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Braids and Nielsen-Thurston Types of Automorphisms of Punctured Surfaces

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Abstract. Let F be a compact, orientable surface with negative Euler characteristic, and let $x_1, \cdots, x_n$ be n fixed but arbitrarily chosen points on int$F$, each of which has a (small) diskal neighborhood $D_i \subset F$. Denote by $\mathcal{S}_n(F)$ a subgroup of Diff(F) consisting of "sliding" maps f each of which satisfies
(1) $f(\{x_1, \cdots, x_n\}) = \{x_1, \cdots, x_n\}$, $f(D_1 \cup \cdots \cup D_n) = D_1 \cup \cdots \cup D_n$ and
(2) f is isotopic to the identity map on F.
Then by restricting such automorphisms to $\hat{F} = F - \mathrm{int}(D_1 \cup \cdots \cup D_n)$, we have automorphisms $\hat{f} : \hat{F} \to \hat{F}$, which form a subgroup $\mathcal{S}_n(\hat{F})$ of Diff$(\hat{F})$. We give a Nielsen-Thurston classification of elements of $\mathcal{S}_n(\hat{F})$ using braids in $F \times I$ which characterize the elements of $\mathcal{S}_n(\hat{F})$.

1. Introduction

An automorphism (i.e., orientation preserving self diffeomorphism) of a compact, orientable surface with possibly non-empty boundary is said to be *periodic* if its some power is equal to the identity map, and is said to be *reducible* if it leaves an *essential* 1-*submanifold* (i.e., a union of pairwise disjoint simple closed curves such that each curve is homotopically non-trivial and not boundary-parallel, and that no two components are properly homotopic) of the surface invariant.

Suppose that the surface has negative Euler characteristic. It is known by [11], [6], [2] that if an automorphism is isotopic to neither a periodic automorphism nor a reducible automorphism, then it is isotopic to a *pseudo-Anosov* automorphism (i.e., an automorphism leaving singular foliations invariant) and vice versa; for the precise definition of a pseudo-Anosov automorphism, see [11], [6, Exposé 11, see also p. 286], [2]. Thus each automorphism is isotopic to an automorphism with (at least) one of the above three types which we refer to as *Nielsen-Thurston types*.

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Let F be a compact orientable surface with negative Euler characteristic, and let $x_1, \cdots, x_n$ be n fixed but arbitrarily chosen points on int$F$; each x_i has a (small) diskal neighborhood $D_i \subset F$. Let f be an automorphism of F which satisfies:
(1) $f(\{x_1, \cdots, x_n\}) = \{x_1, \cdots, x_n\}$ and $f(D_1 \cup \cdots \cup D_n) = D_1 \cup \cdots \cup D_n$, and
(2) f is isotopic to the identity map on F.

We denote $F - \mathrm{int}(D_1 \cup \cdots \cup D_n)$ by $\hat{F}$ and the restriction of f on $\hat{F}$ by $\hat{f}$. Let Diff(F) be the group of all diffeomorphisms of F. Denote a subgroup of Diff(F) consisting of automorphisms each of which satisfies the above two conditions (1) and (2) by $\mathcal{S}_n(F)$. We denote by $\mathcal{S}_n(\hat{F})$ the subgroup of Diff$(\hat{F})$ each of which is the restriction of $f \in \mathcal{S}_n(F)$.

In [10] Kra gave a classification of $\mathcal{S}_1(\hat{F})$ from a viewpoint of Teichmüller space theory. Recently Imayoshi, Ito and Yamamoto gave a classification for the subgroup of $\mathcal{S}_2(\hat{F})$ consisting of automorphisms leaving each ∂D_i $(i = 1, 2)$ invariant [8]. Very recently, in [9], they announce a classification for the subgroup of $\mathcal{S}_n(\hat{F})$ consisting of automorphisms leaving ∂D_i invariant for each i $(i = 1, \cdots, n)$.

The purpose of this paper is to give a Nielsen-Thurston classification for $\mathcal{S}_n(\hat{F})$ (Theorem 1.2) using purely topological methods. To state the result we need some terminologies.

DEFINITION 1 (braids associated to f). Let f be an automorphism in $\mathcal{S}_n(F)$ and Φ an isotopy from f to the identity map: $\Phi : F \times I \to F \times I$, $\Phi(x, 0) = (f(x), 0)$ and $\Phi(x, 1) = (x, 1)$. Suppose that $x_i = f(x_j)$. Define $t_i^f : I \to F \times I$ as $t_i^f(t) = \Phi(x_j, t)$. We call $t_i^f(I)$ an *i-th string*, which is a monotone arc connecting $(f(x_j), 0) = (x_i, 0)$ and $(x_j, 1)$. Then $t_1^f, \cdots, t_n^f$ define a braid $b^f = (t_1^f(I), \cdots, t_n^f(I);\ F \times I)$ in $F \times I$, which we call a *braid associated to f*. In the following, by abuse of notation, we also use the same symbol to denote a map and its image; for instance t_i^f denotes also the image of t_i^f. We orient each string t_i^f from $t_i^f(0) = (x_i, 0)$ to $t_i^f(1) = (x_j, 1)$.

Denote by $\mathrm{Br}_n(F)$ the set of braids in $F \times I$ such that the i-th string is a monotone arc with endpoints $(x_i, 0)$, $(x_j, 1)$. We say that two braids b and b' in $\mathrm{Br}_n(F)$ are *equivalent* if there is a diffeomorphism G of $F \times I$ level preservingly isotopic to the identity map which is the identity on $F \times \{0, 1\}$ and $G(b) = b'$.

For each automorphism $\hat{f} \in \mathcal{S}_n(\hat{F})$, we have an automorphism $f \in \mathcal{S}_n(F)$, which defines a braid $b^f \in \mathrm{Br}_n(F)$ as in Definition 1. Conversely each braid $b \in \mathrm{Br}_n(F)$ gives an automorphism $f \in \mathcal{S}_n(F)$ such that $b^f = b$, whose restriction $\hat{f}$ belongs to $\mathcal{S}_n(\hat{F})$. Then we establish the following one to one correspondence, which is certainly well-known to specialists and can be found in the literature [1] (when F is closed), see also [10], [9]. For convenience of readers, we will give a sketch of a proof in Appendix.

PROPOSITION 1.1. *The map $\Psi : \mathcal{S}_n(\hat{F}) \to \mathrm{Br}_n(F)$ sending $\hat{f}$ to b^f induces a natural isomorphism $\bar{\Psi} : \mathcal{S}_n(\hat{F})/isotopy \to \mathrm{Br}_n(F)/equivalence$. In particular, the equivalence* *class of the braid b^f does not depend on the choice of an isotopy Φ from f to the identity map.*

DEFINITION 2. Let $b = (t_1, \cdots, t_n;\ F \times I)$ be a braid in $\mathrm{Br}_n(F)$. Assume that each t_i is oriented from $t_i \cap F \times \{0\}$ to $t_i \cap F \times \{1\}$.

(1) b is *trivial* if it is equivalent to the braid $(\{x_1\} \times I, \cdots, \{x_n\} \times I;\ F \times I)$.

(2) A subfamily $\{t_{i_1}, \cdots, t_{i_k}\}$ is *cyclic* if $p((t_{i_1} \cup \cdots \cup t_{i_k}) \cap (F \times \{0\})) = p((t_{i_1} \cup \cdots \cup t_{i_k}) \cap (F \times \{1\}))$ for the natural projection $p : F \times I \to F$ and no proper subfamily of $\{t_{i_1}, \cdots, t_{i_k}\}$ satisfies the above condition. Each cyclic subfamily $\{t_{i_1}, \cdots, t_{i_k}\}$ defines a closed oriented curve c_j on F by taking the product of the oriented paths $p(t_{i_1}), \cdots, p(t_{i_k})$ (each of which has an orientation induced from $t_{i_1}, \cdots, t_{i_k}$ respectively) in a suitable order.

(3) Let $b_1, \cdots, b_m$ be a partition of a braid b into cyclic subfamilies, and let $c_1, \cdots, c_m$ be the corresponding closed curves on F. We call $\mathcal{C} = \{c_1, \cdots, c_m\}$ a *system of closed curves associated to b*.

(4) b is *filling* if the corresponding system of closed curves $\mathcal{C} = \{c_1, \cdots, c_m\}$ is filling (i.e., $c_1 \cup \cdots \cup c_m$ intersects every essential embedded loop on F). If any braid equivalent to b is filling, then b is said to be *stably filling*.

(5) A subset $\{t_{i_1}, \cdots, t_{i_k}\}$ $(k \geq 2)$ is called a *parallel family* of b if there is a level preserving embedding $\eta : D_1^2 \times I \cup \cdots \cup D_{k'}^2 \times I \to F \times I$ such that $\eta(D_1^2 \times I \cup \cdots \cup D_{k'}^2 \times I)$ contains $t_{i_1}, \cdots, t_{i_k}$ and does not intersect any other strings, and $p(\eta(D_1^2 \times \{0\} \cup \cdots \cup D_{k'}^2 \times \{0\})) = p(\eta(D_1^2 \times \{1\} \cup \cdots \cup D_{k'}^2 \times \{1\}))$, see Figure 1 (1).

(6) A subset $\{t_{i_1}, \cdots, t_{i_k}\}$ $(k \geq 1)$ is called a *peripheral family* of b if there is a collar neighborhood N of a component of $(\partial F) \times I$ in $F \times I$ which contains $t_{i_1}, \cdots, t_{i_k}$ and does not intersect any other strings, and $p(N \cap (F \times \{0\})) = p(N \cap (F \times \{1\}))$, see Figure 1 (2).

(7) We say that a subset $\{t_{i_1}, \cdots, t_{i_k}\}$ is a *P-family* if it is either a parallel family or a peripheral family.

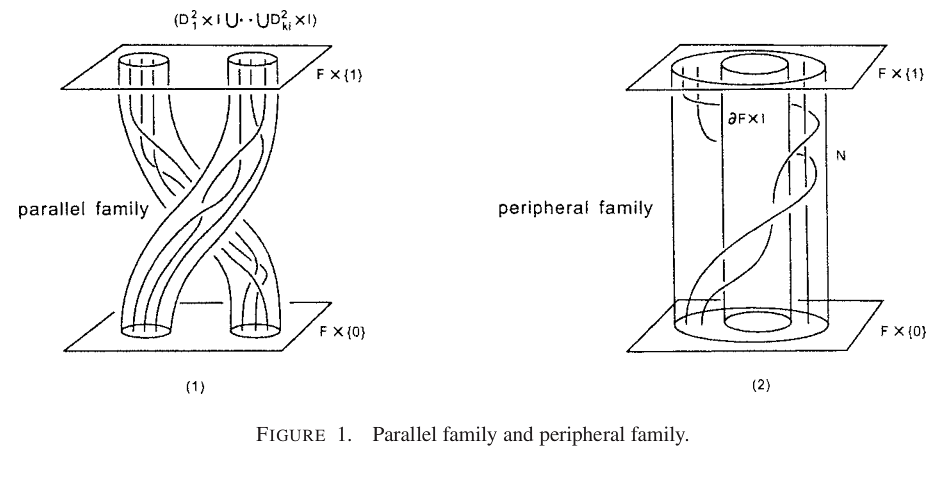

FIGURE 1. Parallel family and peripheral family.

We are ready to state our classification theorem.

THEOREM 1.2 (Nielsen-Thurston types). *Let f be an element in $\mathcal{S}_n(F)$ and b^f an associated braid.*

(i) *$\hat{f}$ is isotopic to a periodic automorphism if and only if the braid b^f is trivial.*

(ii) *$\hat{f}$ is isotopic to a reducible automorphism if and only if the braid b^f has a P-family or is not stably filling.*

(iii) *$\hat{f}$ is isotopic to a pseudo-Anosov automorphism if and only if the braid b^f is stably filling and has no P-families.*

If b^f is trivial, then by Proposition 1.1, $\hat{f}$ is isotopic to the identity map. Hence (i) shows that $\hat{f}$ is isotopic to a periodic automorphism if and only if it is isotopic to the identity map. Since any stably filling braid is necessarily nontrivial, (i) and (ii) imply that there is no periodic automorphism in $\mathcal{S}(\hat{F})$ which is irreducible (i.e. not isotopic to a reducible one). This means that an irreducible automorphism in $\mathcal{S}(\hat{F})$ is isotopic to a pseudo-Anosov automorphism. Thus (ii) can be rephrased as (iii).

Let f be an element in $\mathcal{S}_n(F)$ and b^f an associated braid. We call a system of closed curves associated to b^f *a system of closed curves associated to* f, and denote it by $\mathcal{C}^f$.

We say that two systems of closed curves $\mathcal{C} = \{c_1, \cdots, c_m\}$ and $\mathcal{C}' = \{c'_1, \cdots, c'_m\}$ are *equivalent* if c_i is homotopic to c'_i (as closed curves) for $i = 1, \cdots, m$.

DEFINITION 3. Let $\mathcal{C} = \{c_1, \cdots, c_m\}$ be a system of closed curves.

(1) $\mathcal{C}$ is *stably filling* if any system $\mathcal{C}'$ equivalent to $\mathcal{C}$ is filling.

(2) $\mathcal{C}$ has *property* $(*)$ if it is stably filling and satisfies (i) every c_i is *primitive* (i.e., c_i is not freely homotopic to a closed curve c^p with $p \geq 2$), (ii) c_i and c_j are not freely homotopic (as closed curves) for $i \neq j$, and (iii) c_i cannot be homotoped into ∂F.

In terms of systems of closed curves associated to automorphisms, we have the following result.

COROLLARY 1.3. *Let f be an element in $\mathcal{S}_n(F)$ and $\mathcal{C}^f = \{c_1^f, \cdots, c_m^f\}$ an associated system of closed curves. If the system $\mathcal{C}^f$ has property* $(*)$, *then $\hat{f}$ is isotopic to a pseudo-Anosov automorphism*, *in particular $\hat{f}$ is irreducible.*

REMARK. (a) If f fixes $x_1, \cdots, x_n$ *pointwisely*, then we do not need the condition (i) in property $(*)$ (see the proof of Claim 4.1). In particular, if $n = 1$, then the 1-string braid b^f has no parallel families and the definition of property $(*)$ is simplified to require that $\{c_1\}$ is stably filling and c_1 cannot be homotoped into ∂F. (b) The converse of Corollary 1.3 does not hold if $n \geq 2$. In the case where $n = 1$, adopting the above refinement of property $(*)$, the converse is also true ([10]).

We conclude the introduction with some applications of Corollary 1.3.

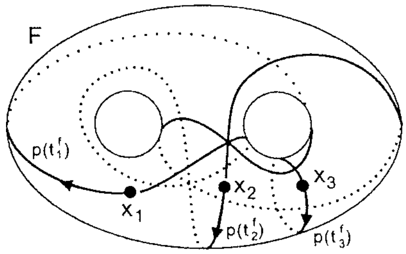

FIGURE 2.

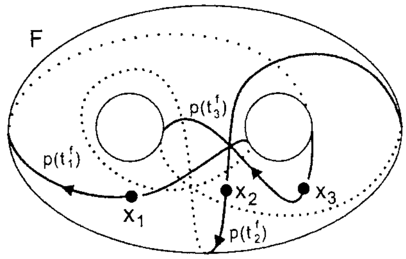

FIGURE 3.

EXAMPLE 1. Let f be an automorphism in $\mathcal{S}_3(F)$ such that $f(x_1) = x_2$, $f(x_2) = x_1$, $f(x_3) = x_3$ and $\mathcal{C}^f$ is given by Figure 2. Then $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$. Note that the automorphism f with the the given system of closed curves $\mathcal{C}^f$ below is not unique, but for each f, $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$.

In fact, by Corollary 1.3, it is sufficient to show that $\mathcal{C}^f = \{c_1, c_2\}$ has property $(*)$, where $c_1 = p(t_1^f) * p(t_2^f)$ and $c_2 = p(t_3^f)$. It is straightforward to check that $\mathcal{C}^f$ satisfies (i), (ii) and (iii). To show that it is stably filling, we first find a hyperbolic structure on F such that the curve c_1 is realized as a closed geodesic. In fact this can be done by decomposing F into a pair of pants. Then it is known that a closed geodesic on a closed hyperbolic surface which cuts the surface into open disks is stably filling. Since $F - c_1$ consists of open disks, $\{c_1\}$ is stably filling, and is also $\{c_1, c_2\}$. This fact can be also checked by [7], in which Hass and Scott gave a combinatorial criteria showing the given system of closed curves are stably filling.

EXAMPLE 2. Let f be an automorphism in $\mathcal{S}_3(F)$ such that $f(x_1) = x_3$, $f(x_2) = x_1$, $f(x_3) = x_2$ and $\mathcal{C}^f$ is given by Figure 3. Then the same argument as above shows that $\mathcal{C}^f = \{c_1\}$ ($c_1 = p(t_1^f) * p(t_2^f) * p(t_3^f)$) has property $(*)$ and $\hat{f}$ is isotopic to a pseudo-Anosov automorphism in $\mathcal{S}_3(\hat{F})$.

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2. Isotopies of essential circles on a surface

In this section we will prove the following result which implies that an isotopy sending a family of circles on a surface F to themselves is essentially unique if F has negative Euler characteristic.

Let F be a compact, orientable surface of negative Euler characteristic and $a_1, \cdots, a_k$ mutually isotopic, pairwise disjoint essential circles on F. Let $A_1, \cdots, A_k$ be pairwise disjoint, monotone (meaning no local maxima and minima) annuli in $F \times I$ such that $p(\partial(A_1 \cup \cdots \cup A_k)) = a_1 \cup \cdots \cup a_k$. Then a map $\sigma : \{1, \cdots, k\} \to \{1, \cdots, k\}$ is determined so that A_i connects $a_{i,0} = a_i \times \{0\}$ and $a_{\sigma(i),1} = a_{\sigma(i)} \times \{1\}$. $A_1 \cup \cdots \cup A_k$ corresponds to an isotopy sending $a_1 \cup \cdots \cup a_k$ to itself. Then we have:

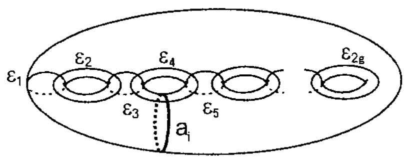
LEMMA 2.1. (1) *$\sigma(i) = i$ for $i = 1, \cdots, k$, i.e., $\partial A_i = a_i \times \{0, 1\}$, and*

(2) *A_i can be isotoped to a vertical annulus $a_i \times I$ by a level preserving isotopy which is the identity on $F \times \{0, 1\}$.*

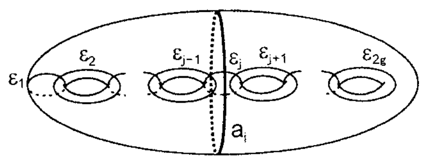
PROOF. First we suppose that F is a closed surface of genus g. Choose a family of $2g$ essential simple closed curves $\varepsilon_1, \cdots, \varepsilon_{2g}$ on F as in Figure 4; $\cup_{k=1}^{2g} \varepsilon_k$ cuts F into a single disk and a_i is homologous to none of $\varepsilon_1, \cdots, \varepsilon_{2g}$. Without loss of generality, we may assume that the curve a_i is precisely as in Figure 4 (1) or (2) depending on whether a_i is non-separating or separating: $a_i \cap \varepsilon_4 = \{z_i\}$, $a_i \cap \varepsilon_j = \{z_i, z_i'\}$. In fact, for a given essential simple loop a_i on F, there is a diffeomorphism $h : F \to F$ sending a_i to the curve as in Figure 4 (1) or (2). Then we have the required situation by applying $h\times$ id. : $F \times I \to F \times I$.

In the following we may relabel the indices and orient a_i so that $a_1, \cdots, a_k$ are homotopic as oriented curves, and if a_i is separating, then a_i intersects ε_j at z_i and z_i' with opposite directions and $(a_1 \cup \cdots \cup a_k) \cap \varepsilon_j$ appears $z_k, \cdots, z_2, z_1, z_1', z_2', \cdots, z_k'$ in circular ordering on ε_j.

Let E_k be the vertical annulus $p^{-1}(\varepsilon_k)$ for $1 \leq k \leq 2g$.



(1) a_i is non-separating



(2) a_i is separating

FIGURE 4.

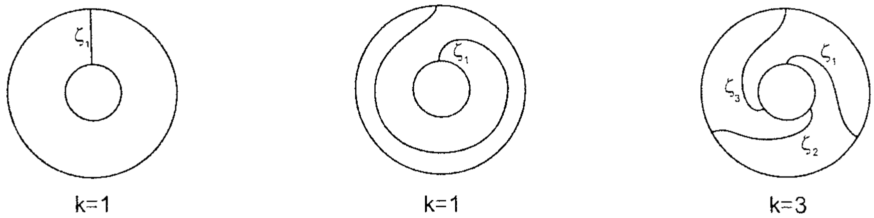


FIGURE 5.

Since a_i is essential, A_i is incompressible. Thus we may assume, by a level preserving isotopy fixing $F \times \{0, 1\}$, that each A_i intersects E_4 (resp. E_j) transversely and that each component of $A_i \cap E_4$ (resp. $A_i \cap E_j$) does not bound a disk in E_4 (resp. E_j). Note that each level preserving isotopy keeps A_i monotone.

We first consider the case where a_i is non-separating.

CLAIM 2.2. *$A_i \cap E_4$ consists of an arc ζ_i isotopic to a vertical segment by a level preserving isotopy leaving its boundary invariant.*

PROOF. Since $a_i \cap \varepsilon_4 = \{z_i\}$, there is no boundary-parallel arc in E_4. Hence $(A_1 \cup \cdots \cup A_k) \cap E_4$ consists of essential monotone arcs, say as in Figure 5.

Take a subfamily $A_1, A_{\sigma(1)}, \cdots, A_{\sigma^{j-1}(1)}$ of the annuli such that $j \in \{1, \cdots, k\}$ satisfies $\sigma^j(1) = 1$ and no proper subfamily satisfies this property. Let T' be a torus obtained from $A_1 \cup A_{\sigma(1)} \cup \cdots \cup A_{\sigma^{j-1}(1)}$ by identifying their boundaries via the identification $(x, 0) = (x, 1)$. Then there is a map p' such that the following diagram commutes:

$$\begin{tikzcd}
A_1 \cup A_{\sigma(1)} \cup \cdots \cup A_{\sigma^{j-1}(1)} \arrow[r] \arrow[d, "p"'] & T' \arrow[dl, "p'"] \\
F &
\end{tikzcd}$$

Connecting the arcs $\zeta_1, \zeta_{\sigma(1)}, \cdots, \zeta_{\sigma^{j-1}(1)}$ in a suitable order, we obtain an essential loop α on T', which satisfies $p'_*([\alpha]) = [\varepsilon_4]^m \in \pi_1(F, z_1)$ for some integer m. Assume for a contradiction that $m > 0$. Then $p'_*([\alpha])$ is nontrivial. The essential loop $a_{1,0}$ also gives an essential loop β on T'. Note that since $[\alpha][\beta] = [\beta][\alpha] \in \pi_1(T')$, $p'_*([\alpha])p'_*([\beta]) = p'_*([\beta])p'_*([\alpha])$ in $\pi_1(F, z_1)$. Furthermore, since $|\varepsilon_4 \cap a_1| = 1$, $p'_*([\alpha])$ and $p'_*([\beta])$ generate a rank two free abelian subgroup in $\pi_1(F, z_1)$. This contradicts that the genus of F is greater than one.

It follows that $m = 0$, hence $p(\partial\zeta_i) = z_i$ and we can isotope A_i by a level preserving isotopy of $F \times I$ fixing $F \times \{0, 1\}$ so that $A_i \cap E_4$ consists of a single vertical segment. □(Claim 2.2)

This claim implies the first assertion of Lemma 2.1 in the case where a_i is non-separating.

Now let us show that A_i can be isotoped to the vertical annulus as required. Since $E_3 \cap E_4$, $E_4 \cap E_5$ (if $g > 2$) and $A_i \cap E_4$ consist of a vertical segment respectively, we can isotope without changing $A_i \cap E_4$ so that $A_i \cap E_3 = \emptyset$ and $A_i \cap E_5 = \emptyset$ (if $g > 2$); here we use also a fact that $a_i \cap \varepsilon_3 = \emptyset$, $a_i \cap \varepsilon_5 = \emptyset$ (if $g > 2$) and an incompressibility of A_i. For other E_s ($s \neq 3, 4, 5$), since $a_i \cap \varepsilon_s = \emptyset$ and A_i is incompressible, we can isotope further by a level preserving isotopy fixing $F \times \{0, 1\}$ so that $A_i \cap E_s$ is empty or consists of essential circles in E_s; each circle is also essential in A_i because E_s is incompressible. In the latter case a_i is homotopic to ε_s, a contradiction. Since A_i intersects only E_4 or E_j in vertical segments and $E_1 \cup \cdots \cup E_{2g}$ cuts $F \times I$ into a [disk]$\times I$, we can isotope A_i to the vertical annulus by a level preserving isotopy as desired.

Next we consider the case where a_i is separating.

In this case, $A_i \cap E_j$ consists of two properly embedded arcs ζ_i and ζ_i' in E_j.

CLAIM 2.3. *$\partial\zeta_i = \{(z_i, 0), (z_i, 1)\}$, and hence $\partial\zeta_i' = \{(z_i', 0), (z_i', 1)\}$.*

PROOF. If ζ_i is boundary-parallel arc, then since A_i is boundary-incompressible and $F \times \{0, 1\}$ are incompressible, there should be a bigon $D \subset F$ with $\partial D = d_1 \cup d_2$ such that $d_1 \subset a_i$ and $d_2 \subset \varepsilon_j$. This is impossible, see Figure 4 (2). Thus the arcs ζ_i's define a bijection τ on $\{z_k, \cdots, z_1, z_1', \cdots, z_k'\}$ so that ζ_i connects the points $(z_i, 0)$ and $(\tau(z_i), 1)$, for otherwise, there must be boundary parallel arcs as illustrated in Figure 6 (1). In fact, since A_i connects $a_i \times \{0\}$ and $a_{\sigma(i)} \times \{1\}$, ζ_i connects $(z_i, 0)$ and $(z_{\sigma(i)}, 1)$ or $(z'_{\sigma(i)}, 1)$; $\tau(z_i)$ equals $z_{\sigma(i)}$ or $z'_{\sigma(i)}$. Remark that since ζ_i' is also a component of $A_i \cap E_j$, if $\tau(z_i) = z_{\sigma(i)}$ (resp. $\tau(z_i) = z'_{\sigma(i)}$), ζ_i' connects the points $(z_i', 0)$ and $(z'_{\sigma(i)}, 1)$ (resp. $(z_{\sigma(i)}, 1)$).

Let us show that $\tau(z_i) = z_i$. Suppose to the contrary that $\tau(z_i) \neq z_i$ for some i. If $\tau(z_i) = z_{\sigma(i)}$, say as in Figure 6 (1) in which $i = 1$ and $\sigma(i) = 2$, then there would be a boundary-parallel arc in $(A_1 \cup \cdots \cup A_k) \cap E_j$, a contradiction. If $\tau(z_i) = z'_{\sigma(i)}$, say as in Figure 6 (2) in which $i = 2$ and $\sigma(i) = 2$, then sliding the oriented closed curve $a_{i,0}$ along the annulus A_i to obtain an oriented closed curve $a_{\sigma(i),1}$. Then since $a_{\sigma(i),0}$ is orientedly

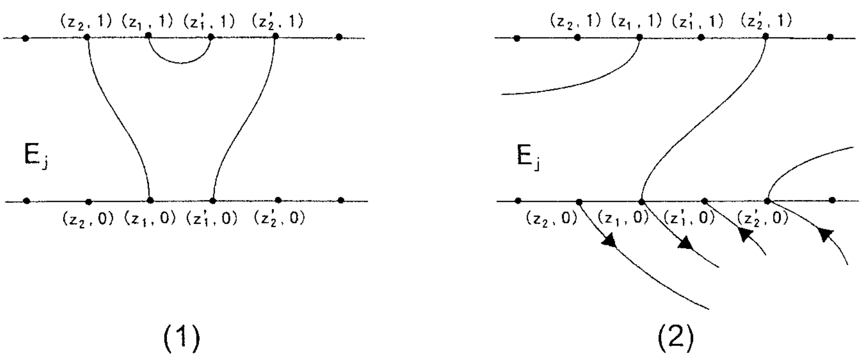


FIGURE 6.

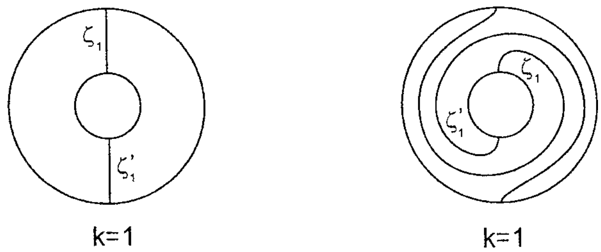


FIGURE 7.

homotopic to $a_{i,0}$, $p(a_{\sigma(i),1})$ and $p(a_{\sigma(i),0})$ have opposite orientations. This implies that $a_{\sigma(i)}$ and $\overline{a_{\sigma(i)}}$ (the closed curve obtained from $a_{\sigma(i)}$ by inverting its orientation) are freely homotopic in F, hence F would be non-orientable, a contradiction. □(Claim 2.3)

Thus we have a situation, say as in Figure 7.

This observation implies the first assertion of Lemma 2.1 in the case where a_i is separating.

Let us show that A_i is also isotoped to the vertical annulus as required in this case. By using the same argument in the proof of Claim 2.2 for a torus T' obtained from single A_i, ζ_i and ζ_i' are shown to be isotopic to vertical segments by a level preserving isotopy leaving their boundaries invariant. Then, as in the above, we can isotope A_i (fixing $F \times \{0, 1\}$) so that $A \cap E_s = \emptyset$ $(s \neq j)$, thus we can isotope A_i to the vertical annulus by a level preserving isotopy as desired.

Finally suppose that F has genus g and d boundary components. We can find a system of properly embedded arcs $\{\varepsilon_1, \cdots, \varepsilon_{2g}, \delta_1, \cdots, \delta_{d-1}\}$ so that they cut F into a single disk. Then the result follows by applying the same argument as above. (The proof is easier, because $p^{-1}(\varepsilon_j)$ and $p^{-1}(\delta_k)$ is a rectangle, not an annulus.) □(Lemma 2.1)

3. Proof of Theorem 1.2

Let f be an element in $\mathcal{S}_n(F)$ and b^f an associated braid.

3.1. Proof of (i). This is certainly well-known, but for completeness, we give a proof. If b^f is trivial, then $\hat{f}$ is isotopic to the identity map, which has period 1. Conversely if $\hat{f}$ is isotopic to a periodic automorphism, then by Proposition 1.1, b^f has a finite order in the braid group. If b^f is nontrivial, then [5, Theorem 8] shows that F would be S^2 or the projective plane $\mathbb{R}P^2$, contradicting our assumption. Thus b^f is trivial.

3.2. Proof of the "only if" part of (ii). We show that if $\hat{f}$ is isotopic to a reducible automorphism, then the braid b^f has a P-family or is not stably filling.

Assume that $\hat{f}$ is isotopic to a reducible automorphism. Then there is an essential 1-submanifold $C = a_1 \cup \cdots \cup a_m \subset F$ such that $f(C)$ is isotopic to C on $\hat{F}$. In the following, we assume that $f(a_{k_i})$ is isotopic to a_i, i.e., a_{k_i} is isotopic to $f^{-1}(a_i)$ $(i = 1, \cdots, m)$ on $\hat{F}$.

The isotopy from $f^{-1}(a_i)$ to a_{k_i} $(1 \leq i \leq m)$ on $\hat{F}$ is realized as a family of monotone annuli $\tilde{A}_1, \cdots, \tilde{A}_m$ in $\hat{F} \times I \subset F \times I$ so that $\partial \tilde{A}_1 = (f^{-1}(a_1) \times \{0\}) \cup (a_{k_1} \times \{1\}), \cdots, \partial \tilde{A}_m = (f^{-1}(a_m) \times \{0\}) \cup (a_{k_m} \times \{1\})$. Note that $\tilde{A}_i \cap (\{x_j\} \times I) = \emptyset$ for $i = 1, \cdots, m$, $j = 1, \cdots, n$. Since f is isotopic to the identity on F, we have a level preserving diffeomorphism of $F \times I$ sending $(x, 0)$ to $(f(x), 0)$ and $(x, 1)$ to $(x, 1)$, which deforms also the vertical segment $\{x_j\} \times I$ to a monotone arc t_i^f with $\partial t_i^f = \{(x_i, 0), (x_j, 1)\}$, where $x_i = f(x_j)$. Then $t_1^f, \cdots, t_n^f$ define a braid b^f in $F \times I$ (see, Definition 1). Simultaneously, the annuli $\tilde{A}_1, \cdots, \tilde{A}_m$ are also deformed to a family of monotone annuli $A_1, \cdots, A_m$ in $F \times I$, each of which is disjoint from the braid b^f and satisfies that $\partial A_i = (a_i \times \{0\}) \cup (a_{k_i} \times \{1\})$. Let us choose annuli $A_1, \cdots, A_k$ (after changing their indices if necessary) so that $p(\partial(A_1 \cup \cdots \cup A_k) \cap (F \times \{0\})) = p(\partial(A_1 \cup \cdots \cup A_k) \cap (F \times \{1\}))$ and no proper subset satisfy this property.

Consider the case where a_i bounds a disk D_i on F. Then since C is an essential 1-submanifold on $\hat{F}$, D_i contains at least two points of $\{x_1, \cdots, x_n\}$. Then for each i $(1 \leq i \leq k)$, $\partial A_i \cap (F \times \{0\})$ bounds a disk $D_{i,0} \subset F \times \{0\}$ and $\partial A_i \cap (F \times \{1\})$ bounds a disk $D_{i,1} \subset F \times \{1\}$. By the irreducibility of $F \times I$, the 2-sphere $A_i \cup D_{i,0} \cup D_{i,1}$ bounds a 3-ball B_i. It turns out that each B_i contains m strings in b^f for some integer $m \geq 2$ independent of i. The collection of strings in b^f each of which is contained in $B_1 \cup \cdots \cup B_k$ give a parallel family in this case.

Next consider the case where a_i does not bound a disk D_i on F. This implies that A_i is incompressible in $F \times I$. Then Lemma 2.1 (1) implies that $k = 1$ and Lemma 2.1 (2) shows that A_1 can be isotoped to the vertical annulus $a_1 \times I$ by a level preserving isotopy fixing $F \times \{0, 1\}$. Under this level preserving isotopy, the braid $b^f = (t_1^f, \cdots, t_n^f;\ F \times I)$ is also isotoped to another braid $b' = (t'_1, \cdots, t'_n;\ F \times I)$ (without moving their endpoints), which is equivalent to b^f; they define equivalent systems of closed curves. Since the annulus A_1 is disjoint from b^f, $a_1 \times I$ does not intersect b' neither, and hence $a_1 \cap (\cup_{i=1}^n p(t'_i)) = p(a_1 \times I) \cap p(b') = p((a_1 \times I) \cap b') = p(\emptyset) = \emptyset$. Therefore if a_1 is essential in F, equivalently, if a_1 is not parallel to a component of ∂F, then b' is not filling. Thus, by definition, b^f is not stably filling.

If a_1 is parallel to a component of ∂F, then the parallelism must contain some specified points x_i, since C is an essential 1-submanifold. Thus $a_1 \times I$, and hence A_1, is the frontier of a collar neighborhood $N(\cong S^1 \times I \times I)$ of a component of $(\partial F) \times I$. Since $N(\cong S^1 \times I \times I)$ contains some strings t_i^f, they give a peripheral family as we desired.

3.3. Proof of the "if" part of (ii). Suppose that the braid b^f has a P-family or is not stably filling. Then by definition, (1) b^f has a parallel family $\{t_{i_1}^f, \cdots, t_{i_k}^f\}$ or (2) b^f has a peripheral family $\{t_{i_1}^f, \cdots, t_{i_k}^f\}$, or (3) b^f is equivalent to a braid $b' = (t_1', \cdots, t_n';\ F \times I)$ such that $p(t_1') \cup \cdots \cup p(t_n')$ does not intersect an essential embedded loop a.

In each case, $C = p(\eta(\partial(D_1^2 \times \{0\} \cup \cdots \cup D_m^2 \times \{0\})))$, the frontier of $p(N \cap (F \times \{0\}))$ in F, or the embedded loop a is an essential 1-submanifold which is isotopic to the image of f on $\hat{F}$. This means that $\hat{f}$ is isotopic to a reducible automorphism.

4. Proof of Corollary 1.3

Corollary 1.3 follows immediately from Theorem 1.2 (iii) and the claim below.

CLAIM 4.1. *If the system of closed curves $\mathcal{C}^f = \{c_1^f, \cdots, c_m^f\}$ has property $(*)$, then the braid $b^f = (t_1^f, \cdots, t_n^f;\ F \times I)$ is stably filling and has no P-families.*

PROOF. Suppose that we have a parallel family $\{t_{i_1}, \cdots, t_{i_k}\}$ $(k \geq 2)$, which consists of some cyclic subfamilies. The cyclic families give a subsystem of closed curves in $\mathcal{C}^f$. Since $k \geq 2$, the subsystem contains a closed curve homotopic to a nontrivial power of a closed curve or a pair of mutually homotopic closed curves. (If f fixes $x_1, \cdots, x_n$ pointwisely, i.e., b^f is a pure braid, then we have the latter possibility.) This contradicts the assumption. If we have a peripheral family $\{t_{i_1}, \cdots, t_{i_k}\}$, then clearly c_i^f is homotoped into a component of ∂F, contradicting the assumption.

Let $b' = (t_1', \cdots, t_n';\ F \times I)$ be a braid equivalent to b^f. Then the system of closed curves $\mathcal{C}'$ corresponding to b' is equivalent to $\mathcal{C}^f$. Since $\mathcal{C}^f$ is stably filling, by definition, $\mathcal{C}'$ is filling. □(Claim 4.1)

Appendix. A sketch of a proof of Proposition 1.1

Here we give a sketch of a proof of Proposition 1.1, for details, see [1].

Let $\mathbf{B}_n(F)$ be the subgroup of Diff(F) consisting of automorphisms f satisfying $f(\{x_1, \cdots, x_n\}) = \{x_1, \cdots, x_n\}$. Let $\mathrm{C}_n(F)$ denote the space of n-tuples $(z_1, \cdots, z_n)$ of distinct points in int$F$ and $\mathrm{B}_n(F)$ the quotient space of $\mathrm{C}_n(F)$ by the symmetry group Σ_n. Then we have an *evaluation map* $\varepsilon_n : \mathrm{Diff}(F) \to \mathrm{B}_n(F)$ defined by the rule $\varepsilon_n(f) = (f(x_1), \cdots, f(x_n))$, which is a fibration with fiber $\mathbf{B}_n(F)$. This fibration gives the following homotopy exact sequence, in which Diff$_0(F)$ denotes the identity component of Diff(F).

$$\to \pi_1(\mathrm{Diff}_0(F)) \xrightarrow{\varepsilon_*} \pi_1(\mathrm{B}_n(F)) \xrightarrow{d_*} \pi_0(\mathbf{B}_n(F)) \xrightarrow{i_*} \pi_0(\mathrm{Diff}(F)) \to \pi_0(\mathrm{B}_n(F)) = \{1\}.$$

Since F has negative Euler characteristic, Diff$_0(F)$ is contractible ([3], [4]), and hence Ker$d_* =$ Im$\varepsilon_* = \{1\}$. Thus d_* is injective. It follows that the connecting homomorphism d_* is an isomorphism between Keri_* and $\pi_1(\mathrm{B}_n(F))$, which is identified with the (full) braid group $\mathrm{Br}_n(F)/equivalence$.

It is obvious that $\mathrm{Ker} i_*$ is just the quotient of the group consisting of automorphisms f of F satisfying $f(\{x_1, \cdots, x_n\}) = \{x_1, \cdots, x_n\}$ and isotopic to the identity on F by isotopies keeping x_i $(i = 1, \cdots, n)$ invariant. This subgroup is naturally isomorphic to the quotient of $\mathcal{S}_n(F)$ by isotopies keeping D_i and x_i $(i = 1, \cdots, n)$ invariant, which is also isomorphic to $\mathcal{S}_n(\hat{F})/isotopy$.

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