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Systematic Singular Triangulations of All Orientable Seifert Manifolds

Taiji TANIGUCHI*, Keiko TSUBOI and Masakatsu YAMASHITA

Keio University, Dokkyo Saitama High School and Toyo University

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1. Introduction

In this paper, we construct singular triangulations [1] of all orientable Seifert manifolds [2]. Especially, we consider singular triangulations with only one vertex, called *one-vertex triangulation*. Our construction is useful to calculate the state sum type invariant, for example, Turaev-Viro invariant, Turaev-Viro-Ocneanu invariant or Dijkgraaf-Witten invariant; this subject will be seen in forthcoming paper [3]. Also our work is made use of the introduction of a new complexity invariant of closed 3-manifold, see [4].

Let $\mathcal{M}$ be a Seifert manifold and P be a special spine [5] of $\mathcal{M}$. Considering a dual complex for $\mathcal{M}$ relative to P, we obtain a one-vertex triangulation of $\mathcal{M}$. Now, how to construct a special spine P of $\mathcal{M}$? Our construction is based on the fact that any orientable Seifert manifold is obtained by gluing M_n, J and $V_{p,q}$, which are homeomorphic to $(S^2 - \coprod_{i=1}^n D_i^2) \times S^1$, $(S^1 \times S^1 - D^2) \times S^1$ and (p, q)-type fibered solid torus respectively.

The first step is to make special spines P_{M_n}, P_J, $P_{V_{p,q}}$ of three compact manifolds M_n, J and $V_{p,q}$ satisfying the following conditions: each connected component of $\partial M_n \cap P_{M_n}$, $\partial J \cap P_J$ and $\partial V_{p,q} \cap P_{V_{p,q}}$ is the theta-curve shown in Figure 1 and the loop $\gamma\bar{\alpha}$ is a fiber, where $\bar{\alpha}$ means the reverse direction of the edge labeled α. As an example, the solid torus $V_{1,1}$

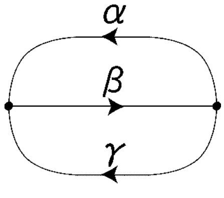

FIGURE 1. A theta-curve θ.

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and a special spine $P_{V_{1,1}}$ is shown in Figure 5. The second step is to define fiber preserving homeomorphisms $\varphi_i : \{\partial J, \partial V_{p,q}\} \to \partial M_n$, see § 5.4. Then, we construct any orientable Seifert manifold $\mathcal{M}$ by gluing J, $V_{p,q}$ and M_n. Furthermore, we show that the polyhedron $P_{\mathcal{M}}$ obtained by gluing P_J, $P_{V_{p,q}}$ and P_{M_n} by $\{\varphi_i\}$ is a special spine of $\mathcal{M}$. Thus, the dual complex for $\mathcal{M}$ concerning $P_{\mathcal{M}}$ is a one-vertex triangulation of $\mathcal{M}$.

2. Word diagram of a solid torus

Let $w = X_1X_2\cdots X_n$ be a sequence of elements $X_i \in \{L, R, \bar{L}, \bar{R}\}$, $i = 1, 2, \cdots, n$. We call such a sequence $w = w(L, R)$ as *a word on the letters* $\{L, R\}$. In this section, for a word $w(L, R)$, we will define a word diagram, denoted by $w(L, R)$-diagram. Then, we show that $w(L, R)$-diagram induces an identification map $f_{w(L,R)}$ on ∂B^3 such that $B^3/f_{w(L,R)}$ is homeomorphic to a solid torus.

In § 2.1, we consider the case when the word $w(L, R)$ is the empty word. We use the notation ϕ for the empty word. The ϕ-diagram plays a role of a pit of $w(L, R)$-diagram of a solid torus. In § 2.2, we consider the case when the word $w(L, R)$ is not the empty word.

2.1. The case $w(L, R) = \phi$. Let D_ϕ be the 2-disc with labels P, A, Q, α, β, γ, u, v, w shown in Figure 2. We call D_ϕ the *ϕ - disc*. It may become clear for the role of labels in D_ϕ, see Remark 2.1.

Suppose the ϕ - disc is embedded in S^2, see Figure 3. Then, the 2-sphere S^2 is separated into two discs by the circle $\alpha\beta\gamma\overline{\alpha}\overline{\beta}\overline{\gamma}$ $(= \partial D_\phi)$, where $\overline{X}$ means the reverse direction of the edge labeled X. We denote by E_ϕ the closed dics $S^2 \setminus \mathrm{Int}(D_\phi)$.

The ϕ - disc gives a natural pasting information f_ϕ on $S^2 (= \partial B^3)$ to obtain a compact 3-manifold B^3/f_ϕ. The ϕ - disc consists of Z_0, Z_1 and Z_2, where Z_0, Z_1 and Z_2 are sets of 0-cells, oriented 1-cells and oriented 2-cells respectively. Now, we will explain Z_0, Z_1 and

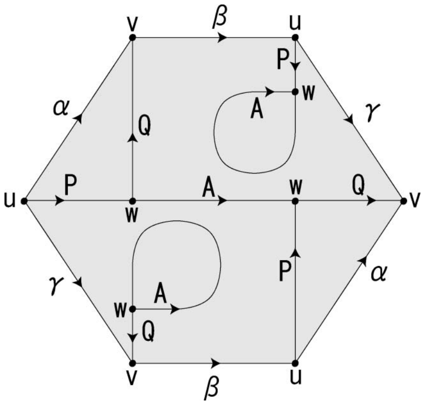

FIGURE 2. ϕ - disc D_ϕ.

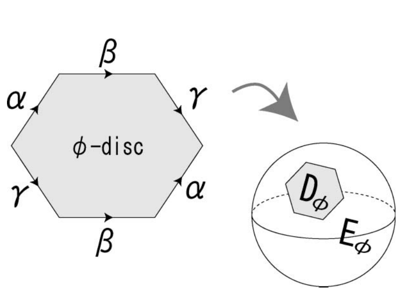

FIGURE 3. E_ϕ.

Z_2 and define a pasting information f_ϕ on S^2. Although it is only an example, but it must give a suggestion of a general method to obtain a pasting information.

In the ϕ-disc, there are some 0-cells with the labels u, v, w. We call a 0-cell a *vertex*. Thus, the set Z_0 consists of ten vertices $\{u^{(i)}, v^{(i)}, w^{(j)} \mid i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4\}$, where $X^{(i)}$ is the vertex with the label X. Then, the map $f_\phi|_{Z_0}$ is defined by satisfying the following conditions:

1. For any $X \in \{u, v\}$ and $i \in \{1, 2, 3\}$, $f_\phi(X^{(i)}) = X$;
2. For any $j \in \{1, 2, 3, 4\}$, $f_\phi(w^{(j)}) = w$,

Now, we observe Z_1. In the ϕ - disc, there are some 1-cells with the labels A, P, Q, α, β, γ, where we mean that a 1-cell is an open arcs. We call a 1-cell an *edge*. We denote the edges in the ϕ - disc with the labels A, P, Q, α, β, γ by $A^{(i)}, P^{(i)}, Q^{(i)}, \alpha^{(j)}, \beta^{(j)}, \gamma^{(j)}$ respectively. Thus, the set Z_1 consists of fifteen edges $\{A^{(i)}, P^{(i)}, Q^{(i)}, \alpha^{(j)}, \beta^{(j)}, \gamma^{(j)} \mid i = 1, 2, 3 \text{ and } j = 1, 2\}$. Then, the map $f_\phi|_{Z_0}$ is extended continuously on $Z_0 \cup Z_1$ satisfying the following conditions.

1. For any $X \in \{A, P, Q\}$ and $i \in \{1, 2, 3\}$, $f_\phi(X^{(i)}) = X$;
2. For any $X \in \{\alpha, \beta, \gamma\}$ and $j \in \{1, 2\}$, $f_\phi(X^{(j)}) = X$;
3. f_ϕ is a homeomorphism on each cell in $Z_0 \cup Z_1$.

At last, we consider Z_2. Let ρ_+, σ_+, τ_+ be oriented 2-cells (open 2-discs) in the upper half disc of D_ϕ with the boundary $\partial\,(\mathrm{cl}(\rho_+)) = A$, $\partial\,(\mathrm{cl}(\sigma_+)) = PQ\overline{\alpha}$, $\partial\,(\mathrm{cl}(\tau_+)) = AQ\overline{\gamma}PA\overline{P}\ \overline{\beta}\ \overline{Q}$ respectively, see Figure 2, where $\mathrm{cl}(X)$ means the closure of X. Similarly, let ρ_-, σ_-, τ_- be oriented 2-cells in the under half disc in D_ϕ such that $\partial\,(\mathrm{cl}(\rho_-)) = A$, $\partial\,(\mathrm{cl}(\sigma_-)) = PQ\overline{\alpha}$, $\partial\,(\mathrm{cl}(\tau_-)) = AQ\overline{\gamma}PA\overline{P}\ \overline{\beta}\ \overline{Q}$ respectively. We call a 2-cell a *face*. Then, the set Z_2 consists of the faces $\{\rho_+, \rho_-, \sigma_+, \sigma_-, \tau_+, \tau_-, \mathrm{Int}(E_\phi)\}$. Then, the map $f_\phi|_{Z_0 \cup Z_1}$ is extended continuously on $Z_0 \cup Z_1 \cup Z_2$ $(\cong S^2)$ satisfying the following conditions:

1. For any $X \in \{\rho, \sigma, \tau\}$, $f_\phi(X_+) = f_\phi(X_-) = X$;
2. f_ϕ is a homeomorphism on each cell in $Z_0 \cup Z_1 \cup Z_2$. Thereby, $f_\phi : S^2 \to S^2/f_\phi$ is determined by the ϕ - disc up to isotopy. We call the pasting information *an identification map* f_ϕ of the ϕ - disc.

REMARK 2.1. Note that the label X of each cell in ϕ - disc is the name of the cell in S^2/f_ϕ.

Let G_ϕ be the connected 3-regular graph with labels on S^2 associated with the underling space of $Z_0 \cup Z_1$. Note that each of $S^2 \setminus G_\phi$ is an open 2 - disc. Thus, D_ϕ, E_ϕ, D_ϕ/f_ϕ and E_ϕ/f_ϕ have cell structure. We call the triple (S^2, G_ϕ, f_ϕ) as *ϕ-diagram*.

Since f_ϕ is a map on $S^2 (= \partial B^3)$, the quotient space $V_\phi := B^3/f_\phi$ is defined. We can confirm that the quotient space V_ϕ is a compact 3-manifold by examining a neighborhood of each point of B^3. Note that the boundary of the 3-manifold V_ϕ is E_ϕ/f_ϕ, and the image $f_\phi(\partial D_\phi) = f_\phi\left(\partial E_\phi\right)$ is the theta-curve θ (Figure 1) embedded in ∂V_ϕ such that $\partial V_\phi \setminus \theta \cong \mathrm{Int}(E_\phi)$.

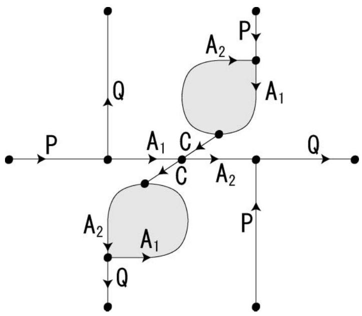

FIGURE 4. D'_ϕ.

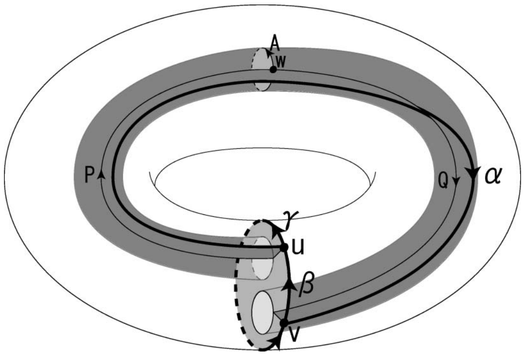

FIGURE 5. The solid torus V_ϕ.

THEOREM 2.2. *The manifold V_ϕ is homeomorphic to a solid torus.*

PROOF. By the definition of f_ϕ, E_ϕ/f_ϕ is the boundary of the manifold V_ϕ, which is a free face of V_ϕ. Thus, the manifold V_ϕ collapses to the cell complex D_ϕ/f_ϕ. And the cell complex D_ϕ/f_ϕ collapses to the cell complex D'_ϕ/f_ϕ, where D'_ϕ is shown in Figure 4.

The cell complex D'_ϕ/f_ϕ collapses to the loop C/f_ϕ, where C is the edge shown in Figure 4. Thus, the manifold V_ϕ collapses to S^1.

Thereby, V_ϕ is homeomorphic to either a solid torus or a solid Klein bottle. By the definition of the map f_ϕ, the manifold V_ϕ is orientable. Thus, it must be homeomorphic to a solid torus. □

The solid torus V_ϕ is shown in Figure 5. The bold lines α, β and γ are embedded in ∂V_ϕ, and the others are embedded in $\mathrm{Int}(V_\phi)$. And shaded faces are D_ϕ/f_ϕ embedded in $\mathrm{Int}(V_\phi)$.

2.2. The case $w(L, R) \neq \phi$. In the previous subsection, we observed the case when $w(L, R)$ is the empty word. In this subsection, we consider the case when $w(L, R)$ is not the empty word. For a word w, we will define w-disc D_w, and show that B^3/f_w is homeomorphic to a solid torus, where f_w is an identification map on ∂B^3 induced by D_w.

Now, we define the two discs D_w and D^*_w for a word w. Denote by $\mathcal{H}_L, \mathcal{H}_{\bar{L}}, \mathcal{H}_R, \mathcal{H}_{\bar{R}}$ the annuli shown in Figure 6, 7, 8, 9 respectively. At first, we consider the case that the length of the word w is 1, that is, $w = X$, where $X \in \{L, R, \bar{L}, \bar{R}\}$.

DEFINITION 2.3. The disc D^*_X is defined by the following steps.

Step 1. Rename the label μ of the vertices and the edges in D_ϕ into μ_ϕ, where $\mu = u, v, w, A, P, Q, \alpha\ \beta, \gamma$.

Step 2. Rename the label μ of the vertices and the edges in $\mathcal{H}_X$ into μ_X, where $\mu = u, v, w, A, P, B, Q, \alpha\ \beta, \gamma$.

Step 3. Rename the label α', β' and γ' of the edges in $\mathcal{H}_X$ into α_ϕ, β_ϕ and γ_ϕ respectively.

Step 4. Glue the ϕ - disc D_ϕ and the annulus $\mathcal{H}_X$ by identifying the corresponding circle $\alpha_\phi \beta_\phi \gamma_\phi \overline{\alpha_\phi}\,\overline{\beta_\phi}\,\overline{\gamma_\phi}$.

See, for example, Figure 10 for D_L^*.

DEFINITION 2.4. The disc D_X is obtained by the more two steps in addition to Definition 2.3.

Step 5. Delete the edges α_ϕ, β_ϕ and γ_ϕ.

Step 6. Change the label $\mu_\phi \nu_X$ (or $\nu_X \mu_\phi$) into ν_X. For example, in the case when $X = L$, there are arcs with label $Q_\phi B_L$, see Figure 10. Then, rename it into B_L. Similarly, change the label $A_L P_\phi$ into A_L.

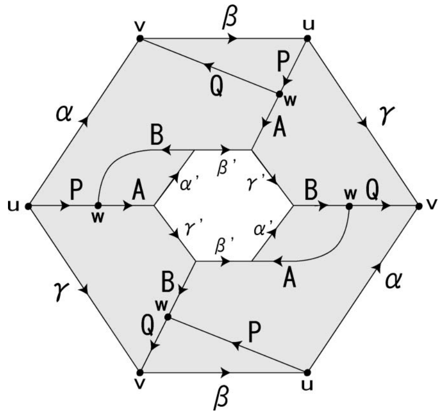

FIGURE 6. $\mathcal{H}_L$.

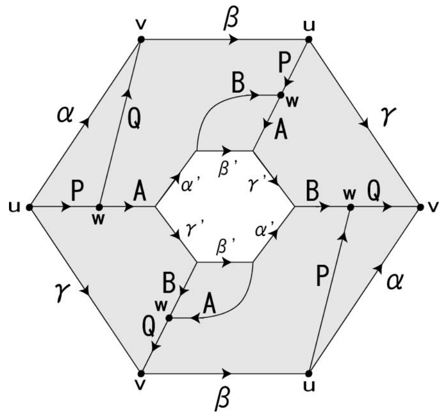

FIGURE 7. $\mathcal{H}_{\bar{L}}$.

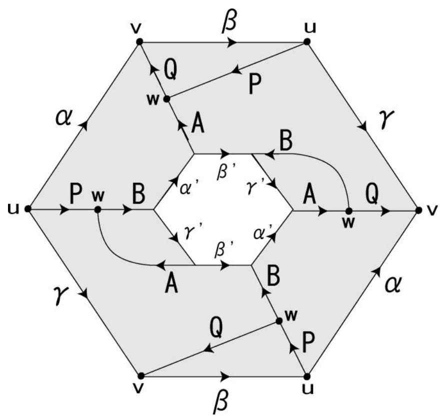

FIGURE 8. $\mathcal{H}_R$.

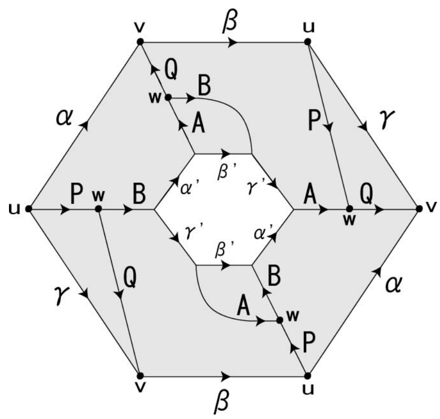

FIGURE 9. $\mathcal{H}_{\bar{R}}$.

See, for example, Figure 11 for D_L.

Now, we define two discs D^*_w and D_w in the case when the length of the word w is greater than 1. Our definition is given by the induction with respect to the length of the word w.

Suppose that two discs D^*_w and D_w are already defined for $w = X_1 X_2 \cdots X_n$, where $X_i \in \{L, R, \bar{L}, \bar{R}\}$. Then, let wX be the word $X_1 X_2 \cdots X_n X$, where $X \in \{L, R, \bar{L}, \bar{R}\}$.

DEFINITION 2.5. The disc D^*_{wX} is defined by the following steps.

Step 1. Rename the label μ of the vertices and the edges in $\mathcal{H}_X$ into μ_{wX}, where $\mu = u, v, w, A, P, B, Q, \alpha\ \beta, \gamma$.

Step 2. Rename the label α', β' and γ' of the edges in $\mathcal{H}_X$ into α_w, β_w and γ_w respectively.

Step 3. Glue the disc D^*_w and the annulus $\mathcal{H}_X$ by identifying the corresponding circle $\alpha_w \beta_w \gamma_w \overline{\alpha_w}\,\overline{\beta_w}\,\overline{\gamma_w}$.

DEFINITION 2.6. The disc D_{wX} is defined by the following steps.

Step 1. The same with Step 1 in Definition 2.5.

Step 2. The same with Step 2 in Definition 2.5.

Step 3. Glue the disc D_w and the annulus $\mathcal{H}_X$ by identifying the corresponding circle $\alpha_w \beta_w \gamma_w \overline{\alpha_w}\,\overline{\beta_w}\,\overline{\gamma_w}$.

Step 4. Delete the edges α_w, β_w and γ_w.

Step 5. Change the label $\mu_w \nu_{wX}$ (or $\nu_{wX} \mu_w$) into ν_{wX}.

Now, we consider an identification map f_w. Assume that the disc D_w is embedded in $S^2 (= \partial B^3)$ as the case $w = \phi$. Denote by E_w the disc $S^2 \setminus \mathrm{Int}(D_w)$. As the case of ϕ-disc, we can define an identification map $f_w : S^2 \to S^2/f_w$ and a 3-manifold $V_w = B^3/f_w$. Then,

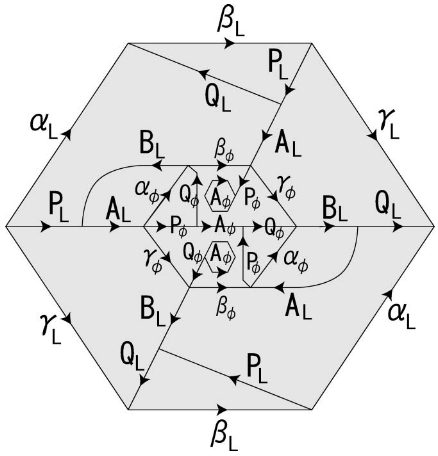

FIGURE 10. D^*_L.

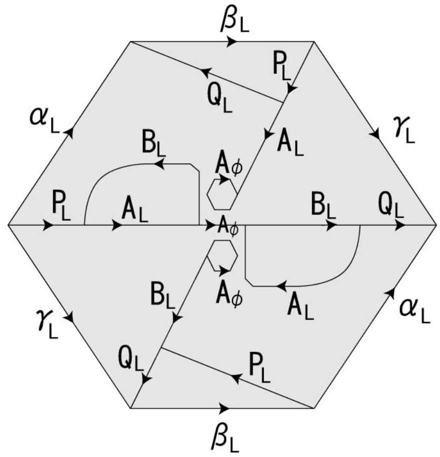

FIGURE 11. D_L.

the image $f_w(\partial D_w) = f_w(\partial E_w)$ is the theta-curve θ (Figure 1) embedded in ∂V_w such that $\partial V_w \setminus \theta \cong \mathrm{Int}(E_w)$.

THEOREM 2.7. *The manifold V_w is a homeomorphic to a solid torus.*

PROOF. At first, we prove that the manifold V_w collapses to S^1 inductively. Since the manifold V_ϕ collapses to S^1 by the proof of Theorem 2.2, it is enough to show that if V_w collapses to S^1 then V_{wX} collapses to S^1 for each $X \in \{L, R, \bar{L}, \bar{R}\}$.

We consider the case $X = L$. Then, the manifold V_{wL} collapses to the cell complex D_{wL}/f_{wL} (Figure 12).

It collapses to the cell complex D'_{wL}/f_{wL} (Figure 13). Furthermore, the cell complex $D'{}_{wL}/f_{wL}$ collapses to the cell complex $D''{}_{wL}/f_{wL}$, (Figure 14). At last, the cell complex $D''{}_{wL}/f_{wL}$ collapses to the cell complex $D'''{}_{wL}/f_{wL}$ (Figure 15). By the definition of f_w, the cell complex D'''_{wL}/f_{wL} coincides with the cell complex D'''_w/f_w. By the assumption of induction, the cell complex D'''_w/f_w collapses to S^1. So, V_{wL} collapses to S^1. Thus, for any word w, the 3-manifold V_w collapses to S^1. In the case $X = R, \bar{L}, \bar{R}$, we can prove similar to the case $X = L$. Since the manifold V_w is orientable, we conclude that V_w is homeomorphic to a solid torus. □

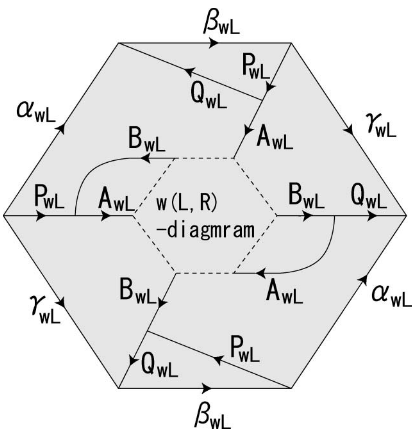

FIGURE 12. D^*_L.

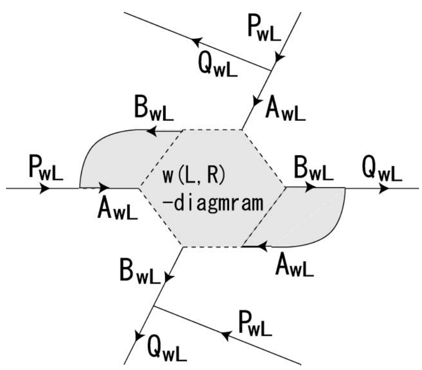

FIGURE 13. D'_{wL}.

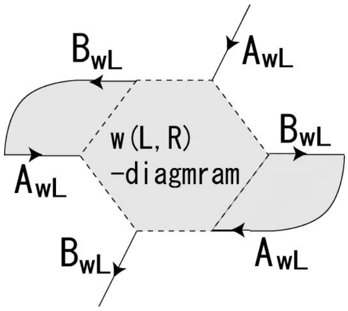

FIGURE 14. $D''_{w(L,R)L}$.

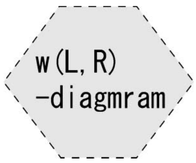

FIGURE 15. $D'''_{w(L,R)L}$.

3. Meridian, longitude and fiber structure of $V_{w(L,R)}$

In this section, we consider a meridian-longitude system (m_w, l_w) of the solid torus V_w, where a meridian-longitude system (m_w, l_w) is a pair of loops m_w and l_w satisfying the following conditions:

1. m_w and l_w are essential loops embedded in ∂V_w
2. m_w and l_w intersect at a point.
3. $\ll m_w \gg = 1$ and $\ll l_w \gg \neq 1$ in $\pi_1(V_w)$, where $\ll\gg$ is a homotopy class.

In § 3.1, we give a presentation of loops embedded in ∂V_w, that is, we define two generators $\ll x_w \gg$ and $\ll y_w \gg$ of $\pi_1(\partial V_w)$ for each word w. Then, any loop embedded in ∂V_w is presented by an element $a_w[x_w] + b_w[y_w]$ in $H_1(\partial V_w)$ uniquely, where a_w and b_w are coprime integers and [ ] is the homology class. In § 3.2 and § 3.3, we consider the coefficients a_w and b_w of the meridian and a longitude of V_w.

3.1. Loops embedded in $\partial V_{w(L,R)}$. In § 2, we showed that E_w/f_w is the boundary of the solid torus V_w. Thus, three loops $\overline{\alpha_w \beta_w}$, $\gamma_w \beta_w$ and $\gamma_w \overline{\alpha_w}$ are embedded in ∂V_w. We denote them by x_w, y_w and z_w respectively.

PROPOSITION 3.1. *For any word w, the two elements $\ll x_w \gg$ and $\ll y_w \gg$ forms a basis of the fundamental group $\pi_1(\partial V_w)$, where $\ll x_w \gg$ and $\ll y_w \gg$ are the homotopy classes of the loops x_w and y_w respectively.*

PROOF. Consider the loop A_w embedded in ∂V_w, where the loop A_w is shown in Figure 16.

Two loops A_w and $\gamma_w \beta_w$ are homotopic in ∂V_w. By the method of cut and paste, we know two loops A_w and $\overline{\alpha_w \beta_w}$ are essential loops and intersect at one point in ∂V_w. □

By Proposition 3.1, any loop embedded in ∂V_w is presented by an element $a_w[\, x_w \,] + b_w[\, y_w \,]$ in $H_1(\partial V_w)$ uniquely, where a_w and b_w is coprime integers.

3.2. The case $w(L, R) = \phi$. In this subsection, we consider a meridian-longitude system of the solid torus V_ϕ.

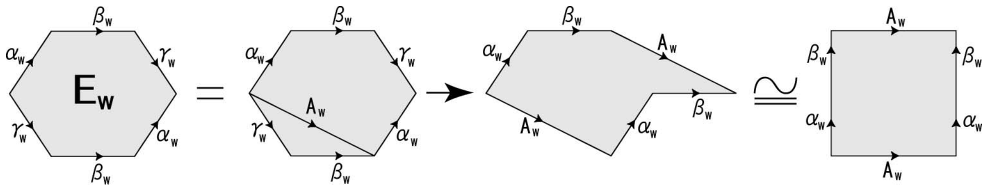

FIGURE 16. E_w.

THEOREM 3.2. *A pair of the loops* (y_ϕ, x_ϕ) *is a meridian-longitude system of the solid torus* V_ϕ.

PROOF. According to the proof of Proposition 3.1, loops x_ϕ and y_ϕ are essential loops in ∂V_ϕ. Furthermore, we can regard x_ϕ and y_ϕ as loops intersecting at a point with the argument of general position. Thus, all we have to do is to show that $\ll x_\phi \gg \neq 1$ and $\ll y_\phi \gg = 1$ in $\pi_1(V_\phi)$. We consider the fundamental group $\pi_1(V_\phi)$. Since V_ϕ collapses to D_ϕ/f_ϕ, the fundamental group $\pi_1(V_\phi)$ is isomorphic to $\pi_1(D_\phi/f_\phi)$. So, we consider $\pi_1(D_\phi/f_\phi)$. We choose a base point of D_ϕ/f_ϕ as the vertex $u = f_\phi(u^{(i)})$, see Figure 2.

The fundamental group $\pi_1(D_\phi/f_\phi)$ is obtained by the following two steps. First, we consider a maximal tree $Q\beta$ of G_ϕ, where $G_\phi := |Z_0 \cup Z_1|$ and Z_0 and Z_1 was defined in § 2.1. Then, any loop embedded in D_ϕ/f_ϕ is homotopic to some loop which is a finite product of the loops $\alpha\beta$, $\gamma\beta$, $PQ\beta$ and $\overline{\beta}\ \overline{Q}AQ\beta$. That is, $\pi_1(D_\phi/f_\phi)$ is generated by the homotopy classes of the loops $\alpha\beta$, $\gamma\beta$, $PQ\beta$ and $\overline{\beta}\ \overline{Q}AQ\beta$.We denote them by $\hat{\alpha}$, $\hat{\gamma}$, $\hat{P}$, $\hat{A}$ respectively. And, any relator of $\pi_1(D_\phi/f_\phi)$ is a sequence of $\{\hat{P}\hat{\alpha}^{-1},\ \hat{A}\hat{\gamma}^{-1},\ \hat{A}\}$ which are homotopy classes of the loops each of which bounds a face of D_ϕ/f_ϕ. Thus, we get

$$\begin{aligned}\pi_1(V_\phi) &= \pi_1(D_\phi/f_\phi,\ u)\\ &=< \hat{\alpha},\ \hat{\gamma},\ \hat{P},\ \hat{A} \mid \hat{P}\hat{\alpha}^{-1},\ \hat{A}\hat{\gamma}^{-1},\ \hat{A}\ >\\ &=< \hat{\alpha},\ \hat{\gamma},\ \hat{P} \mid \hat{P}\hat{\alpha}^{-1},\ \hat{\gamma}^{-1}\ >\\ &=< \hat{\alpha},\ \hat{P} \mid \hat{P}\hat{\alpha}^{-1}\ >\\ &=< \hat{\alpha} \mid -\ >\ .\end{aligned}$$

And the generator of $\pi_1(V_\phi)$ is $\hat{\alpha}$. Since $\hat{\alpha}$ is the homotopy class of the loop $x_\phi = \overline{\alpha_\phi\beta_\phi}$, we get $\ll x_\phi \gg \neq 1$. And $\hat{\gamma}^{-1}$ is a relator of $\pi_1(V_\phi)$. Since $\hat{\gamma}$ is the homotopy class of a loop $y_\phi = \gamma_\phi\beta_\phi$, we have $\ll y_\phi \gg = 1$. □

Now, we consider the loop $z_\phi = \gamma_\phi\overline{\alpha_\phi}$ in ∂V_ϕ.

DEFINITION 3.3. Let V be a solid torus with the meridian-longitude system (m, l). Then, a loop λ in ∂V is the *(p,q) - type* if the condition $[\lambda] = p\,[l] + q\,[m]$ holds in $H_1(\partial V)$.

Then, we consider the type of the loop z_ϕ for a solid torus V_ϕ with the meridian-longitude system (y_ϕ, x_ϕ) given in Theorem 3.2.

THEOREM 3.4. *The loop z_ϕ is the $(1, 1)$-type.*

PROOF. By the definition, the homotopy class of the loops x_ϕ, y_ϕ and x_ϕ satisfy the following condition in $\pi_1(\partial V_\phi)$.

$$\begin{aligned} \ll z_\phi \gg &= \ll \gamma_\phi \overline{\alpha_\phi} \gg \\ &= \ll \gamma_\phi \beta_\phi \overline{\beta_\phi} \overline{\alpha_\phi} \gg \\ &= \ll \gamma_\phi \beta_\phi \gg \ll \overline{\alpha_\phi \beta_\phi} \gg \ . \end{aligned}$$

Thus, the loop z_ϕ satisfies the condition $[z_\phi] = [x_\phi] + [y_\phi]$ in $H_1(\partial V_\phi)$. □

3.3. The case $w(L, R)$ is not the empty. In this subsection, we consider a meridian-longitude system of the solid torus V_w is the case when the word w is not the empty word. In §3.3.1, we give two theorems about meridian-longitude systems of the solid torus V_w. In §3.3.2, we prepare lemmas. In §3.3.3, we give the proofs of the theorems.

3.3.1. Theorems. For a word $w = X_1 X_2 \ldots X_n$, where $X_i \in \{L, R, \bar{L}, \bar{R}\}$, we define the matrix $M_w = U_{X_1} U_{X_2} \cdots U_{X_n}$, where $U_L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $U_R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $U_{\bar{L}} = U_L{}^{-1}$ and $U_{\bar{R}} = U_R{}^{-1}$.

THEOREM 3.5. *There is a meridian-longitude system (m_w, l_w) of V_w satisfying the following equations in $H_1(\partial V_w)$:*

$$\begin{aligned} [m_w] &= -M_{1,2}[x_w] + M_{1,1}[y_w]\,; \\ [\,l_w] &= M_{2,2}[x_w] - M_{2,1}[y_w]\,, \end{aligned}$$

where $M_{i,j}$ is the (i, j)-element of the matrix M_w.

Now, we will see another theorem. We use the following notation for the expansion into continued fraction:

$$[a_1,\ a_2,\ \cdots,\ a_{n-1},\ a_n] := \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}\ .$$

For a pair of natural numbers p, q such that $p > q$, we define an *alternative word* $A(p, q)$ as follows:

$$A(p, q) = \begin{cases} L^{a_1}\, R^{a_2}\, L^{a_3}\, \cdots\, L^{a_{n-2}} R^{a_{n-1}}\, L^{a_n} & (n\ :\ \text{odd}) \\ L^{a_1}\, R^{a_2}\, L^{a_3}\, \cdots\, R^{a_{n-2}} L^{a_{n-1}}\, R^{a_n} & (n\ :\ \text{even}) \end{cases}\ ,$$

where the natural numbers a_i are defined by $q/p = [a_1,\ a_2, \cdots, a_n,\ 1]$.

THEOREM 3.6. *For the solid torus $V_{A(p,q)}$ with the meridian-longitude system $(m_{A(p,q)}, l_{A(p,q)})$ given by Theorem 3.5, the loop $z_{A(p,q)}$ is the (p,q)-type.*

NOTATION 3.7. We call $V_{A(p,q)}$ as *(p,q)-type solid torus* and denote it by $V_{p,q}$.

3.3.2. Lemmas. In this subsection, we show two lemmas. The first one is concerned with the fundamental group of the solid torus V_w. The second one is the calculation of the matrices U_L, U_R, $U_{\bar{L}}$ and $U_{\bar{R}}$.

Now, we see the first one. At first, we consider the solid torus V_L, where V_L is the solid torus V_w in the case $w = L$. Since the solid torus V_L collapses to D_L/f_L, the fundamental group $\pi_1(V_L)$ is isomorphic to $\pi_1(D_L/f_L)$. By the definition, the fundamental group $\pi_1(D_L/f_L)$ is isomorphic to $\pi_1(D_L^*/f_L)$. Thereby, the fundamental group $\pi_1(V_L)$ is isomorphic to $\pi_1(D_L^*/f_L)$.

Thus, we consider $\pi_1(D_L^*/f_L)$. We take a base point u_L of D_L^*/f_L and a maximal tree $\beta_L P_L A_L \beta_\phi P_\phi$ of D_L^*/f_L, see Figure 10, where $\beta_L P_L A_L \beta_\phi P_\phi$ means the union of the edges β_L, P_L, A_L, β_ϕ and P_ϕ. And we consider the generators and relators similar to the case $w = \phi$. Also we use the same notations $\hat{X}$ in § 3.2. For convenience, we denote X_L and X_ϕ by X and X' respectively. Then, we have

$$\begin{aligned}
&\pi_1(V_L)\\
&\quad = \pi_1(D_L^*/f_L,\ u_L)\\
&\quad =< \hat{\alpha'},\ \hat{\gamma'},\ \hat{Q'},\ \hat{A'},\ \hat{B}^{-1},\ \hat{Q},\ \hat{\alpha},\ \hat{\gamma}\\
&\qquad\qquad \mid \hat{Q'}\hat{\alpha'}^{-1},\ \hat{Q'}^{-1}\hat{A'}\hat{Q'}\hat{\gamma'}^{-1}\hat{A'},\ \hat{A'},\ \hat{\alpha'}\hat{B},\ \hat{Q}\hat{\alpha}^{-1}\hat{B}^{-1},\ \hat{Q},\ \hat{\gamma'}\hat{B}\hat{Q}\hat{\gamma}^{-1} >\\
&\quad =< \hat{\alpha'},\ \hat{\gamma'},\ \hat{Q'},\ \hat{B}^{-1},\ \hat{Q},\ \hat{\alpha},\ \hat{\gamma} \mid \hat{Q'}\hat{\alpha'}^{-1},\ \hat{\gamma'}^{-1},\ \hat{\alpha'}\hat{B},\ \hat{Q}\hat{\alpha}^{-1}\hat{B}^{-1},\ \hat{Q},\ \hat{\gamma'}\hat{B}\hat{Q}\hat{\gamma}^{-1} >\\
&\quad =< \hat{\alpha'},\ \hat{\gamma'},\ \hat{B}^{-1},\ \hat{Q},\ \hat{\alpha},\ \hat{\gamma} \mid \hat{\gamma'},\ \hat{\alpha'}\hat{B},\ \hat{Q}\hat{\alpha}^{-1}\hat{B}^{-1},\ \hat{Q},\ \hat{\gamma'}\hat{B}\hat{Q}\hat{\gamma}^{-1} >\\
&\quad =< \hat{\alpha'},\ \hat{\gamma'},\ \hat{Q},\ \hat{\alpha},\ \hat{\gamma} \mid \hat{\gamma'},\ \hat{Q}\hat{\alpha}^{-1}\hat{\alpha'},\ \hat{Q},\ \hat{\gamma'}\hat{\alpha'}^{-1}\hat{Q}\hat{\gamma}^{-1} >\\
&\quad =< \hat{\alpha'},\ \hat{\gamma'},\ \hat{\alpha},\ \hat{\gamma} \mid \hat{\gamma'},\ \hat{\alpha}^{-1}\hat{\alpha'},\ \hat{\gamma'}\hat{\alpha'}^{-1}\hat{\gamma}^{-1} >\\
&\quad =< \hat{\alpha'} \mid - > .
\end{aligned}$$

And we get the two relations $\hat{\alpha}^{-1}\hat{\alpha'} = 1$ and $\hat{\gamma'}\hat{\alpha'}^{-1}\hat{\gamma}^{-1} = 1$ in $\pi_1(V_L)$. Since, the homology class of the loops x_ϕ, y_ϕ, x_L and y_L are $\hat{\alpha'}^{-1}$, $\hat{\gamma'}$, $\hat{\alpha}^{-1}$ and $\hat{\gamma}$ respectively, we get the following relations in $H_1(V_L)$:

$$\begin{cases} [x_L] = [x_\phi] \\ [y_L] = [x_\phi] + [y_\phi] \end{cases} \iff {}^t\begin{pmatrix} [x_L] \\ [y_L] \end{pmatrix} = {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} U_L ,$$

where ${}^t(\quad)$ means the transposition and [ ] means the homology class and U_L is defined in § 3.3.1. Since the generator of $\pi_1(V_L)$ is $\hat{\alpha}'$, we get $[x_\phi] \neq 0$ in $H_1(V_L)$. And by the relation of $\pi_1(V_L)$, we have $[y_\phi] = 0$ in $H_1(V_L)$.

In the case $X = \bar{L}, R, \bar{R}$, we take a base point u_X of $\pi_1(D^*_X/f_X)$ and a maximal tree $\beta_{\bar{L}} P_{\bar{L}} A_{\bar{L}} \beta_\phi P_\phi$, $\beta_R Q_R A_R \beta_\phi P_\phi$ and $\beta_{\bar{R}} Q_{\bar{R}} A_{\bar{R}} \beta_\phi P_\phi$ respectively. Then, we get the relations ${}^t\begin{pmatrix} [x_X] \\ [y_X] \end{pmatrix} = {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} U_X$, $[x_\phi] \neq 0$ and $[y_\phi] = 0$ in $H_1(V_X)$, where the matrix $U_{\bar{L}}$, U_R and $U_{\bar{R}}$ are defined in § 3.3.1.

At last, for any word $w = X_1 X_2 \cdots X_n$, where $X_i \in \{L, R, \bar{L}, \bar{R}\}$, we consider $H_1(V_w)$. We take a base point u_w of D^*_w/f_w and a maximal tree $(\beta_\phi D_\phi) \cup (\coprod_{i=1}^n m_i)$, where $m_i = \begin{cases} \beta_{X_i} P_{X_i} A_{X_i} & (X_i = L, \bar{L}); \\ \beta_{X_i} Q_{X_i} A_{X_i} & (X_i = R, \bar{R}). \end{cases}$ Then, we get the equation ${}^t\begin{pmatrix} [x_{wX}] \\ [y_{wX}] \end{pmatrix} = {}^t\begin{pmatrix} [x_w] \\ [y_w] \end{pmatrix} U_X$ for any $X = L, R, \bar{L}, \bar{R}$. Thus, we have the following lemma.

LEMMA 3.8. *The following two relations* 1 *and* 2 *hold in* $H_1(V_w)$, *where the loops* x_w *and* y_w *are embedded in* ∂V_w *and the loops* x_ϕ *and* y_ϕ *are embedded in Int* (V_w).

1. ${}^t\begin{pmatrix} [x_w] \\ [y_w] \end{pmatrix} = {}^t\begin{pmatrix} [x_\phi] \\ [y_\phi] \end{pmatrix} M_w$, *where the matrix* M_w *is defined in Theorem* 3.5 *and* [ ] *means a homology class.*
2. $[x_\phi] \neq 0$ *and* $[y_\phi] = 0$.

Now, we observe another lemma. For n natural numbers a_i $(1 \leq i \leq n)$, two matrices $\mathcal{A}$ and $\mathcal{B}$ are defined as follows:

$$\mathcal{A} = \begin{cases} U_L^{a_1} U_R^{a_2} U_L^{a_3} \cdots U_L^{a_{n-2}} U_R^{a_{n-1}} U_L^{a_n} & (n : odd) \\ U_L^{a_1} U_R^{a_2} U_L^{a_3} \cdots U_R^{a_{n-2}} U_L^{a_{n-1}} U_R^{a_n} & (n : even) \end{cases},$$

$$\mathcal{B} = \begin{cases} U_R^{a_1} U_L^{a_2} U_R^{a_3} \cdots U_R^{a_{n-2}} U_L^{a_{n-1}} U_R^{a_n} & (n : odd) \\ U_R^{a_1} U_L^{a_2} U_R^{a_3} \cdots U_L^{a_{n-2}} U_R^{a_{n-1}} U_L^{a_n} & (n : even) \end{cases}.$$

We denote by $A_{i,j}$ and $B_{i,j}$ the (i, j)-element of the matrix $\mathcal{A}$ and $\mathcal{B}$ respectively.

LEMMA 3.9. *Two matrices* $\mathcal{A}$ *and* $\mathcal{B}$ *satisfy the following conditions*:

1. $\dfrac{A_{2,1} + A_{2,2}}{A_{1,1} + A_{1,2}} = \dfrac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}} = [a_1, a_2, \cdots, a_n, 1]$, *where* [ ] *means the expansion into continued fraction defined in* § 3.3.1;
2. (a) *Two natural numbers* $A_{1,1}$ *and* $A_{1,2}$ *are coprime*, *and two non-negative integers* $A_{2,1}$ *and* $A_{2,2}$ *are coprime.*
   (b) *Two natural numbers* $A_{1,1} + A_{1,2}$ *and* $A_{2,1} + A_{2,2}$ *are coprime*;

PROOF.

1. Let us consider a proof by induction on the *length* of the matrix $\mathcal{A}$ and $\mathcal{B}$, where the *length* is the natural number n in the matrix $U_{X_1}^{a_1} U_{X_2}^{a_2} U_{X_3}^{a_3} \cdots U_{X_n}^{a_n}$.

If $n = 1$, then $\mathcal{A} = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}$ and $\mathcal{B} = \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix}$. So, we have $\dfrac{A_{2,1} + A_{2,2}}{A_{1,1} + A_{1,2}} = \dfrac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}} = \dfrac{1}{a_1 + 1} = [a_1,\ 1]$. Now, assume that the statement holds in the case $n - 1$. Then, we will consider the case n.

(i) In the case n is odd

$$\begin{aligned}\mathcal{A} &= U_L^{a_1} U_R^{a_2} U_L^{a_3} \cdots U_L^{a_n} = U_L^{a_1} \mathcal{B} \\ &= \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} B_{1,1} + a_1 B_{2,1} & B_{1,2} + a_1 B_{2,2} \\ B_{2,1} & B_{2,2} \end{pmatrix},\end{aligned}$$

where $\mathcal{B}$ is the matrix with respect to $n - 1$ natural numbers $a_2, a_3, \cdots, a_n$, that is, $\mathcal{B} = U_R^{a_2} U_L^{a_3} \cdots U_L^{a_n}$. Then, we obtain

$$\begin{aligned}\frac{A_{2,1} + A_{2,2}}{A_{1,1} + A_{1,2}} &= \frac{B_{2,1} + B_{2,2}}{B_{1,1} + a_1 B_{2,1} + \left(B_{1,2} + a_1 B_{2,2}\right)} \\ &= \frac{1}{a_1 + \dfrac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}}}\,.\end{aligned}$$

By the assumption of the induction, we have the following equation about the matrix $\mathcal{B}$.

$$\frac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}} = [a_2,\ a_3, \cdots,\ a_n,\ 1]\,.$$

Thus, we have

$$\frac{1}{a_1 + \dfrac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}}} = [a_1,\ a_2, \cdots,\ a_n,\ 1]\,.$$

Also, we have that

$$\begin{aligned}\mathcal{B} &= U_R^{a_1} U_L^{a_2} U_R^{a_3} \cdots U_R^{a_n} = U_R^{a_1} \mathcal{A} \\ &= \begin{pmatrix} 1 & 0 \\ a_1 & 1 \end{pmatrix} \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \\ &= \begin{pmatrix} A_{1,1} & A_{1,2} \\ a_1 A_{1,1} + A_{2,1} & a_1 A_{1,2} + A_{2,2} \end{pmatrix}\end{aligned}$$

and

$$\frac{B_{1,1} + B_{1,2}}{B_{2,1} + B_{2,2}} = \frac{A_{1,1} + A_{1,2}}{a_1 A_{1,1} + A_{2,1} + \left(a_1 A_{1,2} + A_{2,2}\right)}$$

$$= \cfrac{1}{a_1 + \cfrac{A_{2,1}+A_{2,2}}{A_{1,1}+A_{1,2}}} = [a_1,\ a_2, \cdots,\ a_n,\ 1]\,.$$

(ii) In the case n is even

We can prove similar to the case (i).

2. (a) We have $\det U_X = 1$ for all $X \in \{L, R, \bar{L}, \bar{R}\}$ by the definition of the matrix U_X. Since the property of the determinant: $\det AB = \det A \det B$, we get $\det \mathcal{A}$=1. It completes the proof.

2. (b) By the property of the determinant, we have $\det \mathcal{A} = \det \begin{pmatrix} A_{1,1}+A_{1,2} & A_{1,2} \\ A_{2,1}+A_{2,2} & A_{2,2} \end{pmatrix}$.

Since $A_{1,1}+A_{1,2}$, $A_{1,2}$, $A_{2,1}+A_{2,2}$ and $A_{2,2}$ are natural numbers and $\det \mathcal{A} = 1$, we get $A_{1,1}+A_{1,2}$ and $A_{2,1}+A_{2,2}$ are coprime. □

3.3.3. Proof of main theorems.

PROOF OF THEOREM 3.5. There is a representative y of homology class $[y_w]$ such that y is an essential loop on a solid torus V_w and y intersect with x_w at one point each other, where x_w and y_w are defined in § 3.1. Since $\det M_w = 1$ and two non-negative integers $M_{i,1}$ and $M_{i,2}$ are coprime for $i = 1, 2$, there are two essential loops m_w and l_w on ∂V_w with one common point such that

$$[m_w] = -M_{1,2}[x_w] + M_{1,1}[y_w]; \qquad [\,l_w] = M_{2,2}[x_w] - M_{2,1}[y_w]\,.$$

Thus, the rest of the proof is to show $[\,l_w] \neq 0$ and $[\,m_w] = 0$ in $H_1(V_w)$. By Lemma 3.8, we get the following equation in $H_1(V_w)$:

$$\begin{aligned} {}^t\!\begin{pmatrix} [x_w] \\ [\,y_w] \end{pmatrix} &= {}^t\!\begin{pmatrix} [\,x_\phi] \\ [\,y_\phi] \end{pmatrix} M_w \\ \iff \quad & {}^t\!\begin{pmatrix} [x_w] \\ [\,y_w] \end{pmatrix} \begin{pmatrix} M_{2,2} & -M_{1,2} \\ -M_{2,1} & M_{1,1} \end{pmatrix} = {}^t\!\begin{pmatrix} [\,x_\phi] \\ [\,y_\phi] \end{pmatrix} \\ \iff \quad & \begin{cases} [\,x_\phi] = M_{2,2}[\,x_w] - M_{2,1}[\,y_w] \\ [\,y_\phi] = -M_{1,2}[\,x_w] + M_{1,1}[\,y_w] \end{cases} . \end{aligned}$$

And we have $[x_\phi] \neq 0$ and $[y_\phi] = 0$ in $H_1(V_w)$ by Lemma 3.8. Also we get $[m_w] = [y_\phi]$ and $[l_w] = [x_\phi]$ in $H_1(V_w)$. It completes the proof. □

PROOF OF THEOREM 3.6. Recall that a_i is defined by the equation $q/p = [a_1,\ a_2, \cdots,\ a_n,\ 1]$. Then, we define the matrix $\mathcal{A}^{(p,q)}$ as follows:

$$\mathcal{A}^{(p,q)} \quad = \quad \begin{cases} U_L^{a_1}\, U_R^{a_2}\, U_L^{a_3} \cdots U_L^{a_{n-2}} U_R^{a_{n-1}}\, U_L^{a_n} & (n\ :\ odd) \\ U_L^{a_1}\, U_R^{a_2}\, U_L^{a_3} \cdots U_R^{a_{n-2}} U_L^{a_{n-1}}\, U_R^{a_n} & (n\ :\ even) \end{cases} .$$

For convenience, we denote the word $A(p,q)$ by A. By Theorem 3.5, we have the following relations in $H_1(\partial T_A)$:

$$[m_A] = -A_{1,2}[x_A] + A_{1,1}[y_A]\,; \quad [l_A] = A_{2,2}[x_A] - A_{2,1}[y_A]\,,$$

where $A_{i,j}$ is the (i,j)-element of the matrix $\mathcal{A}^{(p,q)}$. Thus, we have

$$[x_A] = A_{1,1}[l_A] + A_{2,1}[m_A]\,; \qquad [y_A] = A_{1,2}[l_A] + A_{2,2}[m_A]\,.$$

Then, the following equation holds in $H_1(\partial T_A)$:

$$\begin{aligned}[z_A] &= [x_A] + [y_A] \\ &= A_{1,1}[l_A] + A_{2,1}[m_A] + A_{1,2}[l_A] + A_{2,2}[m_A] \\ &= \left(A_{1,1} + A_{1,2}\right)[l_A] + \left(A_{2,1} + A_{2,2}\right)[m_A]\,.\end{aligned}$$

By Lemma 3.9 and the definition of a_i, we have

$$\frac{A_{2,1} + A_{2,2}}{A_{1,1} + A_{1,2}} = [a_1,\ a_2,\cdots,a_n,\ 1] = \frac{q}{p}\,.$$

And, two natural numbers $A_{2,1} + A_{2,2}$ and $A_{1,1} + A_{1,2}$ are coprime by Lemma 3.9. Thus, we have $A_{1,1} + A_{1,2} = p$ and $A_{2,1} + A_{2,2} = q$. □

3.4. b-type solid torus. In this subsection, we define the *b-type solid torus* for an integer b. In § 5, it will be appeared that b-type solid torus corresponds to an obstruction class of Seifert manifold.

For an integer b, the word $w(b)$ is defined as $LR^b\bar{L}$. Then, we consider a meridian-longitude system of the solid torus $V_{w(b)}$.

COROLLARY 3.10. *For the solid torus $V_{w(b)}$, there is a meridian-longitude system $(m_{w(b)}, l_{w(b)})$ satisfying the following conditions in $H_1(\partial V_{w(b)})$:*

$$\begin{aligned}[m_{w(b)}] &= b\,[x_{w(b)}] + (b+1)\,[y_{w(b)}]\,; \\ [l_{w(b)}] &= (-b+1)\,[x_{w(b)}] - b\,[y_{w(b)}]\,.\end{aligned}$$

PROOF. We have $U_L U_R^b U_{\bar{L}} = \begin{pmatrix} -b+1 & b \\ -b & 1+b \end{pmatrix}$. Theorem 3.5 completes the proof. □

THEOREM 3.11. *For the solid torus $V_{w(b)}$ with the meridian-longitude system $(m_{w(b)}, l_{w(b)})$ given by Corollary* 3.10, *the loop $z_{w(b)}$ is the* $(1,1)$*-type.*

PROOF. The following relation holds in $H_1(\partial V_{w(b)})$:

$$\begin{aligned}[z_{w(b)}] &= [y_{w(b)}] + [x_{w(b)}] \\ &= \{b + (-b+1)\}\,[x_{w(b)}] + \{(b+1) + (-b)\}\,[y_{w(b)}] \\ &= b[x_{w(b)}] + (b+1)[y_{w(b)}] + (-b+1)[x_{w(b)}] + (-b)[y_{w(b)}]\end{aligned}$$

$$= [m_{w(b)}] + [l_{w(b)}] .$$

□

NOTATION 3.12. We call $V_{w(b)}$ as the *b-type solid torus* and denote it by $V_{1,b}$.

4. One-vertex triangulation of Lens spaces

In this section, we construct systematically one-vertex triangulations of lens spaces. In § 4.1, we will define a homeomorphism $\varphi = \varphi(w, w') : \partial V_w \to \partial V_{w'}$ for two words w and w', and consider the manifold obtained by gluing two solid tori V_w and $V_{w'}$ by φ, denoted by $V_w \cup_\varphi V_{w'}$. In § 4.2, we will show the following fact: for any pair of coprime natural numbers p and q such that $p > q$, we can choose two words w and w' such that the lens space $L(p, q)$ is homeomorphic to the manifold $V_w \cup_\varphi V_{w'}$. Then, we will consider a one-vertex triangulation of the lens space $L(p, q)$.

4.1. Gluing map φ. For $i = 1, 2$, we denote by U_i the boundary tori $S^1 \times S^1$. Let h_i be an embedding such that $U_i \setminus h_i(\theta) \cong \mathrm{Int}\,(D^2)$, where θ is a theta-curve shown in Figure 1.

LEMMA 4.1. *Two embedding h_1 and h_2 induces a homeomorphism $\varphi_{1,2} : U_1 \to U_2$ such that $\varphi_{1,2}\,(h_1(X)) = h_2(X)$ for any edge $X \in \{\alpha, \beta, \gamma\}$ of θ.*

PROOF. We define the map $\varphi'' : h_1(\theta) \to h_2(\theta)$ by identifying $h_1(X)$ and $h_2(X)$, where $X = \alpha, \beta, \gamma$. Denote by $N(h_i(\theta))$ a regular neighborhood of $h_i(\theta)$ in U_i. Then, the map φ'' can be extended to a homeomorphism $\varphi' : N\,(h_1(\theta)) \to N\,(h_2(\theta))$ such that $\varphi'\,(h_1(X)) = h_2(X)$, where $X = \alpha, \beta, \gamma$. Since $U_i \setminus \varphi_i(\theta) \cong \mathrm{Int}\,(D^2)$, the homeomorphism φ' can be extended to a homeomorphism $\varphi : U_1 \to U_2$ such that $\varphi\,(h_1(X)) = h_2(X)$, where $X = \alpha, \beta, \gamma$. □

Now, we define a gluing map φ of solid torus V_w and $V_{w'}$. Recall that for any solid torus V_w, a theta-curve θ (Figure 1) is embedded in ∂V_w such that $\partial V_w \setminus \theta \cong \mathrm{Int}(D^2)$. Thus, for two words w and w', a homeomorphism $\varphi(w, w') : \partial V_w \to \partial V_{w'}$ is given by Lemma 4.1. So, we define the manifold obtained by gluing V_w and $V_{w'}$ by $\varphi(w, w')$, denoted by $V_w \cup_\varphi V_{w'}$. In the following subsection, we consider the manifold $V_w \cup_\varphi V_{w'}$.

4.2. The manifold obtained by gluing 0 V_w and $V_{w'}$ by φ. Let p, q be a pair of coprime natural numbers such that $p > q$. There are many pairs of two words (w, w') such that $V_w \cup_\varphi V_{w'} \cong L(p, q)$, where $L(p, q)$ is a lens space. Then, the following theorem gives us a pair (w, w'), say $w = \bar{L}$ and $w' = A(p, q)$, where the word $A(p, q)$ is defined in Theorem 3.6.

THEOREM 4.2. *The manifold $V_{\bar{L}} \cup_\varphi V_{A(p,q)}$ is homeomorphic to the lens space $L(p, q)$. In particular, $V_{\bar{L}} \cup_\varphi V_\phi \cong S^3$ and $V_{\bar{L}} \cup_\varphi V_{\bar{L}} \cong S^2 \times S^1$.*

PROOF. For convenience, we denote the word $A(p,q)$ by A. By Theorem 3.5, there are meridian-longitude systems $(m_{\bar{L}}, l_{\bar{L}})$ and (m_A, l_A) of the solid tori $V_{\bar{L}}$ and V_A satisfying the following relations in $H_1(\partial V_L)$ and $H_1(\partial V_A)$ respectively:

$$[m_{\bar{L}}] = -[x_{\bar{L}}] + [y_{\bar{L}}]; \tag{1}$$

$$[m_A] = -A_{1,2}[x_A] + A_{1,1}[y_A]; \tag{2}$$

$$[l_A] = A_{2,2}[x_A] - A_{2,1}[y_A], \tag{3}$$

where $A_{i,j}$ is the (i,j)-element of the matrix $\mathcal{A}^{(p,q)}$, where $\mathcal{A}^{(p,q)}$ is defined in the proof of Theorem 3.6. According to the definition of the gluing map $\varphi : \partial V_{\bar{L}} \to \partial V_A$, we obtain

$$\varphi(x_{\bar{L}}) = x_A; \quad \varphi(y_{\bar{L}}) = y_A. \tag{4}$$

Let $\varphi^{\#} : H_1(\partial V_{\bar{L}}) \to H_1(\partial V_A)$ be the isomorphism induced from the homeomorphism φ. Using the equations (1), (2), (3) and (4), we have the following relation in $H_1(\partial V_A)$.

$$\begin{aligned}[\varphi(m_{\bar{L}})] &= \varphi^{\#}([m_{\bar{L}}]) = \varphi^{\#}([y_{\bar{L}}]) + \varphi^{\#}([x_{\bar{L}}]) = [\varphi(y_{\bar{L}})] + [\varphi(x_{\bar{L}})] \\ &= [y_A] + [x_A] = A_{1,1}[l_A] + A_{2,1}[m_A] + (A_{1,2}[l_A] + A_{2,2}[m_A]) \\ &= (A_{1,1} + A_{1,2})[l_A] + (A_{2,1} + A_{2,2})[m_A].\end{aligned}$$

By Lemma 3.9, two natural numbers $A_{1,1} + A_{1,2}$ and $A_{2,1} + A_{2,2}$ are coprime and

$$\frac{A_{2,1} + A_{2,2}}{A_{1,1} + A_{1,2}} = [a_1,\ a_2, \cdots, a_n,\ 1] = \frac{q}{p}.$$

Thereby, the manifold $V_{\bar{L}} \cup_\varphi V_A$ is homeomorphic to the lens space $L(p,q)$.

At last, we will show $V_{\bar{L}} \cup_\varphi V_\phi \cong S^3$ and $V_{\bar{L}} \cup_\varphi V_{\bar{L}} \cong S^2 \times S^1$. By theorem 3.5, we get $[\varphi(m_{\bar{L}})] = [m_\phi] + [l_\phi]$ and $[\varphi(m_{\bar{L}})] = [m_{\bar{L}}]$. It completes the proof. □

Recall the definition D_w, see § 2.1 and 2.2. Let P be the 2-manifold obtained by gluing $(D_{\bar{L}}/f_{\bar{L}})$ and $(D_{A(p,q)}/f_{A(p,q)})$ with φ. Then, P is a special spine of the lens space $L(p,q) \cong V_{\bar{L}} \cup_\varphi V_{A(p,q)}$, Thus, the dual complex of P in $V_{\bar{L}} \cup_\varphi V_{A(p,q)}$ is a one-vertex triangulation of $L(p,q)$.

5. One-vertex triangulation of Seifert manifolds

In this section, we construct systematically one-vertex triangulations of all orientable Seifert manifolds with orientable base. Our construction is based on the following fact: any orientable Seifert manifold is obtained by gluing the compact manifolds M_n, J and $V_{p,q}$ which are homeomorphic to $(S^2 - \coprod_{i=1}^n \mathrm{Int}\,(D_i^2)) \times S^1$, $(S^1 \times S^1 - \mathrm{Int}(D^2)) \times S^1$ and (p,q)-type fibered solid torus.

In § 5.2 and 5.3, we consider the manifolds J and M_n respectively. In § 5.4, we define the manifold $\mathcal{M}$ obtained by gluing V_{p_i,q_i}, V_b, J and M_n, see Notation 3.7 and 3.12 about V_{p_i,q_i} and V_b, and consider a fiber structure and a one-vertex triangulation of $\mathcal{M}$.

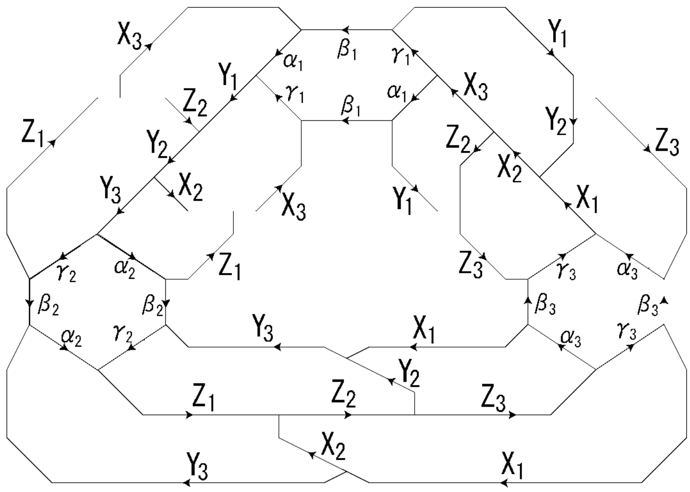

FIGURE 17. $\mathcal{K}$.

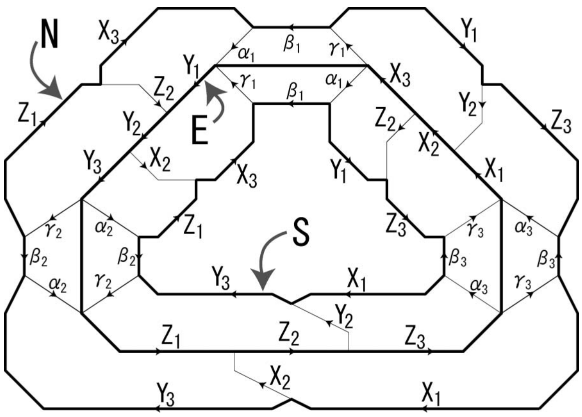

FIGURE 18. $E := \partial B^3 \cap \{z = 0\}$, $S := \partial B^3 \cap \{z = \frac{1}{\sqrt{2}}\}$, $N := \partial B^3 \cap \{z = -\frac{1}{\sqrt{2}}\}$.

5.1. $(S^2 - \coprod_{i=1}^{3} \mathbf{Int}\,(D_i^2)) \times S^1$**.** We use the notation $\mathcal{K}$ for the labeled 3-regular graph shown in Figure 17. Suppose that $\mathcal{K}$ is embedded in ∂B^3. We denote by D_i the 2-disc bounded by the circle $\alpha_i \beta_i \gamma_i \overline{\alpha_i} \overline{\beta_i} \overline{\gamma_i}$ in $S^2 (= \partial B^3)$. Then, by removing D_1, D_2 and D_3 from $S^2 (= \partial B^3)$, we get a 2-sphere with three holes. We denote it by $D_{\mathcal{K}}$. Analyzing the notion of an identification map for a word diagram in § 2, we can consider an identification map $f_{\mathcal{K}}$ on ∂B^3 and a compact 3-manifold $K = B^3 / f_{\mathcal{K}}$.

PROPOSITION 5.1. *The manifold K is homeomorphic to* $(S^2 - \coprod_{i=1}^{3} Int\,(D_i^2)) \times S^1$.

PROOF. Without loss of generality, we assume the following three conditions (a),(b),(c) about the embedding of the graph $\mathcal{K}$ in $S^2(=\partial B^3)$:

(a) $B^3 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 \leq 1\}$;

(b) $\partial B^3 \cap \{z = 0\}$, $\partial B^3 \cap \{z = \frac{1}{\sqrt{2}}\}$ and $\partial B^3 \cap \{z = -\frac{1}{\sqrt{2}}\}$ are the bold circles E, N and S shown in Figure 18 respectively.

We denote by $A_N(a, b)$, $A_S(a, b)$ the arcs in ∂B^3 defined by

$$A_+(a, b) := \left\{(x, y, z) \mid x = \frac{1}{\sqrt{2}}\cos\theta,\ y = \frac{1}{\sqrt{2}}\sin\theta,\ z = \frac{1}{\sqrt{2}},\ a < \theta < b\right\}$$

$$A_-(a, b) := \left\{(x, y, z) \mid x = \frac{1}{\sqrt{2}}\cos\theta,\ y = \frac{1}{\sqrt{2}}\sin\theta,\ z = -\frac{1}{\sqrt{2}},\ a < \theta < b\right\}.$$

(c) The edges in the circles S and N satisfy the following conditions:

$$Y_1 = A_\pm\left(0, \frac{2}{9}\pi\right); \qquad Z_3 = A_\pm\left(\frac{2}{9}\pi, \frac{4}{9}\pi\right); \qquad \beta_3 = A_\pm\left(\frac{4}{9}\pi, \frac{6}{9}\pi\right);$$

$$X_1 = A_\pm\left(\frac{6}{9}\pi, \frac{8}{9}\pi\right); \qquad Y_3 = A_\pm\left(\frac{8}{9}\pi, \frac{10}{9}\pi\right); \qquad \beta_2 = A_\pm\left(\frac{10}{9}\pi, \frac{12}{9}\pi\right);$$

$$Z_1 = A_\pm\left(\frac{12}{9}\pi, \frac{14}{9}\pi\right); \quad X_3 = A_\pm\left(\frac{14}{9}\pi, \frac{16}{9}\pi\right); \quad \beta_1 = E_\pm\left(\frac{16}{9}\pi, 2\pi\right).$$

Consider the flow generated by the vector field $\partial/\partial z$ on B^3. A point a_0 in Figure 19 is mapped by the identification map $f_{\mathcal{K}}$ to a_1. And it is moved by the flow $\partial/\partial z$ and arrives at a_2. After that, it is mapped to a_3 by $f_{\mathcal{K}}$ and it turns back to the same point a by the flow $\partial/\partial z$. Also, any point in Int(A_1) turns back to the same point. Similarly, any point in Int(B_1), Int(C_1), Int(D_1) and Int(E_1) turns back to the same point respectively.

Then, by the method shown in § 4 in [6], we know that $D'_{\mathcal{K}}/f_{\mathcal{K}}$ is the orbit space obtained by $f_{\mathcal{K}}$ and $\partial/\partial z$, see Figure 20, and it is embedded in K. Since $D'_{\mathcal{K}}/f_{\mathcal{K}}$ is homeomorphic to $S^2 - \coprod_{i=1}^3 \mathrm{Int}(D_i^2)$, we get the manifold $(S^2 - \coprod_{i=1}^3 \mathrm{Int}\,(D_i^2)) \times I$ by cutting K along $D'_{\mathcal{K}}/f_{\mathcal{K}}$, Thus, the manifold K is homeomorphic to $(S^2 - \coprod_{i=1}^3 \mathrm{Int}\,(D_i^2)) \times S^1$. □

We define a fiber structure of K by the vector field $\partial/\partial z$. Thus, the loop $\{(x, y, z) \mid x = y = 0,\ -1 \leq z \leq 1\}/f_{\mathcal{K}}$ is a fiber of K. So, the loops $\gamma_k\overline{\alpha_k}$ are homotopic to fibers of K, where $k = 1, 2, 3$.

5.2. $(S^1 \times S^1 - \mathbf{Int}(D^2)) \times S^1$. We consider the manifold $(S^1 \times S^1 - \mathrm{Int}\,(D^2)) \times S^1$. We use the notation $D_{\mathcal{J}}$ for the disc shown in Figure 21. Assume that $D_{\mathcal{J}}$ is embedded in $S^2(=\partial B^3)$. Then, an identification map $f_{\mathcal{J}}$ on $S^2(=\partial B^3)$ is induced by identifying the directed labeled edges of the $D_{\mathcal{J}}$ similar to § 5.1. Then, we consider a compact 3-manifold $J = B^3/f_{\mathcal{J}}$.

PROPOSITION 5.2. *The manifold J is homeomorphic to $(S^1 \times S^1 - \mathit{Int}\,(D^2)) \times S^1$.*

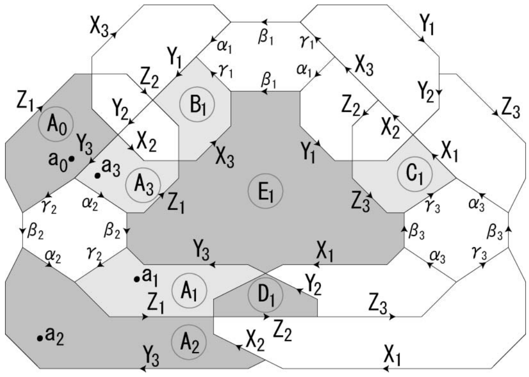

FIGURE 19. A, B, C, D and E.

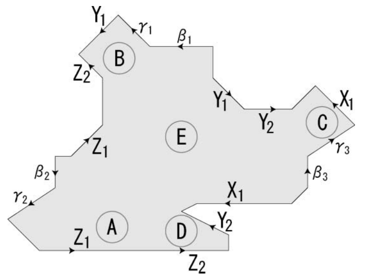

FIGURE 20. $D'_{\mathcal{K}}$.

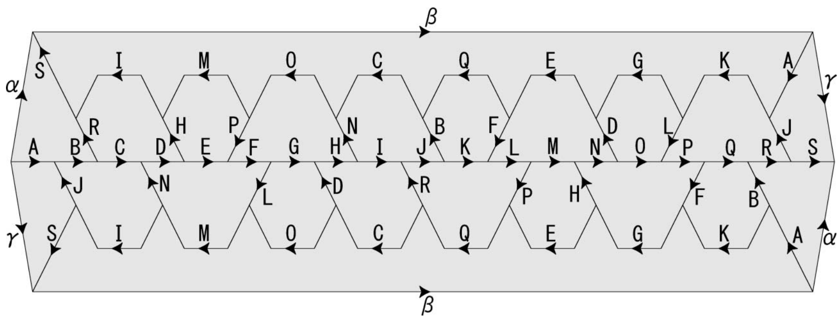

FIGURE 21. $D_{\mathcal{J}}$.

PROOF. The proof is similar to Proposition 5.1. Suppose that the $D_{\mathcal{J}}$ is embedded in the boundary of the unit ball. We consider the orbit space generated by the flow $\partial/\partial z$. Then, we have $D'_{\mathcal{J}}/f_{\mathcal{J}}$ is homeomorphic to $S^1 \times S^1 - \mathrm{Int}\,(D^2)$. Thus, J is homeomorphic to $(S^1 \times S^1 - \mathrm{Int}\,(D^2)) \times S^1$. □

5.3. $(S^2 - \coprod_{i=1}^{n} \mathbf{Int}\,(D_i^2)) \times S^1$**.** For a natural number $n \geq 3$, we define the manifold M_n. We prepare $(n-2)$ - copies $K^{(3)}, K^{(4)}, \cdots, K^{(n)}$ of K, where K is homeomorphic to $(S^2 - \coprod_{i=1}^{3} \mathrm{Int}\,(D_i^2)) \times S^1$ defined in § 5.1. For $j = 3, 4, \cdots, n$, $(C_1^{(j)}, \theta_1)$, $(C_2^{(j)}, \theta_2)$ and $(C_3^{(j)}, \theta_3)$ are pairs of a boundary component $C_i^{(j)}$ of $K^{(j)}$ and the theta-curve θ_i (Figure 22) embedded in $C_i^{(j)}$, where $C_i^{(j)}$ is homeomorphic to $S^1 \times S^1$. Since the theta-curve θ_i is embedded in $C_i^{(j)}$ such that $C_i^{(j)} \setminus \theta_i \cong \mathrm{Int}(D^2)$, we can define homeomorphisms $\varphi^{(j)} : C_3^{(j)} \to C_1^{(j+1)}$ by Lemma 4.1 for $j = 3, 4, \cdots, n-1$. By using $\varphi^{(j)}$, M_n is defined as $K^{(3)} \cup_{\varphi^{(3)}} K^{(4)} \cup_{\varphi^{(4)}} \cdots \cup_{\varphi^{(n-1)}} K^{(n)}$.

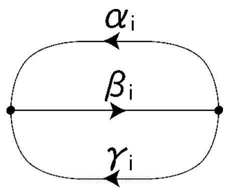

FIGURE 22. θ_i.

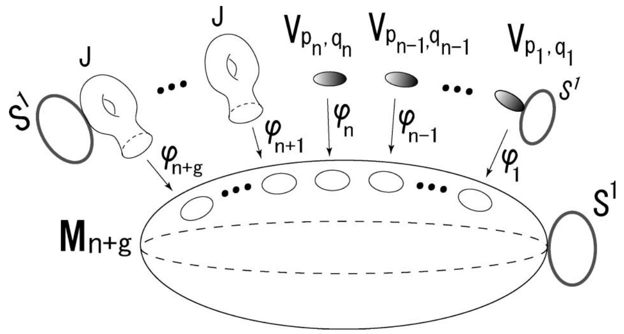

FIGURE 23. Gluing maps $\{\varphi_i\}$.

Note that the homeomorphisms $\varphi^{(j)}$ satisfy the conditions $\varphi^{(j)}(\alpha_3) = \alpha_1$, $\varphi^{(j)}(\beta_3) = \beta_1$ and $\varphi^{(j)}(\gamma_3) = \gamma_1$, where $\theta_i = \alpha_i \cup \beta_i \cup \gamma_i$. Thus, the homeomorphism $\varphi^{(j)}$ is a fiber preserving, and the manifold M_n is homeomorphic to $(S^2 - \coprod_{i=1}^{n} \mathrm{Int}\,(D_i^2)) \times S^1$.

5.4. The manifold obtained by gluing J, V_b, $V_{p,q}$ and M_n. Let g and n be integers such that $g \geq 0$ and $n \geq 3$. Then, we consider the manifold M_{n+g}. As shown in § 5.3, a theta-curve θ_i (Figure 22) is embedded in each boundary component C_i of M_{n+g} such that $C_i \setminus \theta_i \cong \mathrm{Int}\,(D^2)$, where $i = 1, 2, \cdots, n+g$. Recall the manifold J, V_b and $V_{p,q}$ defined in § 5.2, § 3.4 and § 3.3 respectively. The boundary of each manifolds J, V_b and $V_{p,q}$ is homeomorphic to $S^1 \times S^1$ and the theta-curve θ (Figure 1) is embedded in such that $\partial X \setminus \theta \cong \mathrm{Int}(D_i^2)$, where $X = J, V_b, V_{p,q}$. Thus, by Lemma 4.1 there are $n + g$ homeomorphisms $\{\varphi_i\}_{i=1}^{n+g}$

$$\varphi_i := \begin{cases} \partial\, V_{p_i,q_i} \to C_i & (1 \leq i \leq n)\,, \\ \partial\, J \to C_i & (n+1 \leq i \leq n+g)\,, \end{cases}$$

see Figure 23. Thus, we define the manifold obtained by gluing $\coprod_{i=1}^{n} V_{p_i,q_i}$, $\coprod_{i=n+1}^{n+g} J$ and M_{n+g}, denoted by $M(F_g, (p_1, q_1), \cdots, (p_n, q_n))$. Then, we consider fiber structure of it.

1. In the case $1 \leq i \leq n$
   (a) $p_i \neq 1$

Recall V_{p_i,q_i}, see Notation 3.7. By the definition of the gluing map φ_i, we get $\varphi_i^{-1}(\gamma_i\overline{\alpha_i}) = \gamma\overline{\alpha}$. Since the loop $\gamma_i\overline{\alpha_i}$ is a fiber of the manifold M_{n+g}, we regard the loop $\gamma\overline{\alpha}$ in $\partial V_{p_i,q_i}$ as a fiber of the solid torus V_{p_i,q_i}. That is, we decide a fiber structure of V_{p_i,q_i} by the loop $\gamma\overline{\alpha}$. By Theorem 3.6, we have $[\gamma\overline{\alpha}] = p_i[l_i]+q_i[m_i]$ in $H_1(\partial V_{p_i,q_i})$. So, V_{p_i,q_i} is (p_i, q_i)-type fibered solid torus. Thus, the core of the solid torus V_{p_i,q_i} is the (p_i, q_i)-type singular fiber.

(b) $p_i = 1$

Recall $V_{1,b}$, see Notation 3.12. We get the following equation in $H_1(C_i)$ by Corollary 3.10, where C_i is a boundary component of the manifold K_{n+g}.

$$\begin{aligned}
[\varphi_i\,(m_{w(b)})] &= \varphi_i^{\#}\,([m_{w(b)}]) \\
&= \varphi_i^{\#}(b\,[x_{w(b)}] + (b+1)\,[y_{w(b)}]) \\
&= b\,\varphi^{\#}([x_{w(b)}]) + (b+1)\,\varphi^{\#}([\,y_{w(b)}]) \\
&= b\,\varphi^{\#}\,([\overline{\alpha\,\beta}]) + (b+1)\,\varphi^{\#}\,([\gamma\,\beta]) \\
&= b\,\varphi^{\#}\,([\overline{\beta}\;\overline{\gamma}\gamma\overline{\alpha}]) + (b+1)\,\varphi^{\#}\,([\gamma\,\beta]) \\
&= \varphi^{\#}\,([\gamma\,\beta]) + b\,\varphi^{\#}\,([\gamma\,\overline{\alpha}]) \\
&= [\gamma_i\,\beta_i] + b\,[\gamma_i\,\overline{\alpha_i}]\,.
\end{aligned}$$

Recall the following fact shown in the proof of Proposition 5.1: the loop $\gamma_i\beta_i$ is the intersection of C_i and a cross section of M_{n+g}. Since the loop $\gamma_i\bar{\alpha}_i$ is a fiber of M_{n+g}, the core of V_b is a regular fiber corresponding to the obstruction class b.

2. In the case $n+1 \le i \le n+g$

The intersection of cross sections of J and M_{n+g} and ∂J and ∂M_{n+g} are the loops $\beta\gamma$ and $\beta_i\gamma_i$ respectively. Since the conditions $\varphi_i(\gamma\beta) = \gamma_i\beta_i$ holds, the genus of base space of $M(F_g, (p_1, q_1), \cdots, (p_n, q_n))$ is equal to g.

A one-vertex triangulation of $M := M(F_g, (p_1, q_1), \cdots, (p_n, q_n))$ is constructed similar to the lens space in § 4.2. That is, a one-vertex triangulation of M is the dual complex of the 2-manifold obtained by gluing $\coprod_{i=1}^{n} D_{w(p_i,q_i)}/f_{w(p_i,q_i)}$, $\coprod_{i=1}^{g} D_{\mathcal{J}}/f_{\mathcal{J}}$ and $\coprod_{i=3}^{n+g} D_{\mathcal{K}}^{(i)}/f_{\mathcal{K}}$.

EXAMPLE.

1. Quaternionic space

$$S(S^2,\ -1\,;\ (2,1), (2,1), (2,1)) \cong M(F_0,\ (2,1), (2,1), (2,1))$$

2. Brieskorn manifold $\Sigma(2,3,5)$

$$S(S^2,\ -1\,;\ (2,1), (3,1), (5,1)) \cong M(F_0,\ (2,1), (3,1), (5,1))$$

3. $S(S^2,\ -2\,;\ (2,1), (5,3), (7,5)) \cong M(F_0,\ (2,1), (5,2), (7,3))$

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Present Addresses:
TAIJI TANIGUCHI
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY, KEIO UNIVERSITY,
KOHOKU-KU, YOKOHAMA, 223–8522 JAPAN.
e-mal: taiji@math.keio.ac.jp

KEIKO TSUBOI
DEPARTMENT OF MATHEMATICS, DOKKYO SAITAMA HIGH SCHOOL / JUNIOR HIGH SCHOOL,
ONMASHINDEN, KOSHIGAYA, SAITAMA, 343–0037 JAPAN.
e-mal: tsubokei@hotmail.com

MASAKATSU YAMASHITA
DEPARTMENT OF COMPUTATIONAL SCIENCE AND ENGINEERING, FACULTY OF ENGINEERING,
TOYO UNIVERSITY,
KAWAGOE-SHI, SAITAMA, 350–8585 JAPAN.
e-mal: yamasita@eng.toyo.ac.jp