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The Gauss-Bonnet and Chern-Lashof Theorems in a Simply Connected Symmetric Space of Compact Type

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Abstract. In this paper, we prove the theorems of the Gauss-Bonnet and Chern-Lashof types for low dimensional compact submanifolds in a simply connected symmetric space of compact type. In particular, in the case where the ambient space is a sphere, we need not to give the restriction for the dimension of the submanifold. Those proofs are performed by applying the Morse theory to squared distance functions.

1. Introduction

For an n-dimensional compact immersed submanifold M in the m-dimensional Euclidean space $\mathbf{R}^m$ $(m > n)$, it is well-known that the following Gauss-Bonnet and Chern-Lashof theorems hold:

$$\frac{(-1)^n}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}\det A_\xi\,\omega_{U^\perp M} = \chi(M)\,, \tag{1.1}$$

$$\frac{1}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^\perp M}|\det A_\xi|\,\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F}) \tag{1.2}$$

(see [1], [2], [3]), where $\mathrm{Vol}(S^{m-1}(1))$ is the volume of the $(m-1)$-dimensional unit sphere, A is the shape tensor of M, $\omega_{U^\perp M}$ is the standard volume element on the unit normal bundle $U^\perp M$ of M, $\chi(M)$ is the Euler characteristic of M and $b_k(M,\mathbf{F})$ is the k-th Betti number of M with respect to an arbitrary coefficient field $\mathbf{F}$. These relations are proved by applying the Morse theory to height functions h_v $(v \in \mathbf{R}^m)$. The topology of a submanifold in a general complete and simply connected Riemannian manifold should be determined by both the extrinsic curvature A of the submanifold and the curvature R of the ambient space. So we [5] proposed the following problem:

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PROBLEM. *Find functions $F^i_{A,R}$ $(i=1,2)$ on $U^\perp M$ determined by both A and R such that*

$$\int_{\xi\in U^\perp M} F^1_{A,R}(\xi)\omega_{U^\perp M} = \chi(M)$$

and

$$\int_{\xi\in U^\perp M} F^2_{A,R}(\xi)\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F})$$

hold for each n-dimensional compact immersed submanifold M in an arbitrary complete and simply connected Riemannian manifold N.

It is conjectured that the functions $F^i_{A,R}$ $(i=1,2)$ are rather complex. Hence we will obtain the equality and the inequality for practical use in some special cases. By applying the Morse theory to squared distance functions, we [5] proved the theorems of Gauss-Bonnet and Chern-Lashof types for a compact immersed submanifold in a simply connected symmetric space of non-positive curvature. As conjectured, the functions corresponding to $F^i_{A,R}$ $(i=1,2)$ were rather complex. In this paper, we prove the theorems of such types for a low dimensional compact immersed submanifold M in a simply connected symmetric space $N=G/K$ of compact type. We prepare to state those theorems. Define the functions $\mathfrak{K}_1$ and $\mathfrak{K}_2$ on $U^\perp M$ as follows:

$$\mathfrak{K}_1(\xi) := \frac{1}{\mathrm{Vol}(N)}\int_0^{r_\xi} \det\left(\mathrm{pr}_T\circ\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})} - A_\xi\right)\det\frac{\sin(s\sqrt{R_\xi})}{\sqrt{R_\xi}}ds\,, \tag{1.3}$$

$$\mathfrak{K}_2(\xi) := \frac{1}{\mathrm{Vol}(N)}\int_0^{r_\xi} \left|\det(\mathrm{pr}_T\circ\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})} - A_\xi)\right|\det\frac{\sin(s\sqrt{R_\xi})}{\sqrt{R_\xi}}ds \tag{1.4}$$

$(\xi\in U^\perp M)$, where A is the shape tensor of M, $\sqrt{R_\xi}$ is the positive operator with $(\sqrt{R_\xi})^2 = R(\cdot,\xi)\xi$ (R : the curvature tensor of N), r_ξ is the first conjugate radius of direction ξ, $\mathrm{Vol}(N)$ is the volume of N and pr_T is the orthogonal projection of $TN|_M$ onto TM.

REMARK 1.1. (i) In case of $N=S^m(c)$ (the m-dimensional sphere of constant curvature c), we have

$$\mathfrak{K}_1(\xi) := \frac{1}{\mathrm{Vol}(S^m(1))}\int_0^{\pi} \det(\sqrt{c}\cot s\cdot\mathrm{id} - A_\xi)\sin^{m-1} s ds\,, \tag{1.5}$$

$$\mathfrak{K}_2(\xi) := \frac{1}{\mathrm{Vol}(S^m(1))}\int_0^{\pi} |\det(\sqrt{c}\cot s\cdot\mathrm{id} - A_\xi)|\sin^{m-1} s ds\,, \tag{1.6}$$

where id is the identity transformation of TM. Substituting $c = 0$ into (1.5) (resp. (1.6)) formally and using $\int_0^\pi \sin^{m-1} s ds = \frac{\mathrm{Vol}(S^m(1))}{\mathrm{Vol}(S^{m-1}(1))}$, we have $\mathfrak{K}_1(\xi) = \frac{(-1)^n}{\mathrm{Vol}(S^{m-1}(1))}\det A_\xi$ (resp. $\mathfrak{K}_2(\xi) = \frac{1}{\mathrm{Vol}(S^{m-1}(1))}|\det A_\xi|$).

(ii) The first conjugate radius r_ξ is explicitly described as $r_\xi = \frac{\pi}{|\alpha_0(g_*^{-1}\xi)|}$ (see Lemma 2.1), where g is a representative element of the base point of ξ and α_0 is the highest root in the positive root system with respect to a maximal abelian subspace (equipped with some lexicographical ordering) containing $g_*^{-1}\xi$.

We prove the following theorems of Gauss-Bonnet and Chern-Lashof types for compact submanifolds in a simply connected symmetric space of compact type.

THEOREM A. *Let M be an n-dimensional compact immersed submanifold in a simply connected symmetric space N of compact type. If $C_M := \bigcup_{x\in M} C_x$ (C_x : the cut locus of x in N) is of measure zero, then we have*

$$\int_{\xi\in U^\perp M} \mathfrak{K}_1(\xi)\omega_{U^\perp M} = \chi(M)\,, \tag{1.7}$$

$$\int_{\xi\in U^\perp M} \mathfrak{K}_2(\xi)\omega_{U^\perp M} \geq \sum_{k=0}^{n} b_k(M,\mathbf{F})\,. \tag{1.8}$$

In particular, if M is taut in the sense of this paper (see §3) *, then the equality sign holds in the inequality* (1.8).

REMARK 1.2. Let m_N be the maximal dimension of the cut locus C_x in N. If $\dim M \leq \dim N - m_N - 1$, then C_M is of measure zero. See Table 1 about m_N and $\dim N - m_N - 1$ for irreducible simply connected symmetric space N's of compact type, where we note that $\dim N - m_N - 1$ is equal to the multiplicity of the highest root in the root system associated with N (see Lemma 2.1). For the product $N := N_1 \times \cdots \times N_l$ of

TABLE 1.

N	m_N	$\dim N - m_N - 1$
$S^m(c)$	0	$m-1$
$Sp(m)/Sp(l)\times Sp(m-l)$ $(1\leq l\leq[\frac{m}{2}])$	$4l(m-l)-4$	3
$\mathbf{O}P^2$	8	7
$SU(2m)/Sp(m)$	$2m^2-m-6$	4
E_6/F_4	17	8
G	—	2
other	—	1

(G : an irreducible simply connected compact Lie group) irreducible simply connected symmetric space N_i's ($i = 1, \cdots, l$) of compact type, we have $\dim N - m_N - 1 = \min_{1\le i\le l}(\dim N_i - m_{N_i} - 1)$.

In the case where the ambient space is the m-dimensional sphere $S^m(c)$ of constant curvature c, we obtain the following result.

THEOREM B. (i) *Let M be a $2n$-dimensional compact immersed submanifold in $S^m(c)$, where $1 \le n \le m-1$. Then we have*

$$\begin{aligned}\frac{1}{v_m}\sum_{i=0}^{n}\left(\sum_{k=0}^{n-i}(-1)^k\binom{2n}{2i}\binom{n-i}{k}\frac{v_{m-2n+2i+2k}}{v_{m-2n+2i+2k-1}}\right)&\\ \times c^{n-i}\int_{\xi\in U^\perp M}H_{2i}(\xi)\omega_{U^\perp M}=\chi(M)\,,&\end{aligned}\tag{1.9}$$

where $H_{2i}(\xi)$ is the $2i$-th mean curvature of direction ξ of M and $v_i := \mathrm{Vol}(S^i(1))$ $(i \ge 1)$ and $v_0 = 2$.

(ii) *Let M be an n-dimensional compact immersed submanifold in $S^m(c)$. Then we have*

$$\begin{aligned}&\frac{1}{v_m}\sum_{i=0}^{[\frac{n}{2}]}a_i c^i\int_{\xi\in U^\perp M}|H_{n-2i}(\xi)|\omega_{U^\perp M}\\ &+\frac{2}{v_m}\sum_{i=0}^{[\frac{n-1}{2}]}b_i\sqrt{c}^{2i+1}\int_{\xi\in U^\perp M}|H_{n-2i-1}(\xi)|\omega_{U^\perp M}\ge\sum_{k=0}^{n}b_k(M,\mathbf{F})\,,\end{aligned}\tag{1.10}$$

where $H_{n-2i}(\xi)$ and $H_{n-2i-1}(\xi)$ are as in (i), $a_i = \sum_{k=0}^{i}(-1)^k\binom{n}{2i}\binom{i}{k}\times\frac{v_{m-2i+2k}}{v_{m-2i+2k-1}}$ *and*

$b_i = \sum_{k=0}^{i}(-1)^k\binom{n}{2i+1}\binom{i}{k}\frac{1}{m-2i+2k-1}$.

REMARK 1.3. The relation (1.9) for $c = 1$ coincides with the relation (1.7) of [4] obtained by T. Ishihara because $v_i = \frac{(i+1)\pi^{\frac{i+1}{2}}}{\Gamma(\frac{i+3}{2})}$ $(i \ge 1)$, where Γ is the Gamma function. The proof of T. Ishihara is entirely different from the proof in this paper.

In particular, when $\dim M = 2$, we obtain the following relations.

COROLLARY C. *Let M be a 2-dimensional compact immersed submanifold in $S^m(c)$. Then we have*

$$\frac{1}{v_{m-1}}\int_{\xi\in U^\perp M}K(\xi)\omega_{U^\perp M}+\left(\frac{v_{m-2}}{v_m}-\frac{v_{m-3}}{v_{m-1}}\right)c\mathrm{Vol}(M)=2-2g\,,\tag{1.11}$$

$$\begin{aligned}(1.12)\qquad &\frac{1}{v_{m-1}}\int_{\xi\in U^\perp M}|K(\xi)|\omega_{U^\perp M}+\frac{4\sqrt{c}}{(m-1)v_m}\int_{\xi\in U^\perp M}|H(\xi)|\omega_{U^\perp M}\\ &+\left(\frac{v_{m-2}}{v_m}-\frac{v_{m-3}}{v_{m-1}}\right)c\mathrm{Vol}(M)\geq 2+2g\,,\end{aligned}$$

where $K(\xi)$ (*resp.* $H(\xi)$) *is the Gaussian curvature* (*resp. the mean curvature*) *of direction* ξ *of* M *and* g *is the genus of* M.

Also, we have the following inequality for closed curves in $S^m(c)$.

COROLLARY D. *Let* $\gamma : [0,l] \to S^m(c)$ *be a closed curve in* $S^m(c)$ $(m \geq 2)$ *parametrized by the arclength* s *and* $\kappa : [0,l] \to \mathbf{R}_+ \cup \{0\}$ *be the curvature of* γ. *Then we have*

$$\frac{v_{m-3}}{(m-2)v_{m-1}}\int_0^l \kappa ds+\frac{\sqrt{c}v_{m-2}l}{(m-1)v_m}\geq 1\quad (m\geq 3)$$

and

$$\frac{1}{v_{m-1}}\int_0^l \kappa ds+\frac{\sqrt{c}v_{m-2}l}{(m-1)v_m}\geq 1\quad (m=2)\,.$$

REMARK 1.4. These inequalities are different from the inequality of Proposition 1 in [6] proved by E. Teufel.

Also, we have the following inequality for closed curves in a simply connected rank one symmetric space $\mathbf{F}P^m(c)$ of compact type, where $\mathbf{F}$ implies the complex number field $\mathbf{C}$, the quaternion algebra $\mathbf{Q}$ or the Cayley algebra $\mathbf{O}$, $m = 2$ when $\mathbf{F} = \mathbf{O}$ and c is the maximal sectional curvature of the space.

COROLLARY E. *Let* $\gamma : [0,l] \to \mathbf{F}P^m(c)$ *be a closed curve in* $\mathbf{F}P^m(c)$ *parametrized by the arclength* s *and* $\kappa : [0,l] \to \mathbf{R}_+ \cup \{0\}$ *be the curvature of* γ. *Then we have*

$$\begin{aligned}&\frac{4\alpha_{qm-1,q-1}v_{qm-1}v_{qm-3}}{v_{qm-2}}\int_0^l \kappa ds+\sqrt{c}v_{q-2}v_{qm-q-1}(a\beta_{m,q}+b\alpha_{qm-2,q})l\\ &\geq \frac{1}{2^{qm-2}}\mathrm{Vol}(\mathbf{F}P^m(c))\sqrt{c}^{qm+1}\,,\end{aligned}$$

where $q = 2$ (*when* $\mathbf{F} = \mathbf{C}$), 4 (*when* $\mathbf{F} = \mathbf{Q}$) *or* 8 (*when* $\mathbf{F} = \mathbf{O}$), $\alpha_{i,j} = \int_0^{\frac{\pi}{2}} t\sin^i t\cos^j t dt$, $\beta_{m,q} = \int_0^{\frac{\pi}{2}} t|\cos 2t|\sin^{qm-2} t\cos^{q-2} t dt$, $a = \sum_{k=0}^{\frac{q}{2}}(-1)^k\binom{\frac{q}{2}}{k}\frac{v_{qm-q+2k}}{v_{qm-q+2k-1}}$, $b = \sum_{k=0}^{\frac{q-2}{2}}(-1)^k\binom{\frac{q-2}{2}}{k}\frac{v_{qm-q+2k+2}}{v_{qm-q+2k+1}}$ *and* v_i *is as in Theorem B.*

2. Basic notions and facts

In this section, we recall the basic notions and facts. Let $N = G/K$ be a simply connected symmetric space of compact type. Let $w \in U_pN$, where U_pN is the unit tangent sphere of N at p. Denote by γ_w the (non-extendable) geodesic in N with $\dot\gamma_w(0) = w$ and denote by exp the exponential map of N. If there exists a non-zero Jacobi field J along γ_w with $J(0) = 0$ and $J(s_0) = 0$ ($s_0 > 0$), then we call s_0 a *conjugate radius of direction* w and call $\exp(s_0w)$ ($= \gamma_w(s_0)$) a *conjugate point of direction* w. Also, we call the minimum of conjugate radii of direction w *the first conjugate radius of direction* w and denote it by r_w. We call $\exp(r_ww)$ *the first conjugate point of direction* w. Set $\tilde{C}_p := \{r_ww \mid w \in U_pN\}$ and $C_p := \exp(\tilde{C}_p)$. This set C_p is called *the first conjugate locus of* p, which coincides with the cut locus of p because N is a simply connected symmetric space of compact type. For $w \in TN$ with $||w|| < r_w$, we set

$$D_w^{co} := \cos\sqrt{R_w}\,, \quad D_w^{si} := \frac{\sin\sqrt{R_w}}{\sqrt{R_w}}\,, \quad D_w^{ct} := \frac{\sqrt{R_w}}{\tan\sqrt{R_w}}$$

for simplicity, where $\sqrt{R_w}$ is the positive operator with $\sqrt{R_w}^2 = R(\cdot, w)w$ (R : the curvature tensor of N). Note that $D_0^{si} = D_0^{ct} = \mathrm{id}$. A Jacobi field J along a geodesic γ in N is described as

$$J(s) = P_{\gamma|_{[0,s]}}(D^{co}_{s\dot\gamma(0)}J(0) + sD^{si}_{s\dot\gamma(0)}J'(0))\,, \tag{2.1}$$

where $P_{\gamma|_{[0,s]}}$ is the parallel translation along $\gamma|_{[0,s]}$ with respect to the Levi-Civita connection $\tilde\nabla$ of N, $\dot\gamma(0)$ is the velocity vector of γ at 0 and $J'(0) = \tilde\nabla_{\dot\gamma(0)}J$. Let $\mathfrak{g}$ (resp. $\mathfrak{f}$) be the Lie algebra of G (resp. K) and $\mathfrak{g} = \mathfrak{f}+\mathfrak{p}$ be the Cartan decomposition. The subspace $\mathfrak{p}$ is identified with the tangent space $T_{eK}N$ of N at eK, where e is the identity element of G. Denote by ad the adjoint representation of $\mathfrak{g}$. Take a maximal abelian subspace $\mathfrak{h}$ of $\mathfrak{p}$. For each $\alpha \in \mathfrak{h}^*$ (the dual space of $\mathfrak{h}$), we set $\mathfrak{p}_\alpha := \{X \in \mathfrak{p} \mid \mathrm{ad}(a)^2(X) = -\alpha(a)^2X \text{ for all } a \in \mathfrak{h}\}$. If $\mathfrak{p}_\alpha \neq \{0\}$, then the linear function α is called a *root for* $\mathfrak{h}$. Let $\triangle_+$ be the positive root system with respect to some lexicographical ordering of $\mathfrak{h}$. Then we have $\mathfrak{p} = \mathfrak{h} + \sum_{\alpha\in\triangle_+}\mathfrak{p}_\alpha$. Note that $D_w^{co} = g_*\circ\cosh(\mathrm{ad}(g_*^{-1}w))\circ g_*^{-1}$, $D_w^{si} = g_*\circ\frac{\sinh(\mathrm{ad}(g_*^{-1}w))}{\mathrm{ad}(g_*^{-1}w)}\circ g_*^{-1}$ and $D_w^{ct} = g_*\circ\frac{\mathrm{ad}(g_*^{-1}w)}{\tanh(\mathrm{ad}(g_*^{-1}w))}\circ g_*^{-1}$ ($w \in T_{gK}N$) because of $\sqrt{R_w}^2 = -g_*\circ\mathrm{ad}(g_*^{-1}w)^2\circ g_*^{-1}$. From (2.1), we can show the following fact for the first conjugate radius.

LEMMA 2.1. *Take* $w \in U_{gK}N$, $\triangle_+$ *be the positive root system with respect to a maximal abelian subspace* $\mathfrak{h}$ (*equipped with some lexicographical ordering*) *containing* $g_*^{-1}w$ *and* α_0 *be the highest root in* $\triangle_+$. *Then we have* $r_w = \frac{\pi}{|\alpha_0(g_*^{-1}w)|}$ *and* $\mathrm{rank}(\exp|_{T_{gK}N})_{*r_ww} = \dim N - \sum_{\alpha\in\triangle_w}\dim\mathfrak{p}_\alpha$, *where* $\triangle_w := \{\alpha \in \triangle_+ \mid |\alpha(g_*^{-1}w)| = \frac{\pi}{r_w}\}$. *In particular*, *if* w *is a point of a Weyl chamber*, *then we have* $\mathrm{rank}(\exp|_{T_{gK}N})_{*r_ww} = \dim N - \dim\mathfrak{p}_{\alpha_0}$.

PROOF. Let s_0 be a conjugate radius of direction w. Then there exists a non-trivial Jacobi field J along γ_w with $J(0)=0$ and $J(s_0)=0$. From (2.1), we have $s_0 D^{si}_{s_0 w} J'(0)=0$. On the other hand, we have $s_0 D^{si}_{s_0 w} J'(0) = s_0 J'(0)_{\mathfrak{h}} + \sum_{\alpha\in\triangle_+} \frac{\sin(s_0\alpha(g_*^{-1}w))}{\alpha(g_*^{-1}w)} J'(0)_\alpha = 0$, where $J'(0)_{\mathfrak{h}}$ (resp. $J'(0)_\alpha$) is the $g_*\mathfrak{h}$-component (resp. $g_*\mathfrak{p}_\alpha$-component of $J'(0)$). Hence we see that $s_0\alpha_0(g_*^{-1}w) \equiv 0 \pmod{\pi}$ and $J'(0)_{\alpha_0} \neq 0$ for some $\alpha_0 \in \triangle_+$ because $J'(0)_\alpha$ vanishes for each $\alpha \in \triangle_+$ with $s_0\alpha(g_*^{-1}w) \not\equiv 0 \pmod{\pi}$ and $J'(0)_{\mathfrak{h}} = 0$. It follows from this fact that $r_w = \frac{\pi}{\max_{\alpha\in\triangle_+}|\alpha(g_*^{-1}w)|} = \frac{\pi}{|\alpha_0(g_*^{-1}w)|}$ and that $\mathrm{rank}(\exp|_{T_{gK}N})_{*r_w w} = \dim N - \sum_{\alpha\in\triangle_w}\dim\mathfrak{p}_\alpha$. In particular, if w is a point of a Weyl chamber, then we have $\triangle_w = \{\alpha_0\}$. Hence the last part of the statement follows. q.e.d.

From this lemma, the fact of Table 1 is deduced.

3. Squared distance functions

In this section, we prepare some lemmas for squared distance functions. Let M be an n-dimensional compact immersed submanifold in an m-dimensional symmetric space $N = G/K$ of compact type. We omit the notation of the immersion. For two points p and q of N with $q \notin C_p$, we denote the shortest geodesic from p to q by γ_{pq} (i.e., $\gamma_{pq}(0) = p,\ \gamma_{pq}(1) = q,\ \|\dot\gamma_{pq}\| = d(p,q)$). Also, we denote $\dot\gamma_{pq}(0)$ by $\overrightarrow{pq}$. For the squared distance function d_p^2 $(x\in M \to d(p,x)^2)$ $(p\in N)$, we have the following fact.

LEMMA 3.1. *Let x be a critical point of d_p^2 with $x\notin C_p$. Then the following statements* (i) *and* (ii) *hold*:

(i) *$\overrightarrow{xp}$ is normal to M*,

(ii) *The Hessian* $(\mathrm{Hess}\, d_p^2)_x$ *of d_p^2 at x is given by*

$$(\mathrm{Hess}\, d_p^2)_x(X,Y) = 2\langle X,\ (\mathrm{pr}_T \circ D^{ct}_{\overrightarrow{xp}} - A_{\overrightarrow{xp}})Y\rangle\,, \tag{3.1}$$

where $X, Y \in T_xM$.

PROOF. The statement (i) is trivial. We shall show the statement (ii). Take tangent vectors X and Y to M at x. Take a two-parameter map $\bar\delta$ to M with $\bar\delta_*(\frac{\partial}{\partial u}|_{u=t=0}) = X$ and $\bar\delta_*(\frac{\partial}{\partial t}|_{u=t=0}) = Y$, where u (resp. t) is the first (resp. the second) parameter of $\bar\delta$. We may assume that $\mathrm{Im}\,\bar\delta \cap C_p = \emptyset$ by restricting the domain of $\bar\delta$ to a neighborhood of $(0,0)$ if necessary. Define a three-parameter map δ into N by $\delta(u,t,s) = \gamma_{\overrightarrow{\bar\delta(u,t)p}}(s)$. For simplicity, we denote $\delta_*(\frac{\partial}{\partial u})$, $\delta_*(\frac{\partial}{\partial t})$ and $\delta_*(\frac{\partial}{\partial s})$ by $\frac{\partial}{\partial u}$, $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$, respectively. Set $J_t(s) := \frac{\partial}{\partial u}|_{u=0}$, which is a Jacobi field along $\gamma_{\overrightarrow{\delta(0,t,0)p}}$. From (2.1), we have $J_t(s) = P_{\gamma_{\overrightarrow{\delta(0,t,0)p}}|_{[0,s]}}(D^{co}_{s\cdot\overrightarrow{\delta(0,t,0)p}} J_t(0) + s D^{si}_{s\cdot\overrightarrow{\delta(0,t,0)p}} J_t'(0))$. This together with $J_t(1) = 0$ deduces $D^{co}_{\overrightarrow{\delta(0,t,0)p}} J_t(0) + D^{si}_{\overrightarrow{\delta(0,t,0)p}} J_t'(0) = 0$. Since $\delta(0,t,0) \notin C_p$ (i.e., $||\overrightarrow{\delta(0,t,0)p}|| <$

$r_{\overrightarrow{\delta(0,t,0)p}/||\overrightarrow{\delta(0,t,0)p}||}$), we have $\max_{\alpha\in\triangle_+}|\alpha(g_*^{-1}\overrightarrow{\delta(0,t,0)p})| < \pi$ in terms of Lemma 2.1, where $\delta(0,t,0) = gK$. This implies that $D^{si}_{\overrightarrow{\delta(0,t,0)p}}$ is non-singular. Hence we obtain $J_t'(0) = -D^{ct}_{\overrightarrow{\delta(0,t,0)p}}J_t(0)$. Also, since x is a critical point of d_p^2, $\frac{\partial}{\partial s}|_{u=t=s=0}$ is normal to M. These facts deduce

$$(\mathrm{Hess}\,d_p^2)_x(X,Y) = -2\left\langle X, \nabla_Y\left(\frac{\partial}{\partial s}|_{u=s=0}\right)_T\right\rangle,$$

where ∇ is the Levi-Civita connection of M and $(\cdot)_T$ is the tangent component of $\cdot$ (see the proof of Lemma 3.1 in [5]). On the other hand, we can show $\nabla_Y(\frac{\partial}{\partial s}|_{u=s=0})_T = -(\mathrm{pr}_T\circ D^{ct}_{\overrightarrow{xp}} - A_{\overrightarrow{xp}})Y$ (see the proof of Lemma 3.1 in [5]). Therefore, we obtain the relation (3.1). q.e.d.

Let $B := \bigcup_{\xi\in U^\perp M}\{s\xi \mid s\in[0,r_\xi)\}$, which is an open potion of $T^\perp M$. Denote by $\exp_B^\perp$ the restriction of the normal exponential map $\exp^\perp$ of M to B. Also, denote by $\tilde{\omega}$ (resp. ω_B) the volume element of N (resp. B induced from the volume element of $T^\perp M$). Then we have the following relation.

LEMMA 3.2. *For each $\xi\in B$, the following relation holds*:

$$((\exp_B^\perp)^*\tilde{\omega})_\xi = \det(\mathrm{pr}_T\circ D^{ct}_\xi|_{T_{\pi(\xi)}M} - A_\xi)\det D^{si}_\xi(\omega_B)_\xi\,,$$

where π is the bundle projection of B.

PROOF. From $\xi\in B$ (i.e., $\|\xi\| < r_{\xi/\|\xi\|}$), we see that D^{si}_ξ is non-singular. By noticing this fact and imitating the proof of Lemma 3.2 in [5], we obtain the desired relation. q.e.d.

Denote by $\beta(\phi)$ the number of non-degenerate critical points of a function ϕ and by $\beta_{even}(\phi)$ (resp. $\beta_{odd}(\phi)$) the number of non-degenerate critical points of even (resp. odd) index of a function ϕ. Denote by F the focal set of M. For $p\in N\setminus F$, we set

$$(\exp_B^\perp)^{-1}(p)_+ := \{\xi\in(\exp_B^\perp)^{-1}(p) \mid (\exp_B^\perp)_{*\xi} \text{ preserves the orientation}\}\,,$$

$$(\exp_B^\perp)^{-1}(p)_- := \{\xi\in(\exp_B^\perp)^{-1}(p) \mid (\exp_B^\perp)_{*\xi} \text{ reverses the orientation}\}\,.$$

Further we prepare the following lemma.

LEMMA 3.3. *Let $p\in N\setminus(F\cup C_M)$. Then we have the following relations:*

$$\beta(d_p^2) = \sharp(\exp_B^\perp)^{-1}(p)\,, \tag{3.2}$$

$$\beta_{even}(d_p^2) = \sharp(\exp_B^\perp)^{-1}(p)_+\,, \tag{3.3}$$

$$\beta_{odd}(d_p^2) = \sharp(\exp_B^\perp)^{-1}(p)_-\,, \tag{3.4}$$

where $\sharp(*)$ *is the number of elements of a set* $*$.

PROOF. The relation (3.2) is directly deduced from (i) of Lemma 3.1. The relations (3.3) and (3.4) are directly deduced from (3.1) and Lemma 3.2, where we use $\det D^{si}_{\xi} > 0$ ($\xi \in B$). q.e.d.

At the end of this section, we define the tautness of a compact submanifold M with $N \setminus (F \cup C_M) \neq \emptyset$ in a complete Riemannian manifold N, where F is the focal set of M and $C_M := \bigcup_{x\in M} C_x$ (C_x : the cut locus of x). If d_p^2 is a perfect Morse function for each $p \in N \setminus (F \cup C_M)$, then we say that M is *taut*.

4. Proofs of Theorems A, B and Corollaries

In this section, we first prove Theorem A in terms of Lemmas 3.2 and 3.3.

PROOF OF THEOREM A. First we prove the relation (1.7) in Theorem A. According to Lemma 3.2, we have

$$\begin{aligned}
\int_{\xi\in B}((\exp_B^{\perp})^*\tilde{\omega})_{\xi} &= \int_{\xi\in B}\det(\mathrm{pr}_T\circ D^{ct}_{\xi}|_{T_{\pi(\xi)}M} - A_{\xi})\det D^{si}_{\xi}\,\omega_B \\
&= \int_{\xi\in U^{\perp}M}\left(\int_0^{r_\xi}\det\left(\mathrm{pr}_T\circ\frac{1}{s}D^{ct}_{s\xi}|_{T_{\pi(\xi)}M} - A_{\xi}\right)\det D^{si}_{s\xi}\cdot s^{m-1}ds\right)\omega_{U^{\perp}M} \qquad (4.1)\\
&= \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(S^{m-1}(1))}\int_{\xi\in U^{\perp}M}\mathfrak{K}_1(\xi)\omega_{U^{\perp}M}\,.
\end{aligned}$$

On the other hand, since C_M is of measure zero, we have

$$\begin{aligned}
\int_{\xi\in B}((\exp_B^{\perp})^*\tilde{\omega})_{\xi} &= \int_{\xi\in B\setminus \exp_B^{\perp -1}(F)}((\exp_B^{\perp})^*\tilde{\omega})_{\xi} \\
&= \int_{p\in N\setminus F}(\sharp(\exp_B^{\perp})^{-1}(p)_+ - \sharp(\exp_B^{\perp})^{-1}(p)_-)\tilde{\omega}_p \\
&= \int_{p\in N\setminus (F\cup C_M)}(\sharp(\exp_B^{\perp})^{-1}(p)_+ - \sharp(\exp_B^{\perp})^{-1}(p)_-)\tilde{\omega}_p \qquad (4.2)\\
&= \int_{p\in N\setminus (F\cup C_M)}(\beta_{even}(d_p^2) - \beta_{odd}(d_p^2))\tilde{\omega}_p \\
&= \chi(M)\mathrm{Vol}(N)\,.
\end{aligned}$$

Therefore, we obtain the equality (1.7). In similar to (4.1) and (4.2), we can show

$$\begin{aligned}
&\int_{\xi\in B}|((\exp_B^{\perp})^*\tilde{\omega})_{\xi}| \\
&= \int_{\xi\in U^{\perp}M}\left(\int_0^{r_\xi}|\det\left(\mathrm{pr}_T\circ\frac{1}{s}D^{ct}_{s\xi}|_{T_{\pi(\xi)}M} - A_{\xi}\right)|\det D^{si}_{s\xi}\cdot s^{m-1}ds\right)\omega_{U^{\perp}M}
\end{aligned}$$

$$= \frac{\mathrm{Vol}(N)}{\mathrm{Vol}(S^{m-1}(1))} \int_{\xi \in U^\perp M} \mathfrak{K}_2(\xi) \omega_{U^\perp M}$$

and

$$\int_{\xi \in B} |((\exp_B^\perp)^* \tilde{\omega})_\xi| = \int_{p \in N \setminus (F \cup C_M)} \beta(d_p^2) \tilde{\omega}_p \geq \sum_{k=0}^{n} b_k(M, \mathbf{F}) \mathrm{Vol}(N) .$$

Therefore, we obtain the inequality (1.8). In particular, if M is taut, then we have $\int_{p \in N \setminus (F \cup C_M)} \beta(d_p^2) \tilde{\omega}_p = \sum_{k=0}^{n} b_k(M, \mathbf{F}) \mathrm{Vol}(N)$. Hence the equality sign holds in (1.8). q.e.d.

Next we prove Theorem B.

PROOF OF THEOREM B. Since the ambient space is $S^m(c)$, we have $r_\xi = \frac{\pi}{\sqrt{c}}$, $D^{ct}_{s\xi} = \sqrt{c}s \cot(\sqrt{c}s)\mathrm{id}$ and $D^{si}_{s\xi} = \frac{\sin(\sqrt{c}s)}{\sqrt{c}s}\mathrm{id}$. Also, since the cut locus C_x consists of one point for each $x \in M$ and $\dim M \leq m-1$, the set C_M is of measure zero. Hence the relations (1.7) and (1.8) in Theorem A hold. The left-hand side of the relation (1.7) in Theorem 1 is written as

$$\frac{1}{\mathrm{Vol}(S^m(c))} \int_{\xi \in U^\perp M} \left(\int_0^{\frac{\pi}{\sqrt{c}}} \det(\sqrt{c} \cot(\sqrt{c}s)\mathrm{id} - A_\xi) \left(\frac{\sin(\sqrt{c}s)}{\sqrt{c}} \right)^{m-1} ds \right) \omega_{U^\perp M} ,$$

which is further written as

$$\frac{1}{\mathrm{Vol}(S^m(1))} \int_{\xi \in U^\perp M} \left(\int_0^{\pi} \det(\sqrt{c} \cot s \cdot \mathrm{id} - A_\xi) \sin^{m-1} s ds \right) \omega_{U^\perp M} .$$

Hence we have

$$\frac{1}{\mathrm{Vol}(S^m(1))} \int_{\xi \in U^\perp M} \left(\int_0^{\pi} \det(\sqrt{c} \cot s \cdot \mathrm{id} - A_\xi) \sin^{m-1} s ds \right) \omega_{U^\perp M} = \chi(M) . \tag{4.3}$$

Similarly we have

$$\begin{aligned} &\frac{1}{\mathrm{Vol}(S^m(1))} \int_{\xi \in U^\perp M} \left(\int_0^{\pi} |\det(\sqrt{c} \cot s \cdot \mathrm{id} - A_\xi)| \sin^{m-1} s ds \right) \omega_{U^\perp M} \\ &\quad \geq \sum_{k=0}^{n} b_k(M, \mathbf{F}) . \end{aligned} \tag{4.4}$$

Let ξ be the fixed unit normal vector field of M. First we show the statement (i). From the definitions of the i-th mean curvature $H_i(\xi)$ of direction ξ ($i = 0, \cdots, 2n$), we have

$$\det(\sqrt{c} \cot s \cdot \mathrm{id} - A_\xi) = \sum_{i=0}^{2n} (-1)^i (\sqrt{c} \cot s)^{2n-i} \binom{2n}{i} H_i(\xi) .$$

Hence we have

$$\begin{aligned}
&\int_0^{\pi} \left(\det(\sqrt{c}\cot s\cdot \mathrm{id}-A_{\xi})+\det(\sqrt{c}\cot s\cdot \mathrm{id}-A_{(-\xi)})\right)\sin^{m-1} s ds\\
&=2\sum_{i=0}^{n}\binom{2n}{2i}H_{2i}(\xi)c^{n-i}\int_0^{\pi}\sin^{m-n+2i-1} s\cos^{2n-2i} s ds\\
&=2\sum_{i=0}^{n}\binom{2n}{2i}H_{2i}(\xi)c^{n-i}\left(\sum_{k=0}^{n-i}(-1)^k\binom{n-i}{k}\frac{v_{m-n+2i+2k}}{v_{m-n+2i+2k-1}}\right),
\end{aligned} \tag{4.5}$$

where we also use $\int_0^{\pi}\sin^j s ds=\frac{v_{j+1}}{v_j}$. From this relation and (4.3), we obtain (1.9). Next we show the statement (ii). In similar way to get (4.5), we have

$$\begin{aligned}
&\int_0^{\pi} \left(|\det(\sqrt{c}\cot s\cdot \mathrm{id}-A_{\xi})|+|\det(\sqrt{c}\cot s\cdot \mathrm{id}-A_{(-\xi)})|\right)\sin^{m-1} s ds\\
&\le 4\sum_{i=0}^{n}\binom{n}{i}|H_i(\xi)|\sqrt{c}^{n-i}\int_0^{\frac{\pi}{2}}\sin^{m-n+i-1} s\cos^{n-i} s ds\\
&=4\sum_{i=0}^{n}\binom{n}{i}|H_{n-i}(\xi)|\sqrt{c}^{i}\int_0^{\frac{\pi}{2}}\sin^{m-i-1} s\cos^{i} s ds\\
&=2\sum_{i=0}^{[\frac{n}{2}]}a_i c^i|H_{n-2i}(\xi)|+4\sum_{i=0}^{[\frac{n-1}{2}]}b_i\sqrt{c}^{2i+1}|H_{n-2i-1}(\xi)|,
\end{aligned}$$

where we use $\int_0^{\frac{\pi}{2}}\sin^i s\cos^{2j} s ds=\frac{1}{2}\sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{v_{i+2k+1}}{v_{i+2k}}$ and $\int_0^{\frac{\pi}{2}}\sin^i s\cos^{2j+1} s ds = \sum_{k=0}^{j}(-1)^k\binom{j}{k}\frac{1}{i+2k+1}$. From this relation and (4.4), we obtain (1.10). q.e.d.

Next we prove Corollary C in terms of Theorem B.

PROOF OF COROLLARY C. From $\dim M=2$, the relations (1.9) (resp. (1.10)) of Theorem B is as follows:

$$\frac{1}{v_{m-1}}\int_{\xi\in U^{\perp}M}K(\xi)\omega_{U^{\perp}M}+\frac{c}{v_m}\left(\frac{v_{m-2}}{v_{m-3}}-\frac{v_m}{v_{m-1}}\right)\mathrm{Vol}(U^{\perp}M)=\chi(M)$$

$$\begin{aligned}
&\left(\text{resp. }\frac{1}{v_{m-1}}\int_{\xi\in U^{\perp}M}|K(\xi)|\omega_{U^{\perp}M}+\frac{4\sqrt{c}}{(m-1)v_m}\int_{\xi\in U^{\perp}M}|H_1(\xi)|\omega_{U^{\perp}M}\right.\\
&\left.+\frac{c}{v_m}\left(\frac{v_{m-2}}{v_{m-3}}-\frac{v_m}{v_{m-1}}\right)\mathrm{Vol}(U^{\perp}M)\ge\sum_{k=0}^{2}b_k(M,\mathbf{F})\right).
\end{aligned}$$

Hence we obtain the relations (1.11) and (1.12) in terms of $\mathrm{Vol}(U^{\perp}M) = v_{m-3}\mathrm{Vol}(M)$, $\chi(M) = 2-2g$ and $\sum_{k=0}^{2} b_k(M,\mathbf{F}) = 2+2g$. q.e.d.

Next we prove Corollary D in terms of Theorem B.

PROOF OF COROLLARY D. We show the statement in the case where κ has no zero point. Let v be the unit principal normal vector of γ. Clearly we have $A_\xi\dot\gamma = \kappa(\pi(\xi))\langle v,\xi\rangle\dot\gamma$ for $\xi \in U^{\perp}\gamma$, where $\dot\gamma = \frac{d\gamma}{ds}$ and π is the projection of the unit normal bundle $U^{\perp}\gamma$ of γ. So we have $|H_1(\xi)| = \kappa(\pi(\xi))|\langle v,\xi\rangle|$. Hence, from (1.10) of Theorem B, we have

$$\frac{1}{2v_{m-1}}\int_{\xi\in U^{\perp}\gamma}\kappa(\pi(\xi))|\langle v,\xi\rangle|\omega_{U^{\perp}\gamma} + \frac{\sqrt{c}v_{m-2}l}{(m-1)v_m} \geq 1\,, \tag{4.6}$$

where we also use $\mathrm{Vol}(U^{\perp}\gamma) = v_{m-2}l$. The first term of the left-hand side of (4.6) is rewritten as

$$\begin{aligned}\frac{1}{2v_{m-1}}\int_{\xi\in U^{\perp}\gamma}\kappa(\pi(\xi))|\langle v,\xi\rangle|\omega_{U^{\perp}\gamma} &= \frac{1}{2v_{m-1}}\int_0^l\left(\kappa(s)\int_{\xi\in U_s^{\perp}\gamma}|\langle v,\xi\rangle|\omega_{U_s^{\perp}\gamma}\right)ds\\ &= \begin{cases}\dfrac{v_{m-3}}{(m-2)v_{m-1}}\displaystyle\int_0^l\kappa(s)ds & (m\geq 3)\\ \dfrac{1}{v_{m-1}}\displaystyle\int_0^l\kappa(s)ds & (m=2)\,.\end{cases}\end{aligned}$$

Hence we obtain the desired relations. Similarly we can show the statement in the case where κ has zero points. q.e.d.

Next we prove Corollary E.

PROOF OF COROLLARY E. We show the statement in the case where κ has no zero point. Let v be the unit principal normal vector of γ. Clearly we have

$$A_\xi\dot\gamma = \kappa(\pi(\xi))\langle v,\xi\rangle\dot\gamma \quad (\xi\in U^{\perp}\gamma)\,, \tag{4.7}$$

where $\dot\gamma = \frac{d\gamma}{ds}$ and π is the projection of the unit normal bundle $U^{\perp}\gamma$ of γ. Fix $\xi \in U_{s_0}^{\perp}\gamma$. Let $W_\xi := \mathrm{Span}\{J_1\xi,\cdots,J_{q-1}\xi\}$ and $W'_\xi := \mathrm{Span}\{\xi, J_1\xi,\cdots,J_{q-1}\xi\}^{\perp}$, where $\{J_1,\cdots,J_{q-1}\}$ is the complex structure of $\mathbf{C}P^m(c)$, the quaternionic structure of $\mathbf{Q}P^m(c)$ or the Cayley structure of $\mathbf{O}P^2(c)$. Denote by pr_ξ, pr_{W_ξ} and $\mathrm{pr}_{W'_\xi}$ the orthogonal projection of $T_{\gamma(s_0)}\mathbf{F}P^m(c)$ onto $\mathrm{Span}\{\xi\}$, W_ξ and W'_ξ, respectively. Since $\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})} = \frac{1}{s}\mathrm{pr}_\xi + \frac{\sqrt{c}}{\tan(s\sqrt{c})}\mathrm{pr}_{W_\xi} + \frac{\sqrt{c}}{2\tan\frac{s\sqrt{c}}{2}}\mathrm{pr}_{W'_\xi}$ and $\dot\gamma(s_0)\in W_\xi\oplus W'_\xi$, we have

$$\left(\mathrm{pr}_T\circ\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})}\right)(\dot\gamma(s_0)) = \frac{\sqrt{c}}{\tan(s\sqrt{c})}(\mathrm{pr}_T\circ\mathrm{pr}_{W_\xi})(\dot\gamma(s_0))$$

$$+\frac{\sqrt{c}}{2\tan\frac{s\sqrt{c}}{2}}(\mathrm{pr}_T\circ\mathrm{pr}_{W'_\xi})(\dot\gamma(s_0))\,.$$

Denote by θ_ξ the angle between $\dot\gamma(s_0)$ and $\mathrm{pr}_{W_\xi}(\dot\gamma(s_0))$. Then we have

$$\left(\mathrm{pr}_T\circ\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})}\right)(\dot\gamma(s_0))=\left(\frac{\sqrt{c}}{\tan(s\sqrt{c})}\cos^2\theta_\xi+\frac{\sqrt{c}}{2\tan\frac{s\sqrt{c}}{2}}\sin^2\theta_\xi\right)\dot\gamma(s_0)\,. \tag{4.8}$$

From (4.7) and (4.8), we have

$$\begin{aligned}&\left|\det\left(\mathrm{pr}_T\circ\frac{\sqrt{R_\xi}}{\tan(s\sqrt{R_\xi})}|_{T_{s_0}\gamma}-A_\xi\right)\right|\\&\quad\le\left|\frac{\sqrt{c}}{\tan(s\sqrt{c})}\right|\cos^2\theta_\xi+\left|\frac{\sqrt{c}}{2\tan\frac{s\sqrt{c}}{2}}\right|\sin^2\theta_\xi+\kappa|\langle v,\xi\rangle|\,.\end{aligned}$$

On the other hand, we have

$$\frac{\sin(s\sqrt{R_\xi})}{\sqrt{R_\xi}}=s\,\mathrm{pr}_\xi+\frac{\sin(s\sqrt{c})}{\sqrt{c}}\mathrm{pr}_{W_\xi}+\frac{2\sin\frac{s\sqrt{c}}{2}}{\sqrt{c}}\mathrm{pr}_{W'_\xi}$$

and hence

$$\begin{aligned}\det\frac{\sin(s\sqrt{R_\xi})}{\sqrt{R_\xi}}&=s\left(\frac{\sin(s\sqrt{c})}{\sqrt{c}}\right)^{q-1}\left(\frac{2\sin\frac{s\sqrt{c}}{2}}{\sqrt{c}}\right)^{qm-q}\\&=\left(\frac{2}{\sqrt{c}}\right)^{qm-1}s\sin^{qm-1}\frac{s\sqrt{c}}{2}\cos^{q-1}\frac{s\sqrt{c}}{2}\,.\end{aligned}$$

Also, the first conjugate radius r_ξ of direction ξ is equal to $\frac{\pi}{\sqrt{c}}$. Hence we have

$$\begin{aligned}\mathfrak{K}_2(\xi)\le\frac{v_{qm-1}}{\mathrm{Vol}(\mathbf{F}P^m(c))}\left(\frac{2}{\sqrt{c}}\right)^{qm-1}&\left\{\frac{2\beta_{m,q}}{\sqrt{c}}\cos^2\theta_\xi+\frac{2\alpha_{qm-2,q}}{\sqrt{c}}\sin^2\theta_\xi\right.\\&\left.+\kappa|\langle v,\xi\rangle|\frac{4\alpha_{qm-1,q-1}}{c}\right\}.\end{aligned}$$

Therefore, from (1.8) of Theorem A, we obtain

$$\begin{aligned}&\frac{1}{\mathrm{Vol}(\mathbf{F}P^m(c))}\left(\frac{2}{\sqrt{c}}\right)^{qm-1}\left\{\frac{2\beta_{m,q}}{\sqrt{c}}\int_{\xi\in U^\perp\gamma}\cos^2\theta_\xi\,\omega_{U^\perp\gamma}\right.\\&\quad\left.+\frac{2\alpha_{qm-2,q}}{\sqrt{c}}\int_{\xi\in U^\perp\gamma}\sin^2\theta_\xi\,\omega_{U^\perp\gamma}+\frac{4\alpha_{qm-1,q-1}}{c}\int_{\xi\in U^\perp\gamma}\kappa|\langle v,\xi\rangle|\omega_{U^\perp\gamma}\right\}\\&\quad\ge b_0(S^1,\mathbf{F})+b_1(S^1,\mathbf{F})=2\,.\end{aligned} \tag{4.9}$$

On the other hand, we have

$$\begin{aligned}\int_{\xi\in U^\perp\gamma}\cos^2\theta_\xi\,\omega_{U^\perp\gamma} &= \int_0^l\left(\int_{\xi\in U_s^\perp\gamma}\cos^2\theta_\xi\,\omega_{U_s^\perp\gamma}\right)ds\\ &= l\int_{[0,\frac{\pi}{2}]\times S^{qm-q-1}(1)\times S^{q-2}(1)}\sin^{qm-q-1}\theta\cos^q\theta\,d\theta\\ &\quad\wedge\omega_{S^{qm-q-1}(1)}\wedge\omega_{S^{q-2}(1)}\\ &= lv_{qm-q-1}v_{q-2}\int_0^{\frac{\pi}{2}}\sin^{qm-q-1}\theta\cos^q\theta\,d\theta\\ &= \frac{lv_{qm-q-1}v_{q-2}}{2}\sum_{k=0}^{\frac{q}{2}}(-1)^k\binom{\frac{q}{2}}{k}\frac{v_{qm-q+2k}}{v_{qm-q+2k-1}}\,,\end{aligned}$$

$$\begin{aligned}\int_{\xi\in U^\perp\gamma}\sin^2\theta_\xi\,\omega_{U^\perp\gamma} &= lv_{qm-q-1}v_{q-2}\int_0^{\frac{\pi}{2}}\sin^{qm-q+1}\theta\cos^{q-2}\theta\,d\theta\\ &= \frac{lv_{qm-q-1}v_{q-2}}{2}\sum_{k=0}^{\frac{q-2}{2}}(-1)^k\binom{\frac{q-2}{2}}{k}\frac{v_{qm-q+2k+2}}{v_{qm-q+2k+1}}\end{aligned}$$

and

$$\int_{\xi\in U^\perp\gamma}\kappa|\langle v,\xi\rangle|\omega_{U^\perp\gamma} = \frac{v_{qm-1}v_{qm-3}}{v_{qm-2}}\int_0^l\kappa(s)ds\,.$$

By substituting these relations into (4.9), we obtain the desired relation. Similarly we can show the statement in the case where κ has zero points. q.e.d.

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