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Exponential Decay and Spectral Structure for Wave Equation with Some Dissipations

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Abstract. Exponential decaying states for the wave equations are considered and some examples are given. Moreover, the spectral structure of the operator with coulomb type dissipation is investigated.

1. Introduction

We consider the exponential decay of the solution for wave equations with dissipation. We give the examples of the equations and their solutions, and investigate the structure of the spectrum of the operator corresponding to the equations. As to the exponential decay of the solution, we have two results: one is for the equation of the form with the Coulomb-type dissipation (Theorem 1), another is for the equation of the form with only the boundary condition (Theorem 7), which is considered as a singular dissipation ([2]). We consider the following equations:

$$w_{tt} - \Delta w + b(x)w_t = 0 \quad \text{in} \quad (0, +\infty) \times \mathbf{R}^N , \tag{1.1}$$

$$w(0, x) = w_0(x) , \quad w_t(0, x) = w_1(x) \quad \text{in} \quad \mathbf{R}^N , \tag{1.2}$$

where $N \geq 1$ and $b(\cdot) \in C^1(\mathbf{R}^N \setminus \{0\})$ is a positive function, and

$$\begin{cases} w_{tt}(t, x) - \Delta w(t, x) = 0 , & (t, x) \in \mathbf{R}_+ \times \mathbf{R}^3 , \\ i\sqrt{\sigma} w(t, 0) - w_r(t, 0) = 0 , & t \in \mathbf{R}_+ , \end{cases} \tag{1.3}$$

with initial data

$$w(0, x) = w_0(x) , \quad w_t(0, x) = w_1(x) , \quad x \in \mathbf{R}^3 ,$$

where $\sigma \in \mathbf{C}$.

In the following we denote $H^m = H^m(\mathbf{R}^N)$, $\dot{H}^m = \dot{H}^m(\mathbf{R}^N)$ $(m \geq 0)$ and $L^2 = H^0$. If we assume $\{w_0, w_1\} \in \dot{H}^1 \times L^2$ then the energy identity holds:

$$\|w(t)\|_E^2 + \int_0^t \|b(\cdot)^{1/2} w_t(\tau)\|_{L^2}^2 d\tau = \|w(0)\|_E^2$$

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where

$$\|w(t)\|_E^2 = \frac{1}{2}(\|w_t(t)\|_{L^2}^2 + \|\nabla w(t)\|_{L^2}^2)$$

is the total energy at time t (≥ 0).

THEOREM 1. *If $b(x) = b_0(x)$ is the following function:*

(1.4)
$$b_0(x) = \begin{cases} (3-N)|x|^{-1} & (N = 1, 2)\,, \\ (N-1)|x|^{-1} & (N \geq 3)\,, \end{cases}$$

then the explicit radial solution of (1.1)–(1.2) *with*

$$\begin{cases} w_0(x) \equiv |x| f(|x|)\,, & (N = 1) \\ f(|x|)\,, & (N \geq 2) \end{cases}, \quad w_1(x) = \partial_{|x|}\{w_0(|x|)\}\,,$$

where $f(|x|) = e^{\beta|x|} g(|x|)$, $\beta < 0$ and $g \in \mathcal{S}'$, is given by

$$w(t, x) = \begin{cases} |x| f(|x| + t)\,, & (N = 1) \\ f(|x| + t)\,. & (N \geq 2) \end{cases}$$

Therefore if $f \in H^1$ then the total energy decays exponentially as t tends to infinity.

REMARK 2. (In the case of $N = 3$ the following decomposition has been already shown by Kadowaki [1]). What the equation (1.1) has the progressive wave solution as in Theorem 1 in the case $N \geq 2$ and $b(x)$ satisfies (1.4) follows from the following decomposition of (1.1). If $N = 2$, then

$$\begin{cases} (\partial_t + \partial_r + r^{-1})(\partial_t - \partial_r) w(t, r) = 0\,, \\ r^{-1}(\partial_t + \partial_r) r (\partial_t - \partial_r) w(t, r) = 0\,, \end{cases}$$

and if $N \geq 3$, then

$$(\partial_t + \partial_r + (N-1)r^{-1})(\partial_t - \partial_r) w(t, r) = 0\,.$$

On the other hand in the case $N = 1$, we have

$$r^{-1}(\partial_t + \partial_r) r (\partial_t - \partial_r + r^{-1}) w(t, r) = 0\,.$$

REMARK 3. The solution obtained in Theorem 1 is an example of *disappearing solution* studied by A. Majda [6].

THEOREM 4. *Let $b(x) = b_0(x)$ in* (1.4) *and*

$$H_b = i \begin{pmatrix} 0 & 1 \\ \Delta & -b \end{pmatrix}$$

with domain

$$\mathcal{D}(H_b) = \{v = (v_1, v_2) \in E \mid H_b v \in E\}\,,$$

where $E = \dot{H}^1(\mathbf{R}^N) \times L^2(\mathbf{R}^N)$ is energy space. Assume $N \geq 3$ and (1.4). *Then we have*

$$\sigma_p(H_b) = \mathbf{C}_- , \quad \sigma_r(H_b) = \emptyset ,$$
$$\sigma_c(H_b) = \mathbf{R} , \quad \rho(H_b) = \mathbf{C}_+ ,$$

where $\sigma_p(A)$, $\sigma_r(A)$, $\sigma_c(A)$ and $\rho(A)$ denote point spectrum, residue spectrum, continuous spectrum and resolvent set of operator A, respectively.

In general case, we find the following result on spectral structure for H_b:

THEOREM 5. *Assume $N \geq 3$ and $|b(x)| \leq b_1|x|^{-1}$ in $\mathbf{R}^N$ for some $b_1 \in (0, N-2)$. Then the following inclusion relation holds:*

$$\sigma_p(H_b) \subset \left\{ \kappa = \alpha + i\beta \in \mathbf{C} \mid \beta^2 \leq \frac{b_1^2}{(N-2)^2 - b_1^2}\alpha^2 \right\}$$

REMARK 6. Under the assumption with the smallness of the dissipation, the solution for (1.1)–(1.2) behave like free solutions, i.e., if there exists a L^1-function $a(r)$ ($r = |x|$) such that

$$|b(x)| \leq a(r) \quad \text{in} \quad \mathbf{R}^N$$

with

$$0 < \left\{ \sum_{j \geq 1} 2^{j-1} a(2^{j-2}) \right\} b_2 < 1 ,$$

$$b_2 = \begin{cases} 1 + \sqrt{5} & (N \geq 3) , \\ \{(1+\sqrt{5})^2 + 1\}^{1/2} & (N = 2) , \end{cases}$$

where $2^{-1} \equiv 0$, then the Møller wave operator and the scattering operator exist ([10]). See also [8]. If we do not assume the smallness of the dissipation then the existence of scattering states is shown, but not the completeness of the wave operator ([7], [10]).

Moreover if $N \geq 3$ and

$$0 \leq b(x) \leq b_3(1+r)^{-1-\delta} \quad \text{in} \quad \mathbf{R}^N$$

for some $\delta \in (0, 1]$ with $0 < b_2 < \frac{2-\sqrt{2}}{2}\delta$, then the limiting absorption principle holds and the spectral structure coinsides with the free operator H_0 ([9]). See also [2], Appendix C.

Finally, we state a result on exponential decaying solution without dissipation in equation (1.3).

THEOREM 7. *Assume that*

$$w_0(x) = f(r) \equiv e^{i\sqrt{\sigma}r} , \quad w_1(x) = 0 ,$$

where $\sigma \in \mathbf{C}$, $Im\,\sigma < 0$, $Im\,\sqrt{\sigma} > 0$. Then the solution of (1.3) *is given by $w(t, x) = f(r+t)$, and the total energy decays exponentially as t goes to infinity.*

REMARK 8. This example can be regarded as the following equation, formally:

$$w_{tt}(x,t) + (-\Delta + \sigma\delta(x))w(x,t) = f(x,t),$$

where $\delta(x)$ is the delta-function on $\mathbf{R}^3$(cf. [2]).

The contents of the present paper will be outlined as follows.

In Section 2, we construct a solution for the stationary equation (2.1) below (Proposition 2.1). Theorem 1, 4, 5 and 7 will be shown in Section 3.

2. Construction of the stationary solution

We construct a solution for the stationary equation corresponding to (1.1) and (1.2).

PROPOSITION 2.1. *If $b(x)$ is the function given by* (1.4), *then the explicit solution of the stationary problem of* (1.1);

$$(-\Delta - i\kappa b(x) - \kappa^2)u(x) = 0, \tag{2.1}$$

is given by

$$u(x) = \begin{cases} |x|e^{-i\kappa|x|}, & (N=1) \\ e^{-i\kappa|x|}, & (N \geq 2) \end{cases}$$

where $\kappa = \alpha + i\beta$ ($\alpha \in \mathbf{R}$, $\beta < 0$).

To find the solution for (2.1) we put (cf. [3], [5])

$$u(x) = e^{p(|x|)} \tag{2.2}$$

with

$$p(|x|) = -i\kappa|x| - \frac{(N-1)}{2}\log|x| + \frac{1}{2}\int_1^{|x|} b(s)ds. \tag{2.3}$$

If b is the solution of

$$2b_{|x|}(|x|) + b(|x|)^2 - \frac{(N-1)(N-3)}{|x|^2} = 0, \tag{2.4}$$

then (2.2) satisfies (2.1).

LEMMA 2.2. *If $b(x)$ satisfies* (1.4), *then* (2.4) *holds.*

PROOF. Put $h(r) = rb(r)$ with $r = |x|$. Then the function h solves the equation:

$$2rh_r(r) + (h(r) - N + 1)(h(r) + N - 3) = 0. \tag{2.5}$$

This is easily solved to obtain the desired result. □

PROOF OF PROPOSITION 2.1. Noting Lemma 2.2, (1.4), (2.2) and (2.3), we easily obtain the desired results. □

3. Proof of Theorems

PROOF OF THEOREM 1. Without loss of generarity, we may assume $N \geq 2$. By Proposition 2.1, we find that the function $u(r;\alpha)$ defined by

$$u(r;\alpha) = e^{-i(\alpha+i\beta)r}$$

is the solution of (2.1). Thus the solution $w(r;\alpha)$ of (1.1) with

$$w_0(r) = u(r;\alpha)\,, \quad w_1(r;\alpha) = u_r(r;\alpha)$$

is given by $w_\alpha(t,r) = u(r+t;\alpha)$. Since

$$g(r) = (2\pi)^{-N/2} \int_{\mathbf{R}} \check{g}(\alpha) e^{-i\alpha r} d\alpha\,,$$

where we extend g to the following $\tilde{g}$:

$$\tilde{g}\,(s) = \begin{cases} g(s) & (s \geq 0)\,, \\ 0 & (s < 0)\,, \end{cases}$$

we obtain

$$(2\pi)^{-N/2} \int_{\mathbf{R}} \check{\tilde{g}}\,(\alpha) w_\alpha(t,r) d\alpha = e^{\beta(r+t)} \tilde{g}\,(r+t) = f(r+t)$$

is the solution of (1.1)–(1.2). □

To show Theorem 4, we prepare the following

LEMMA 3.1. *Under the same assumption as in Theorem* 4, *the operator H_b is maximal dissipative and*

$$\mathcal{R}(H_b - i) = E\,,$$

therefore the operator H_b generates the contraction semi-group in energy space E.

PROOF. Since b is non-negative, we easily obtain

$$\text{(3.1)} \qquad \operatorname{Im}(H_b v, v)_E = -\|\sqrt{b(\cdot)} v_2\|^2 \leq 0\,,$$

where $(\cdot,\cdot)_E$ denote the inner product in E. Hence H_b is dissipative. We prove that

$$\mathcal{R}(H_b - i) = E\,,$$

where $\mathcal{R}(A)$ denotes the range of operator A. We prove that for any $h = \{h_0, h_1\} \in E$ there exists a $v = \{v_0, v_1\} \in \mathcal{D}(H_b)$ such that

$$(H_b - i)v = h\,.$$

By the above system (equations) we have

$$(-\Delta + b + 1)v_1 = i(\Delta h_0 + h_1)\,, \quad v_0 = v_1 + i h_0\,.$$

We consider the operator $(-\Delta+b+1)$ in the L^2-theory. Taking account that b is $-\Delta$-bounded with the Δ-bound smaller than 1(infinitesimal small), we know that $\mathcal{D}(-\Delta+b+1) = H^2(\mathbf{R}^N)$ (cf. [4, chapter 6]). By rewriting the former equation we have

$$v_1 = i[(-\Delta + b + 1)^{-1}\nabla] \cdot \nabla h_0 + i(-\Delta + b + 1)^{-1}h_1 .$$

Hence we see that the first term is of H^1 and that the second term is of H^2. Therefore $v \in \mathcal{D}(H_b)$. □

PROOF OF THEOREM 4. By Proposition 2.1, $\mathbf{C}_- \subset \sigma_p(H_b)$ holds. If $\kappa \in \mathbf{C}_+$, then we obtain by (3.1)

$$(\mathrm{Im}\,\kappa)\|v\|_E \le \|(H_b - \kappa)v\|_E .$$

From this, it follows that $\sigma_p(H_b) \cap \mathbf{C}_+ = \emptyset$. Assume $\kappa \in \mathbf{R}$. Multiplying $\overline{u}$ on (2.1) and integrating by parts, we have

$$\|\sqrt{b(\cdot)}u\|^2 = 0$$

and from this, it holds $\sigma_p(H_b) \cap \mathbf{R} = \emptyset$. These arguments show $\sigma_p(H_b) = \mathbf{C}_-$. Since $H_b^* = H_{-b}$, we have $\sigma_p(H_b^*) = \mathbf{C}_+$. Therefore we have $\sigma_r(H_b) = \emptyset$ if we note the relation

$$\kappa \in \sigma_r(H_b) \quad \Leftrightarrow \quad \overline{\kappa} \in \sigma_p(H_b^*) \;\;\&\;\; \kappa \notin \sigma_p(H_b) .$$

On resolvent set, we have $\mathbf{C}_+ \subset \rho(H_b) \subset \mathbf{R} \cap \mathbf{C}_+$ by Lemma 3.1. Since the resolvent set is open in $\mathbf{C}$, we obtain $\rho(H_b) = \mathbf{C}_+$. Thus $\sigma_c(H_b) = \mathbf{R}$ holds. □

PROOF OF THEOREM 5. Assume that $u \neq 0$. Multiplying $\overline{u}$ on (2.1), integrating by parts on $\mathbf{R}^N$ and taking the real part, we have

$$\|\nabla u\|^2 + \beta \int_{\mathbf{R}^N} b(x)|u(x)|^2 dx + \beta^2\|u\|^2 = \alpha^2\|u\|^2 .$$

Using Hardy inequality

$$\left\| \frac{u}{|\cdot|} \right\| \le \frac{2}{N-2}\|\nabla u\|^2 ,$$

we obtain

$$\left|\beta \int_{\mathbf{R}^N} b(x)|u(x)|^2 dx\right| \le |\beta| b_1 \int_{\mathbf{R}^N} \frac{|u(x)|}{r} \cdot |u(x)| dx \le |\beta| b_1 \frac{2}{N-2}\|\nabla u\|\|u\| .$$

Putting

$$f(|\beta|) \equiv \|u\|^2|\beta|^2 - \frac{2b_1}{N-2}\|\nabla u\|\|u\||\beta| + \|\nabla u\|^2 ,$$

we find

$$f(|\beta|) \le \alpha^2\|u\|^2 . \tag{3.2}$$

On the other hand, we have by the assumption on b,

$$
\begin{cases}
f(|\beta|) &= \|u\|^2\left(|\beta|^2 - 2\dfrac{b_1}{N-2}\dfrac{\|\nabla u\|}{\|u\|}|\beta|\right) + \|\nabla u\|^2 \\
&= \|u\|^2\left(|\beta| - \dfrac{b_1}{N-2}\dfrac{\|\nabla u\|}{\|u\|}\right)^2 + \left\{1 - \left(\dfrac{b_1}{N-2}\right)^2\right\}\|\nabla u\|^2 \\
&\geq \left\{1 - \left(\dfrac{b_1}{N-2}\right)^2\right\}\|\nabla u\|^2 .
\end{cases}
\tag{3.3}
$$

Therefore we find that from (3.2), (3.3)

$$
\left\{1 - \left(\frac{b_1}{N-2}\right)^2\right\}\|\nabla u\|^2 \leq \alpha^2\|u\|^2 . \tag{3.4}
$$

Multiplying $\overline{u}$ on (1.4), integrating by parts on $\mathbf{R}^N$ and taking imaginary part, we have

$$
\int_{\mathbf{R}^N} b(x)|u(x)|^2 dx + 2\beta\|u\|^2 = 0 .
$$

Using Hardy inequality again, we obtain

$$
2|\beta|\|u\|^2 \leq \frac{2b_1}{N-2}\|\nabla u\|\|u\| \Leftrightarrow \|u\| \leq \frac{b_1}{N-2}\frac{\|\nabla u\|}{|\beta|} . \tag{3.5}
$$

It follows from (3.4), (3.5) that

$$
\begin{aligned}
&\left\{1 - \left(\frac{b_1}{N-2}\right)^2\right\}\|\nabla u\|^2 \leq \alpha^2\|u\|^2 \leq \alpha^2\left(\frac{b_1}{N-2}\right)^2\frac{\|\nabla u\|^2}{|\beta|^2} \\
&\Leftrightarrow \left(\left\{1 - \left(\frac{b_1}{N-2}\right)^2\right\}\beta^2 - \left(\frac{b_1}{N-2}\right)^2\alpha^2\right)\|\nabla u\|^2 \leq 0 .
\end{aligned}
\tag{3.6}
$$

So, if

$$
\kappa \notin \left\{ \kappa = \alpha + i\beta \in \mathbf{C} \mid \beta^2 \leq \frac{b_1^2}{(N-2)^2 - b_1^2}\alpha^2 \right\}
$$

then we find $u = 0$ by (3.6) and (3.5). □

PROOF OF THEOREM 7. Direct computation. □

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References

[1] M. KADOWAKI, Resolvent estimate and scattering states for dissipative systems, Talk presented at the Workshop in Hachinohe National College of Technology (2000).

[2] M. KADOWAKI, H. NAKAZAWA and K. WATANABE, On the asymptotics of solutions for some Schrödinger equations with dissipative perturbations of rank one, (submitted) (2002).

[3] T. KATO, Growth properties of solutions of the reduced wave equation with variable coefficient, Comm. Pure. Appl. Math. **12** (1959), 403–425.

[4] T. KATO, *Perturbation Theory for Linear Operators*, 2nd-ed., Springer-Verlag (1976).

[5] M. KAWASHITA, H. NAKAZAWA and H. SOGA, Non decay of the total energy for the wave equation with the dissipative term of spatial anisotropy, (to appear) (2002).

[6] A. MAJDA, Disappearing solution for dissipative wave equation, Indiana Univ. Math. J. **24** (1975), 1119–1133.

[7] K. MOCHIZUKI, Scattering theory for wave equations with dissipative terms, Publ. RIMS Kyoto Univ. **12** (1976), 383–390.

[8] K. MOCHIZUKI, Inverse scattering for a small nonselfadjoint perturbation of the wave equation, to appear in Kluwer Ser. Proceeding ISSAC '01.

[9] H. NAKAZAWA, Principle of limiting absorption for the non-selfadjoint Schrödinger operator with energy dependent potential, Tokyo. J. Math bf 23 (2000), 519–536.

[10] H. NAKAZAWA, Scattering theory for the nonconservative wave equations in $\mathbf{R}^N$ $(N \geq 2)$, (submitted) (2002).

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