

(ii) For $i \geq 1$, l_i is the least positive integer such that $f^{l_i+1}(m_i)$ is odd.

(iii) We put $m_{i+1} = f^{l_i+1}(m_i)$.

Received July 12, 2004

If $m = m_1 = m_{k+1}$, then we call k “odd-cycle-sequence”. We write

$$m_1 = \langle l_1 + 1, l_2 + 1, \cdots, l_k + 1 \rangle \quad (l_i \geq 0)$$

We can easily see that

$$m_i = \langle l_i + 1, l_{i+1} + 1, \cdots, l_k + 1, l_1 + 1, \cdots, l_{i-1} + 1 \rangle . \quad (i = 1, \cdots, k)$$

We can write trivial-cycle

$$1 = \langle 2 \rangle .$$

If m is a cycle-number, and $f^n(m) = m$, then we call n a “cycle-sequence”. We can easily see that

$$n = \sum_{i=1}^{k} (l_i + 1) .$$

THEOREM 1.1. *Let $m = \langle l_1 + 1, l_2 + 1, \cdots, l_k + 1 \rangle$ and $l_0 = -1$. Then we have*

$$m = \frac{\sum_{i=1}^{k} 3^{k-i} \cdot 2^{\sum_{j=0}^{i-1}(l_j+1)}}{2^n - 3^k} .$$

Theorem 1.1 was proved in [1].

THEOREM 1.2. *Suppose $m = \langle 1, \cdots, 1, l_k + 1 \rangle$ is a cycle-number, then $m = 1 = \langle 2 \rangle$.*

Theorem 1.2 was proved in [2]. We call $\langle 1, \cdots, 1, l_k + 1 \rangle$ “circuit”.

THEOREM 1.3. *The total number of positive integer of $m_1 = \langle 1, \cdots, 1, l+1, \cdots, l+1 \rangle$ is finite.*

Theorem 1.3 was proved in [3]. This theorem has necessary condition $n < 22033$, where n is cycle-sequence.

The conjectuer has been verified with a computer up to $m = 2^{40} \simeq 1.1 \times 10^{12}$ by N. Yoneda (Stated in [7]).

THEOREM 1.4. *Let m be a positive cycle-number, $min\{m_1, m_2, \cdots m_k\} > 2^{40}$, n be cycle-sequence of m_1. We have*

$$n = 301994a + 17087915b + 85137581c ,$$

where a, b, c are nonnegative integers, $b > 0$, $ac = 0$. In particular, the smallest admissible values for n is 17087915.

Combining Theorem 1.3, Theorem 1.4 and computing check, we have

COROLLARY 1.5. *$l \geq 1, m_1 = \langle 1, \cdots, 1, l + 1, \cdots, l + 1 \rangle$ is not a positive integer.*

We shall prove the next theorem in Section 3.

THEOREM 1.6. *$l \geq 1, m = \langle 1, \cdots, 1, l_1+1, 1, \cdots, 1, l_2+1 \rangle$ is not a positive integer.*

This theorem is a generalization of Theorem 1.2.
We call $m = \langle 1, \cdots, 1, l_1 + 1, 1, \cdots, 1, l_2 + 1 \rangle$ "crossing-circuit".

2. Some lemmas

Let $1/2 < 3^k/2^n < 1$, then we have

$$(n-1)\log_3 2 < k < n \log_3 2$$
$$k \log_2 3 < n < k \log_2 3 + 1\,.$$

LEMMA 2.1. *Let $1/2 < 3^k/2^n < 1$, then*

$$k = \lfloor n \log_3 2 \rfloor = n \log_3 2 + c_1 \quad (-\log_3 2 < c_1 < 0)$$
$$n = \lceil k \log_2 3 \rceil = k \log_2 3 + c_2 \quad (0 < c_2 < 1)$$

$\lfloor x \rfloor$ means the greatest integer not exceeding x, and $\lceil x \rceil$ means the smallest integer exceeding x.

THEOREM 2.2. *Let $\alpha_1, \alpha_2 > 1$ be multiplicatively independent real algebraic numbers, and $D = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}]$. Let A_1, A_2 denote real numbers > 1 such that*

$$\log A_j \geq \max\left\{h(\alpha_j), \frac{\log \alpha_j}{D}, \frac{1}{D}\right\}, \quad (j = 1, 2)\,,$$

where $h(\alpha)$ is absolute logarithmic height of α. Let b_1, b_2 be positive integers, and put

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2\,.$$

Then

$$\log |\Lambda| \geq -32.31 D^4 \left(\max\left\{\log B + 0.18, \frac{10}{D}, \frac{1}{2}\right\}\right)^2 (\log A_1)(\log A_2)\,,$$

where

$$B = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}\,.$$

Theorem 2.2 was proved in [6]. Let $1/2 < 3^k/2^n < 1$, then

$$\Lambda = b_1 \log \alpha_1 - b_2 \log \alpha_2 = n \log 2 - k \log 3\,,$$

by putting $\alpha_1 = 2, \alpha_2 = 3, b_1 = n, b_2 = k$. Using the inequality

$$\frac{|\log x|}{2} < 1 - x\,,$$

for $1/2 < x < 1$, we have

$$\frac{|\Lambda|}{2} = \frac{1}{2}|k\log 3 - n\log 2| = \frac{1}{2}\left|\log\frac{3^k}{2^n}\right| < 1 - \frac{3^k}{2^n}\,.$$

And, it follows from Theorem 2.2 that

$$\log|\Lambda| \geq -32.31H^2\log 3\,.$$

Hence we have

$$\left|1 - \frac{3^k}{2^n}\right| > 2^{-32.31H^2\log_2 3 - 1}$$

where $H = \max\{\log B + 0.18, 10\}$, and

$$B = \frac{n}{\log 3} + k > \frac{n}{\log 3} + (n-1)\log_3 2 = n\frac{1+\log 2}{\log 3} - \log_3 2$$

for Lemma 2.1. We assume $H = 10$. Then $9.82 > \log B$. The inequality

$$9.82 > \log B > \log\left(n\frac{1+\log 2}{\log 3} - \log_3 2\right)$$

says

$$n \leq 11938\,.$$

LEMMA 2.3. *Let* $1/2 < 3^k/2^n < 1$, $n > 11938$. *Then*,

$$2^{-51.2102H^2-1} < \left|1 - \frac{3^k}{2^n}\right|$$

where H *is* $\log B + 0.18$.

We consider the denominator of Theorem 1.1. *Let* $n \geq 3$, *then* $2^n - 3^k \equiv -3^k \equiv -3$ *or* $-1 \not\equiv 1$ *(mod.*8*). It follows that*;

LEMMA 2.4. *The exponential indeterminate equation* $2^n - 3^k = 1$ *has only one positive integral solution* $(n, k) = (2, 1)$.

$(n, k) = (2, 1)$ *means trivial-cycle* $m = 1 = \langle 2\rangle$.

3. Proof of Theorem 1.6

Let $m_1 = \langle 1, \cdots, 1, l_1 + 1, 1, \cdots, 1, l_2 + 1\rangle$ be positive crossing-circuit, $m_2 = f(m_1) = \langle 1, \cdots, 1, l_1 + 1, 1, \cdots, 1, l_2 + 1, 1\rangle$. x_1, x_2 satisfy $f^{x_1+l_1}(m_1) = \langle 1, \cdots, 1, l_2 + 1, 1, \cdots, 1, l_1 + 1\rangle$, $x_2 = k - x_1$. Hence we have $n = x_1 + l_1 + x_2 + l_2$. Let $l_1 \geq 1, l_2 \geq 1$ for corollary 1.5, and without loss of generality, $x_1 \geq x_2$. Let $n \geq 8$. Then, since Theorem 1.1,

$$\langle l_1 + 1, l_2 + 1\rangle = \frac{3 + 2^{n-l_2-1}}{2^n - 9}\,, \quad \langle 1, l_1 + 1, 1, l_2 + 1\rangle = \frac{5\cdot 2^{n-l_2-2} + 45}{2^n - 81}\,,$$

$$\langle 1, l_1+1, l_2+1\rangle = \frac{2^{n-l_2-1}+15}{2^n-27}$$

are not positive integers. Hence we have $x_1 \geq 3$, and

$$m_1 = \frac{3^{k-1}+\cdots+2^{x_1-2}\cdot 3^{k-x_1+1}+2^{x_1-1}\cdot 3^{k-x_1}+2^{x_1+l_1}\cdot 3^{k-x_1-1}+\cdots+2^{x_1+x_2+l_1-1}}{2^n-3^k}$$

$$m_2 = \frac{3^{k-1}+\cdots+2^{x_1-2}\cdot 3^{k-x_1+1}+2^{x_1+l_1-1}\cdot 3^{k-x_1}+2^{x_1+l_1}\cdot 3^{k-x_1-1}+\cdots+2^{x_1+x_2+l_1+l_2-1}}{2^n-3^k}$$

for Theorem 1.1. Since $m_2 > m_1$,

$$m_2 - m_1 = \frac{2^{x_1-1}\{3^{k-x_1}(2^{l_1}-1)+2^{x_2+l_1}(2^{l_2}-1)\}}{2^n-3^k}$$

Now, $m_2 - m_1$ is integral, $2^n - 3^k > 1$ and $(2^n-3^k, 2^{x_1-1}) = 1$. It follows that

$$(2^n-3^k) | \{3^{k-x_1}(2^{l_1}-1)+2^{x_2+l_1}(2^{l_2}-1)\}.$$

We consider the right hand. Since $n = x_1+l_1+x_2+l_2$, $k - x_1 = x_2$, $x_1 \geq x_2$,

$$1 \leq \frac{3^{k-x_1}(2^{l_1}-1)+2^{x_2+l_1}(2^{l_2}-1)}{2^n-3^k} < \frac{2^{-x_1\log_2(4/3)-l_2}+2^{-x_1}}{1-3^k/2^n} < \frac{2^{-x_1\log_2(4/3)}}{1-3^k/2^n}. \qquad (*)$$

First, we assume $3^k/2^n \leq 1/2$. Then, $m_2 - m_1$ is not a positive integer for $(*)$ and $x_1 \geq 3$.

Next, we assume $1/2 < 3^k/2^n < 1$. Since Lemma 2.1 and $x_1 \geq k/2$ (for $x_1 \geq x_2$ and $k = x_1 + x_2$), then $x_1 > (n-1)(\log_3 2)/2$. It follows that

$$2^{-x_1\log_2(4/3)} < 2^{-(n-1)\log_3(2/\sqrt{3})}.$$

Hence we have

$$2^{-51.2102H^2-1} < 2^{-(n-1)\log_3(2/\sqrt{3})}$$

for $(*)$ and Lemma 2.3. It means

$$n < 51371.$$

It is a contradiction to Theorem 1.4.

References

[1] C. Bohm and G. Sontacchi, On the existence of cycles of given length in integer sequence like $x_{n+1} = x_n/2$ if x_n even, and $x_{n+1} = 3x_n + 1$ otherwise, Atti Accad. Naz Lincei Rend. Sci. Fis. Mat. Natur. (8) **64** (1978), 260–264.

[2] R. P. STEINER, A Thorem on the Syracuse Problem, Proc. 7th Manitoba Conf. Numerical Mathematics and Computing 1977 Winnipeg (1978), 553–559.
[3] T. MIMURO, On certain simple cycles of the Collatz conjecture, SUT. J. Math. **37** (2001), 79–89.
[4] S. ELIAHOU, The $3x + 1$ problem: new lower bounds on nontrivial cycle lengths, Discrete Math. 118 (1993), 45–56.
[5] M. LAURENT, M. Mignotte et Y. Nesterenko, Formes linéairies en deux logarithmes et déterminante d'interpolation, J. Number Theory **55** (1995), 258–321.
[6] G. J. WIRSCHING, *The Dynamical System Generated by the* $3n + 1$ *Function*, Lecture Notes in Mathematics **1681**, Springer-Verlag (1998).
[7] S. WAGON, The Collatz Problem, Math. Intelligencer **7** (1985) 72–76.

Present Address:
DEPARTMENT OF SYSTEM ENGINEERING, HOSEI UNIVERSITY,
KOGANEI, TOKYO, 184–8584 JAPAN.
e-mail: mimuro@cc9.ne.jp