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Spectral Geometry of Kähler Hypersurfaces in a Complex Grassmann Manifold

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1. Introduction

Let M be a compact C^∞-Riemannian manifold, $C^\infty(M)$ the space of all smooth functions on M, and Δ the Laplacian on M. Then Δ is a self-adjoint elliptic differential operator acting on $C^\infty(M)$, which has an infinite discrete sequence of eigenvalues:

$$Spec(M) = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \uparrow \infty\}.$$

Let $V_k = V_k(M)$ be the eigenspace of Δ corresponding to the k-th eigenvalue λ_k. Then V_k is finite-dimensional. We define an inner product $(\ ,\ )_{L^2}$ on $C^\infty(M)$ by

$$(f,\ g)_{L^2} = \int_M f\,g\,dv_M\,,$$

where dv_M denotes the volume element on M. Then $\sum_{t=0}^{\infty} V_t$ is dense in $C^\infty(M)$ and the decomposition is orthogonal with respect to the inner product $(\ ,\ )_{L^2}$. Thus we have

$$C^\infty(M) = \sum_{t=0}^{\infty} V_t(M) \quad (\text{in } L^2\text{-sense})\,.$$

Since M is compact, V_0 is the space of all constant functions which is 1-dimensional.

In this point of view, it is one of the simplest and the most interesting problems to estimate the first eigenvalue. In [13], A. Ros gave the following sharp upper bound for the first eigenvalue of Kähler submanifold of a complex projective space.

THEOREM 1.1. *Suppose that M is a complex m-dimensional compact Kähler submanifold of the complex projective space $\mathbf{C}P^n$ of constant holomorphic sectional curvature c. Then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c(m+1)\,.$$

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The equality holds if and only if M is congruent to a totally geodesic Kähler submanifold $\mathbf{C}P^m$ *of* $\mathbf{C}P^n$.

If M is not totally geodesic, J-P. Bourguignon, P. Li and S. T. Yau in [3] gave the following sharper estimate. (See also [11].)

THEOREM 1.2. *Suppose that M is a complex m-dimensional compact Kähler submanifold of $\mathbf{C}P^n$, which is fully immersed and not totally geodesic. Then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c\,m\,\frac{n+1}{n}\,.$$

It is not known when the equality holds in this inequality.

The purpose of this paper is to give the upper bound for the first eigenvalue of Kähler hypersurfaces of a complex Grassmann manifold.

Denote by $G_r(\mathbf{C}^n)$ the complex Grassmann manifold of r-planes in $\mathbf{C}^n$, equipped with the Kähler metric of maximal holomorphic sectional curvature c. In the case that M is a complex hypersurface of $G_r(\mathbf{C}^n)$, we obtain the following result, which is a generalization of Theorem 1.1.

THEOREM A. *Suppose that M is a compact connected Kähler hypersurface of $G_r(\mathbf{C}^n)$. Then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c\left(n - \frac{n-2}{r(n-r)-1}\right).$$

The equality holds if and only if $r = 1$ or $n-1$, and M is congruent to the totally geodesic complex hypersurface $\mathbf{C}P^{n-2}$ of the complex projective space $\mathbf{C}P^{n-1}$.

The 2-plane Grassmann manifold $G_2(\mathbf{C}^n)$ admits the quaternionic Kähler structure $\mathfrak{J}$. For the normal bundle $T^{\perp}M$ of a Kähler hypersurface M of $G_2(\mathbf{C}^n)$, $\mathfrak{J}T^{\perp}M$ is a vector bundle of real rank 6 over M which is a subbundle of the tangent bundle of $G_2(\mathbf{C}^n)$. We consider a Kähler hypersurface M of $G_2(\mathbf{C}^n)$ satisfying the property that $\mathfrak{J}T^{\perp}M$ is a subbundle of the tangent bundle TM of M. In Section 5, we will provide examples satisfying this property.

For a Kähler hypersurface of $G_2(\mathbf{C}^n)$ satisfying this property, we obtain the following upper bound of the first eigenvalue.

THEOREM B. *Suppose that M is a compact connected Kähler hypersurface of $G_2(\mathbf{C}^n)$, $n \geqq 4$. If M satisfies the condition $\mathfrak{J}\ T^{\perp}M \subset TM$, then the first eigenvalue λ_1 satisfies*

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right).$$

The equality holds if and only if $n = 4$ and M is congruent to the totally geodesic complex hypersurface Q^3 of the complex quadric $Q^4 = G_2(\mathbf{C}^4)$.

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NOTATIONS. $M_{r,s}(\mathbf{C})$ denotes the set of all $r \times s$ matrices with entries in $\mathbf{C}$, and $M_r(\mathbf{C})$ stands for $M_{r,r}(\mathbf{C})$. I_r and O_r denote the identity r-matrix and the zero r-matrix.

2. Preliminaries

In this section, we discuss geometries of the complex r-plane Grassmann manifold and its first standard imbedding.

Let $M_r(\mathbf{C}^n)$ be the complex Stiefel manifold which is the set of all unitary r-systems of $\mathbf{C}^n$, i.e.,

$$M_r(\mathbf{C}^n) = \{Z \in M_{n,r}(\mathbf{C}) \mid Z^* Z = I_r\}.$$

The complex r-plane Grassmann manifold $G_r(\mathbf{C}^n)$ is defined by

$$G_r(\mathbf{C}^n) = M_r(\mathbf{C}^n)/U(r).$$

The origin o of $G_r(\mathbf{C}^n)$ is defined by $\pi(Z_0)$, where $Z_0 = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$ is an element of $M_r(\mathbf{C}^n)$, and $\pi : M_r(\mathbf{C}^n) \to G_r(\mathbf{C}^n)$ is the natural projection.

The left action of the unitary group $\tilde{G} = SU(n)$ on $G_r(\mathbf{C}^n)$ is transitive, and the isotropy subgroup at the origin o is

$$\begin{aligned} \tilde{K} &= S(\,U(r) \cdot U(n-r)\,) \\ &= \left\{ \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \,\middle|\, U_1 \in U(r),\ U_2 \in U(n-r),\ \det U_1 \det U_2 = 1 \right\}, \end{aligned}$$

so that $G_r(\mathbf{C}^n)$ is identified with a homogeneous space $\tilde{G}/\tilde{K}$.

Set $\tilde{\mathfrak{g}} = \mathfrak{su}(n)$ and

$$\begin{aligned} \tilde{\mathfrak{k}} &= \mathbf{R} \oplus \mathfrak{su}(r) \oplus \mathfrak{su}(n-r) \\ &= \left\{ \begin{pmatrix} u_1 & 0 \\ 0 & u_2 \end{pmatrix} + a \begin{pmatrix} -\frac{1}{r}\sqrt{-1} I_r & 0 \\ 0 & \frac{1}{n-r}\sqrt{-1} I_{n-r} \end{pmatrix} \,\middle|\, a \in \mathbf{R}, \begin{array}{c} u_1 \in \mathfrak{su}(r) \\ u_2 \in \mathfrak{su}(n-r) \end{array} \right\}, \end{aligned}$$

then $\tilde{\mathfrak{g}}$ and $\tilde{\mathfrak{k}}$ are the Lie algebras of $\tilde{G}$ and $\tilde{K}$, respectively. Define a linear subspace $\tilde{\mathfrak{m}}$ of $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{m}} = \left\{ \begin{pmatrix} 0 & -\xi^* \\ \xi & 0 \end{pmatrix} \,\middle|\, \xi \in M_{n-r,r}(\mathbf{C}) \right\}.$$

Then $\tilde{\mathfrak{m}}$ is identified with the tangent space $T_o(G_r(\mathbf{C}^n))$. The $\tilde{G}$-invariant complex structure J of $G_r(\mathbf{C}^n)$ and the $\tilde{G}$-invariant Kähler metric $\tilde{g}_c$ of $G_r(\mathbf{C}^n)$ of the maximal holomorphic sectional curvature c are given by

$$J\begin{pmatrix}0 & -\xi^* \\ \xi & 0\end{pmatrix} = \begin{pmatrix}0 & \sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0\end{pmatrix},$$

$$\tilde{g}_{c_o}(X, Y) = -\frac{2}{c} trXY, \quad X, Y \in \tilde{\mathfrak{m}}. \tag{2.1}$$

Notice that $\tilde{g}_c$ satisfies

$$\tilde{g}_{c_o} = -\frac{2}{c}\frac{1}{2n}B_{\tilde{\mathfrak{g}}} = -\frac{2}{c}\frac{L(\tilde{\mathfrak{g}})}{2}B_{\tilde{\mathfrak{g}}} \tag{2.2}$$

on $\tilde{\mathfrak{m}}$, where $B_{\tilde{\mathfrak{g}}}$ is the Killing form of $\tilde{\mathfrak{g}}$, and $L(\tilde{\mathfrak{g}})$ is the squared length of the longest root of $\tilde{\mathfrak{g}}$ relative to the Killing form.

In the case of $r = 2$, the complex 2-plane Grassmann manifold $G_2(\mathbf{C}^n)$ admits another geometric structure named the quaternionic Kähler structure $\mathfrak{J}$. $\mathfrak{J}$ is a $\tilde{G}$-invariant subbundle of $End(T(G_2(\mathbf{C}^n)))$ of rank 3, where $End(T(G_2(\mathbf{C}^n)))$ is the $\tilde{G}$-invariant vector bundle of all linear endmorphisms of the tangent bundle $T(G_2(\mathbf{C}^n))$. Under the identification of $T_o(G_r(\mathbf{C}^n))$ with $\tilde{\mathfrak{m}}$, the fiber $\mathfrak{J}_o$ at the origin o is given by

$$\mathfrak{J}_o = \{J_{\tilde{\varepsilon}} = ad(\tilde{\varepsilon}) \mid \tilde{\varepsilon} \in \tilde{\mathfrak{k}}_q\},$$

where $\tilde{\mathfrak{k}}_q$ is an ideal of $\tilde{\mathfrak{k}}$ defined by

$$\tilde{\mathfrak{k}}_q = \left\{\begin{pmatrix}u_1 & 0 \\ 0 & 0\end{pmatrix} \;\middle|\; u_1 \in \mathfrak{su}(2)\right\} \cong \mathfrak{su}(2).$$

Define a basis $\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ of $\mathfrak{su}(2)$ by

$$\varepsilon_1 = \begin{pmatrix}\sqrt{-1} & 0 \\ 0 & -\sqrt{-1}\end{pmatrix}, \quad \varepsilon_2 = \begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}, \quad \varepsilon_3 = \begin{pmatrix}0 & \sqrt{-1} \\ \sqrt{-1} & 0\end{pmatrix}.$$

Then ε_1, ε_2 and ε_3 satisfy

$$[\varepsilon_1, \varepsilon_2] = 2\varepsilon_3, \quad [\varepsilon_2, \varepsilon_3] = 2\varepsilon_1, \quad [\varepsilon_3, \varepsilon_1] = 2\varepsilon_2.$$

Set $\tilde{\varepsilon}_i = \begin{pmatrix}\varepsilon_i & 0 \\ 0 & 0\end{pmatrix}$ and $J_i = J_{\tilde{\varepsilon}_i}$ for $i = 1, 2, 3$. Then the basis $\{J_1, J_2, J_3\}$ is a canonical basis of $\mathfrak{J}_o$, satisfying

$$J_i^2 = -id_{\tilde{\mathfrak{m}}} \quad \text{for } i = 1, 2, 3,$$

$$J_1J_2 = -J_2J_1 = J_3, \quad J_2J_3 = -J_3J_2 = J_1, \quad J_3J_1 = -J_1J_3 = J_2,$$

$$\tilde{g}_{c_o}(J_iX, J_iY) = \tilde{g}_{c_o}(X, Y), \quad \text{for } X, Y \in \tilde{\mathfrak{m}} \text{ and } i = 1, 2, 3.$$

Since J is given by

$$J = ad(\tilde{\varepsilon}_{\mathbf{C}}), \quad \tilde{\varepsilon}_{\mathbf{C}} = \frac{r(n-r)}{n}\begin{pmatrix}-\frac{1}{r}\sqrt{-1}I_r & 0 \\ 0 & \frac{1}{n-r}\sqrt{-1}I_{n-r}\end{pmatrix}$$

on $\mathfrak{m}$, and since $\tilde{\varepsilon}_{\mathbf{C}}$ is an element of the center of $\tilde{\mathfrak{k}}$, J is commutable with $\mathfrak{J}$.

Let $HM(n, \mathbf{C})$ be the set of all Hermitian (n, n)-matrices over $\mathbf{C}$, which can be identified with $\mathbf{R}^{n^2}$. For $X, Y \in HM(n, \mathbf{C})$, the natural inner product is given by

$$(X, Y) = \frac{2}{c} \, trXY \, . \tag{2.3}$$

$GL(n, \mathbf{C})$ acts on $HM(n, \mathbf{C})$ by $X \mapsto BXB^*$, $B \in GL(n, \mathbf{C})$, $X \in HM(n, \mathbf{C})$. Then the action of $SU(n)$ leaves the inner product (2.3) invariant. Define two linear subspaces of $HM(n, \mathbf{C})$ as follows:

$$HM_0 = \{X \in HM(n, \mathbf{C}) \mid trX = 0\} \, ,$$
$$HM_{\mathbf{R}} = \{aI \mid a \in \mathbf{R}\} \, ,$$

where I is the n-identity matrix. Both of them are invariant under the action of $SU(n)$, and irreducible. We get the orthogonal decomposition of $HM(n, \mathbf{C})$ as follows:

$$HM(n, \mathbf{C}) = HM_0 \oplus HM_{\mathbf{R}} \, .$$

It is well-known that HM_0 (resp. $HM_{\mathbf{R}}$) is identified with the first eigenspace $V_1(G_r(\mathbf{C}^n))$ (resp. the set of all constant functions, i.e. $V_0(G_r(\mathbf{C}^n))$).

The first standard imbedding Ψ of $G_r(\mathbf{C}^n)$ is defined by

$$\Psi(\pi(Z)) = ZZ^* \in HM(n, \mathbf{C}) \, , \quad Z \in M_r(\mathbf{C}^n) \, .$$

Ψ is $SU(n)$-equivariant and the image N of $G_r(\mathbf{C}^n)$ under Ψ is given by

$$N = \Psi(G_r(\mathbf{C}^n)) = \{A \in HM(n, \mathbf{C}) \mid A^2 = A, \ trA = r\} \, , \tag{2.4}$$

so that it is contained fully in a hyperplane

$$HM_r = \{A \in HM(n, \mathbf{C}) \mid trA = r\} = \left\{ A + \frac{r}{n} I \;\middle|\; A \in HM_0 \right\}$$

of $HM(n, \mathbf{C})$. The tangent bundle TN and the normal bundle $T^{\perp}N$ are given by

$$\begin{aligned} T_A N &= \{X \in HM(n, \mathbf{C}) \mid XA + AX = X\} \subset HM_0 \, , \\ T_A^{\perp} N &= \{Z \in HM(n, \mathbf{C}) \mid ZA = AZ\} \, . \end{aligned} \tag{2.5}$$

In particular, at the origin $A_o = \Psi(o) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$, we can obtain

$$\begin{aligned} T_{A_o} N &= \left\{ \begin{pmatrix} 0 & \xi^* \\ \xi & 0 \end{pmatrix} \;\middle|\; \xi \in M_{n-r,r}(\mathbf{C}) \right\} \, , \\ T_{A_o}^{\perp} N &= \left\{ \begin{pmatrix} Z_1 & 0 \\ 0 & Z_2 \end{pmatrix} \;\middle|\; Z_1 \in HM(r, \mathbf{C}), \ Z_2 \in HM(n-r, \mathbf{C}) \right\} \, . \end{aligned} \tag{2.6}$$

The complex structure J acts on $T_{A_o}N$ as

$$J\begin{pmatrix}0 & \xi^* \\ \xi & 0\end{pmatrix} = \begin{pmatrix}0 & -\sqrt{-1}\xi^* \\ \sqrt{-1}\xi & 0\end{pmatrix}. \tag{2.7}$$

If $r = 2$, then the quaternionic Kähler structure $\mathfrak{J}$ acts on $T_{A_o}N$ as

$$J_{\tilde{\varepsilon}}\begin{pmatrix}0 & \xi^* \\ \xi & 0\end{pmatrix} = \begin{pmatrix}0 & \varepsilon\xi^* \\ -\xi\varepsilon & 0\end{pmatrix}, \quad \varepsilon \in \mathfrak{su}(2). \tag{2.8}$$

Let $\tilde{\sigma}$ and $\tilde{H}$ denote the second fundamental form and the mean curvature vector of Ψ, respectively. Then, for $A \in N$ and $X, Y \in T_AN$, we can see

$$\tilde{\sigma}_A(X, Y) = (XY + YX)(I - 2A), \tag{2.9}$$

$$\tilde{H}_A = \frac{c}{2r(n-r)}(rI - nA) \tag{2.10}$$

and $\tilde{\sigma}$ satisfies the following:

$$\tilde{\sigma}_A(JX, JY) = \tilde{\sigma}_A(X, Y), \tag{2.11}$$

$$(\tilde{\sigma}_A(X, Y), A) = -(X, Y). \tag{2.12}$$

Denote by $S^{n^2-2}(\frac{c}{2}\frac{n}{r(n-r)})$ the hypersphere in HM_r centered at $\frac{r}{n}I$ with radius $\sqrt{\frac{2}{c}\frac{r(n-r)}{n}}$. Then we see that Ψ is a minimal immersion of $G_r(\mathbf{C}^n)$ into $S^{n^2-2}(\frac{c}{2}\frac{n}{r(n-r)})$, and that the center of mass of $\Psi(G_r(\mathbf{C}^n))$ is $\frac{r}{n}I$. In fact, Ψ satisfies the equation $\Delta\Psi = cn(\Psi - \frac{r}{n}I)$. Moreover, all coefficients of $\Psi - \frac{r}{n}I$ span the first eigenspace $V_1(G_r(\mathbf{C}^n))$.

Let's assume that M is a submanifold of $G_r(\mathbf{C}^n)$ with an immersion φ. Then $F = \Psi \circ \varphi$ is an immersion of M into $HM(n, \mathbf{C})$, and the set of all coefficients of $F - \frac{r}{n}I$ spans the pull-back $\varphi^* V_1(G_r(\mathbf{C}^n))$.

3. Examples

One of the simplest typical examples of submanifolds of $G_r(\mathbf{C}^n)$ is a totally geodesic submanifold. B. Y. Chen and T. Nagano in [5, 6] determined maximal totally geodesic submanifolds of $G_2(\mathbf{C}^n)$. I. Satake and S. Ihara in [14, 9] determined all (equivariant) holomorphic, totally geodesic imbeddings of a symmetric domain into another symmetric domain. When an ambient symmetric domain is of type $(\mathrm{I})_{p,q}$, taking a compact dual symmetric space, we obtain the complete list of maximal totally geodesic Kähler submanifolds of $G_r(\mathbf{C}^n)$.

Let M be a maximal totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$ given by a Kähler immersion $\varphi : M \to G_r(\mathbf{C}^n)$. Since M is a symmetric space, denote by (G, K) the compact symmetric pair of M, and denote by $(\mathfrak{g}, \mathfrak{k})$ its Lie algebra. Then there exists a certain unitary representation $\rho : G \to \tilde{G} = SU(n)$, such that $\varphi(M)$ is given by the orbit of $\rho(G)$ through the origin $o = \{\tilde{K}\}$ in $G_r(\mathbf{C}^n)$.

Let $L(\mathfrak{g})$ be the squared length of the longest root of $\mathfrak{g}$ relative to the Killing form $B_{\mathfrak{g}}$. Tables of the $L(\mathfrak{g})$ constants appear in [8]. The Kähler metric induced by φ is a G-invariant metric corresponding to an $Ad(G)$-invariant inner product

$$\rho^*\left(-\frac{2}{c}\frac{L(\tilde{\mathfrak{g}})}{2}B_{\tilde{\mathfrak{g}}}\right) = -\frac{2}{c}\frac{L(\mathfrak{g})}{2}\, l_\rho\, B_{\mathfrak{g}} \tag{3.1}$$

on $\mathfrak{g}$, where l_ρ is the index of a linear representation ρ defined by Dynkin. Tables of indices of basic representations of simple Lie algebras appear in [7].

Using Freudenthal's formula with respect to the inner product (3.1), we can calculate the first eigenvalue of the Laplacian of M. (cf. [17])

Summing up these results, we obtain the following.

THEOREM 3.1. *Let $M = G/K$ be a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$, ρ a corresponding unitary representation of G to $SU(n)$, and λ_1 the first eigenvalue of the Laplacian with respect to the induced Kähler metric. Then, M, ρ and λ_1 are one of the following (up to isomorphism).*

(1) $M_1 = G_r(\mathbf{C}^{n-1}) \hookrightarrow G_r(\mathbf{C}^n)$, $\quad 1 \leqq r \leqq n-2$,
$\rho_1 =$ *natural inclusion* and $\lambda_1 = c(n-1)$

(2) $M_2 = G_{r-1}(\mathbf{C}^{n-1}) \hookrightarrow G_r(\mathbf{C}^n)$, $\quad 2 \leqq r \leqq n-1$,
$\rho_2 =$ *natural inclusion* and $\lambda_1 = c(n-1)$

(3) $M_3 = G_{r_1}(\mathbf{C}^{n_1}) \times G_{r_2}(\mathbf{C}^{n_2}) \hookrightarrow G_{r_1+r_2}(\mathbf{C}^{n_1+n_2})$, $\quad 1 \leqq r_i \leqq n_i - 1,\ i = 1, 2$,
$\rho_3 =$ *natural inclusion* and $\lambda_1 = c\min\{n_1, n_2\}$

(4) $M_4 = M_{4,p} = Sp(p)/U(p) \hookrightarrow G_p(\mathbf{C}^{2p})$, $\quad p \geqq 2$,
$\rho_4 =$ *natural inclusion* and $\lambda_1 = c(p+1)$

(5) $M_5 = M_{5,p} = SO(2p)/U(p) \hookrightarrow G_p(\mathbf{C}^{2p})$, $\quad p \geqq 4$,
$\rho_5 =$ *natural inclusion* and $\lambda_1 = c(p-1)$

(6) $M_{6,m} = \mathbf{C}P^p \hookrightarrow G_r(\mathbf{C}^n)$: *the complex projective space*,
$r = \binom{p}{m-1}$, $\quad n = \binom{p+1}{m}$, $\quad 2 \leqq m \leqq p-1$,
$\rho_{6,m} =$ *the exterior representation of degree* m,
and $\lambda_1 = c(p+1)\binom{p-1}{m-1}^{-1}$

(7) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbf{C}^4)$: *the complex quadric*,
$\rho_7 =$ *spin representation* and $\lambda_1 = 3c$

(8) $M_8 = M_{8,2l} = Q^{2l} \hookrightarrow G_r(\mathbf{C}^{2r})$: *the complex quadric*, $\quad r = 2^{l-1}$, $\quad l \geqq 3$,
$\rho_8^{\pm} =$ *(two) spin representations* and $\lambda_1 = c\dfrac{2l}{2^{l-2}}$

In the above list, notice that $M_{4,2} = M_7$ and $M_{5,4} = M_{8,6}$.

Another one of the simplest typical examples of submanifolds of $G_r(\mathbf{C}^n)$ is a homogeneous Kähler hypersurface. K. Konno in [10] determined all Kähler C-spaces embedded as a hypersurface into a Kähler C-space with the second Betti number $b_2 = 1$.

THEOREM 3.2. *Let M be a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbf{C}^n)$, and λ_1 the first eigenvalue of the Laplacian with respect to the induced Kähler metric. Then, M and λ_1 are one of the following (up to isomorphism).*

(1) $M_9 = \mathbf{C}P^{n-2} \hookrightarrow \mathbf{C}P^{n-1} = G_1(\mathbf{C}^n)$ *and* $\lambda_1 = c(n-1)$

(2) $M_{10} = Q^{n-2} \hookrightarrow \mathbf{C}P^{n-1} = G_1(\mathbf{C}^n)$ *and* $\lambda_1 = c(n-2)$

(3) $M_7 = Q^3 \hookrightarrow Q^4 = G_2(\mathbf{C}^4)$ *and* $\lambda_1 = 3c$

(4) $M_{11} = Sp(l)/U(2) \cdot Sp(l-2) \hookrightarrow G_2(\mathbf{C}^{2l})$: *Kähler C-space of type* (C_l, α_2), $l \geqq 2$ *and* $\lambda_1 = c(2l-1)$

M_9 and M_7 are totally geodesic. M_9, M_{10} and M_7 are symmetric spaces. If $l = 2$, then M_{11} is congruent to M_7.

For each l with $l > 2$, M_{11} is not a symmetric space. Then, it is not easy to calculate the first eigenvalue λ_1 of M_{11}. We will calculate λ_1 of M_{11} in the next section.

From these two theorems, we obtain the following proposition:

PROPOSITION 3.3. *Let M be either a proper maximal totally geodesic Kähler submanifold of $G_r(\mathbf{C}^n)$ or a compact, simply connected homogeneous Kähler hypersurface of $G_r(\mathbf{C}^n)$. Then, the first eigenvalue λ_1 of M with respect to the induced Kähler metric satisfies*

$$\lambda_1 \leqq c\,(n-1)\,.$$

Moreover, the equality holds if and only if M is congruent to one of the following:

$$M_1\,, \quad M_2\,, \quad M_{4,2} = M_7\,, \quad M_9\,, \quad M_{11}\,.$$

4. The Kähler C-spaces with $b_2 = 1$

In this section, we will consider the first eigenvalue of the Kähler C-space whose second Betti number is equal to 1. First, we review the general theory of Kähler C-spaces. For details, see [2] and [16].

Let $\mathfrak{g}$ be a compact semisimple Lie algebra and $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{g}$. Denote by $\mathfrak{g}^{\mathbf{C}}$ and $\mathfrak{t}^{\mathbf{C}}$ the complexifications of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. $\mathfrak{t}^{\mathbf{C}}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathbf{C}}$. Let $(\ ,\ )$ be an $Ad(G)$-invariant inner product on $\mathfrak{g}$ defined by $-B_{\mathfrak{g}}$, where $B_{\mathfrak{g}}$ is the Killing form of $\mathfrak{g}$. Let $\Sigma \subset (\mathfrak{t}^{\mathbf{C}})^*$ denote the root system of $\mathfrak{g}$ relative to $\mathfrak{t}$. We have a root space decomposition of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha \in \Sigma} \mathfrak{g}^{\mathbf{C}}_{\alpha}\,, \tag{4.1}$$

where $\mathfrak{g}^{\mathbf{C}}_{\alpha} = \{X \in \mathfrak{g}^{\mathbf{C}} \mid (adH)X = \alpha(H)X$ for any $H \in \mathfrak{t}\}$. Since $\mathfrak{g}$ is compact type, for any $\alpha \in \Sigma$ and $H \in \mathfrak{t}$, $\alpha(H)$ is pure imaginary, so that there exists a unique element $\check{\alpha} \in \mathfrak{t}$ such that, for any $H \in \mathfrak{t}$, the equality $\alpha(H) = \sqrt{-1}(\check{\alpha}, H)$ holds. We identify α with $\check{\alpha}$, so that the root system Σ is identified with a subset $\{\check{\alpha} \mid \alpha \in \Sigma\}$ of $\mathfrak{t}$. Choose a lexicographic order $>$ on Σ and put $\Sigma^+ = \{\alpha \in \Sigma \mid \alpha > 0\}$. Let Π be the fundamental root system of Σ consisting of simple roots with respect to the linear order $>$. Π is identified with its Dynkin diagram. Let $\{\Lambda_\alpha\}_{\alpha\in\Pi} \subset \mathfrak{t}$ be the fundamental weight system of $\mathfrak{g}^{\mathbf{C}}$ corresponding to Π:

$$\frac{2(\Lambda_\alpha, \beta)}{(\beta, \beta)} = \begin{cases} 1 & \text{if } \alpha = \beta\,, \\ 0 & \text{if } \alpha \neq \beta\,. \end{cases}$$

Let Π_0 be a subdiagram of Π. We may suppose that the pair (Π, Π_0) is effective, that is, Π_0 contains no irreducible component of Π. Put $\Sigma_0 = \Sigma \cap \{\Pi_0\}_{\mathbf{Z}}$, where $\{\Pi_0\}_{\mathbf{Z}}$ denote the subgroup of $\mathfrak{t}$ generated by Π_0 over $\mathbf{Z}$. Define a subalgebra $\mathfrak{u}$ of $\mathfrak{g}^{\mathbf{C}}$ by

$$\mathfrak{u} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha\in\Sigma_0\cup\Sigma^+} \mathfrak{g}^{\mathbf{C}}_\alpha\,. \tag{4.2}$$

Let $G^{\mathbf{C}}$ be the connected complex semisimple Lie group without center, whose Lie algebra is $\mathfrak{g}^{\mathbf{C}}$, and U the connected closed complex subgroup of $G^{\mathbf{C}}$ generated by $\mathfrak{u}$. Let G be a compact connected semisimple subgroup of $G^{\mathbf{C}}$ generated by $\mathfrak{g}$ and put $K = G \cap U$. The canonical imbedding $G \to G^{\mathbf{C}}$ gives the diffeomorphism of a compact coset space $M = G/K$ to a simply connected complex coset space $G^{\mathbf{C}}/U$. Therefore, the homogeneous space $M = G/K$ is a complex, compact, simply connected manifold called a *generalized flag manifold* or a *Kähler C-space*. Lie algebra $\mathfrak{k}$ of K is given by

$$\mathfrak{k}^{\mathbf{C}} = \mathfrak{t}^{\mathbf{C}} + \sum_{\alpha\in\Sigma_0} \mathfrak{g}^{\mathbf{C}}_\alpha\,. \tag{4.3}$$

Define a subspace $\mathfrak{c}$ of $\mathfrak{t}$ and a cone $\mathfrak{c}^+$ in $\mathfrak{c}$ by

$$\mathfrak{c} = \sum_{\alpha\in\Pi-\Pi_0} \mathbf{R}\Lambda_\alpha\,,$$

$$\mathfrak{c}^+ = \{\theta \in \mathfrak{c} - \{0\} \mid (\theta, \alpha) > 0 \text{ for each } \alpha \in \Pi - \Pi_0\}\,, \tag{4.4}$$

respectively. Then we have $\mathfrak{c}^+ = \sum_{\alpha\in\Pi-\Pi_0} \mathbf{R}^+\Lambda_\alpha$, where $\mathbf{R}^+$ denotes the set of positive real numbers.

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to (,), so that we have a direct sum decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ as vector space. The subspace $\mathfrak{m}$ is K-invariant under the adjoint action and identified with the tangent space T_oM of M at the origin $o = \{K\}$. Put $\Sigma^+_{\mathfrak{m}} = \Sigma^+ - \Sigma_0$, $\Sigma^-_{\mathfrak{m}} = -\Sigma^+_{\mathfrak{m}}$ and define K-invariant subspaces $\mathfrak{m}^\pm$ of $\mathfrak{g}^{\mathbf{C}}$ by

$$\mathfrak{m}^\pm = \sum_{\alpha\in\Sigma^\pm_{\mathfrak{m}}} \mathfrak{g}^{\mathbf{C}}_{-\alpha}\,. \tag{4.5}$$

Then the complexification $\mathfrak{m}^{\mathbf{C}}$ of $\mathfrak{m}$ is the direct sum $\mathfrak{m}^{\mathbf{C}} = \mathfrak{m}^+ + \mathfrak{m}^-$, and $\mathfrak{m}^\pm$ is the $\pm\sqrt{-1}$-eigenspace of the complex structure J of M at the origin o.

Denote by $X \to \bar{X}$ the complex conjugation of $\mathfrak{g}^{\mathbf{C}}$ with respect to the real form $\mathfrak{g}$. We can choose root vectors $E_\alpha \in \mathfrak{g}^{\mathbf{C}}_\alpha$ for $\alpha \in \Sigma$ with the following properties and fix them once for all:

$$(4.6)\qquad [E_\alpha, E_{-\alpha}] = \sqrt{-1}\alpha\,, \quad (E_\alpha, E_{-\alpha}) = 1\,, \quad \bar{E}_\alpha = E_{-\alpha} \quad \text{for } \alpha \in \Sigma\,.$$

Let $\{\omega^\alpha\}_{\alpha\in\Sigma}$ be the linear forms of $\mathfrak{g}^{\mathbf{C}}$ dual to $\{E_\alpha\}_{\alpha\in\Sigma}$, more precisely, the linear forms defined by

$$\omega^\alpha(\mathfrak{t}^{\mathbf{C}}) = \{0\}\,,$$

$$\omega^\alpha(E_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta\,, \\ 0 & \text{if } \alpha \neq \beta\,. \end{cases}$$

Every G-invariant Kähler metric on M is given by

$$(4.7)\qquad g(\theta) = 2\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} (\theta, \alpha)\, \omega^{-\alpha}\cdot\bar{\omega}^{-\alpha}\,, \quad \omega^{-\alpha}\cdot\bar{\omega}^{-\alpha} = \frac{1}{2}(\omega^{-\alpha}\otimes\bar{\omega}^{-\alpha} + \bar{\omega}^{-\alpha}\otimes\omega^{-\alpha})$$

for $\theta \in \mathfrak{c}^+$. Note that the inner product (,) satisfies

$$(\ ,\ )_{\mathfrak{m}^+\times\overline{\mathfrak{m}^+}} = 2\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} \omega^{-\alpha}\cdot\bar{\omega}^{-\alpha}\,.$$

We define an element $\delta_{\mathfrak{m}} \in \mathfrak{t}$ by

$$\delta_{\mathfrak{m}} = \frac{1}{2}\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} \alpha \quad \in \mathfrak{c}^+\,.$$

Then, for the Kähler metric $g(\theta)$, the Ricci tensor Ric and the scalar curvature τ are given respectively by

$$Ric = 4\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} (\delta_{\mathfrak{m}}, \alpha)\, \omega^{-\alpha}\cdot\bar{\omega}^{-\alpha}\,,$$

$$(4.8)\qquad \tau = 4\sum_{\alpha\in\Sigma_{\mathfrak{m}}^+} \frac{(\delta_{\mathfrak{m}}, \alpha)}{(\theta, \alpha)}\,.$$

If $\Pi - \Pi_0$ consists of only one root, say α_r, then the Kähler C-space M is said to be of type $(\mathfrak{g}, \alpha_r)$. The second Betti number b_2 of M is equal to 1. In this case, we obtain

$$\mathfrak{c}^+ = \mathbf{R}^+\Lambda_{\alpha_r}\,,$$

so that there exists a positive real number b with $2\delta_{\mathfrak{m}} = b\Lambda_{\alpha_r}$. Therefore, $(\mathfrak{g}, \alpha_r)$ is a Kähler-Einstein manifold, and the Ricci tensor and the scalar curvature with respect to a Kähler metric $g(a\Lambda_{\alpha_r})$ are given by

$$Ric = \frac{b}{a} g(a\Lambda_{\alpha_r})\,, \quad \tau = 2\frac{b}{a}\dim_{\mathbf{C}} M\,,$$

respectively.

Y. Matsushima and M. Obata showed the following:

THEOREM 4.1 ([12]). *Let M be an n-dimensional compact Einstein Kähler manifold of positive scalar curvature τ. Then the first eigenvalue $\lambda_1(M)$ of the Laplacian satisfies that*

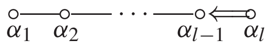
$$\lambda_1(M) \geqq \frac{\tau}{n}\,.$$

The equality holds if and only if M admits a one-parameter group of isometries (i.e., a non-trivial Killing vector field).

This theorem implies the following proposition immediately.

PROPOSITION 4.2. *For the Kähler C-space $M = (\mathfrak{g}, \alpha_r)$ equipped with the Kähler metric $g(a\Lambda_{\alpha_r})$, the first eigenvalue $\lambda_1(M)$ of the Laplacian is given by $\lambda_1(M) = \frac{2b}{a}$.*

From now on, we assume that $\mathfrak{g}$ is a compact semisimple simple Lie algebra of type C_l, $l \geqq 2$, and we consider a Kähler C-space of type $(\mathfrak{g}, \alpha_r)$. Then, Π is identified with the Dynkin diagram of type C_l



and Σ^+ is given by

$$\Sigma^+ = \left\{ \begin{array}{l} \alpha_i + \cdots + \alpha_{j-1} \quad (1 \leqq i < j \leqq l+1)\,, \\ (\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \leqq i \leqq j \leqq l-1) \end{array} \right\}.$$

Therefore, we have

$$\begin{aligned} \Sigma^+_{\mathfrak{m}} &= \Sigma' \cup \Sigma'' \text{: disjoint}\,, \\ \Sigma' &= \{\alpha_i + \cdots + \alpha_r + \cdots + \alpha_j \quad (1 \leqq i \leqq r \leqq j \leqq l)\}\,, \\ \Sigma'' &= \{(\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \leqq i \leqq r,\ i \leqq j \leqq l-1)\}\,. \end{aligned}$$

Immediately, we get

$$\dim_{\mathbf{C}} M = \#\Sigma^+_{\mathfrak{m}} = \frac{r}{2}(4l - 3r + 1)\,.$$

Put

$$\begin{aligned} \Sigma' &= \Sigma'_1 \cup \Sigma'_2 \cup \Sigma'_3 \cup \{\alpha_r\}\,, \\ \Sigma'_1 &= \{\alpha_i + \cdots + \alpha_{r-1} + \alpha_r + \alpha_{r+1} + \cdots + \alpha_j \quad (1 \leqq i \leqq r-1,\ r+1 \leqq j \leqq l)\}\,, \\ \Sigma'_2 &= \{\alpha_i + \cdots + +\alpha_{r-1} + \alpha_r \quad (1 \leqq i \leqq r-1)\}\,, \\ \Sigma'_3 &= \left\{\alpha_r + \alpha_{r+1} + \cdots + \alpha_j \quad (r+1 \leqq j \leqq l)\right\}. \end{aligned}$$

Then a direct computation gives

$$\sum_{\alpha\in\Sigma_1'}\alpha=\sum_{i=1}^{r-1}\sum_{j=r+1}^{l}\alpha_i+\cdots+\alpha_{r-1}+\alpha_r+\alpha_{r+1}+\cdots+\alpha_j$$

$$=(l-r)\sum_{i=1}^{r-1}\alpha_i+\cdots+\alpha_{r-1}$$

$$+(r-1)(l-r)\alpha_r+(r-1)\sum_{j=r+1}^{l}\alpha_{r+1}+\cdots+\alpha_j$$

$$=(l-r)\sum_{i=1}^{r-1}i\alpha_i+(r-1)(l-r)\alpha_r+(r-1)\sum_{j=r+1}^{l}(l-j+1)\alpha_j\,,$$

$$\sum_{\alpha\in\Sigma_2'}\alpha=\sum_{i=1}^{r-1}i\alpha_i+(r-1)\alpha_r\,,\qquad\sum_{\alpha\in\Sigma_3'}\alpha=(l-r)\alpha_r+\sum_{j=r+1}^{l}(l-j+1)\alpha_j\,,$$

so that we have

$$\sum_{\alpha\in\Sigma'}\alpha=(l-r+1)\sum_{i=1}^{r-1}i\alpha_i+r(l-r+1)\alpha_r+r\sum_{j=r+1}^{l}(l-j+1)\alpha_j\,. \tag{4.9}$$

On the other hand, we get

$$\sum_{\alpha\in\Sigma''}\alpha=\sum_{i\leqq r}\sum_{j=i}^{l-1}\{(\alpha_i+\cdots+\alpha_{l-1})+(\alpha_j+\cdots+\alpha_{l-1})+\alpha_l\} \tag{4.10}$$

$$=\sum_{i\leqq r}(l-i)(\alpha_i+\cdots+\alpha_l)+\sum_{i\leqq r}\sum_{j=i}^{l-1}(j-i+1)\alpha_j\,.$$

We have

$$\sum_{i\leqq r}(l-i)(\alpha_i+\cdots+\alpha_l) \tag{4.11}$$

$$=\sum_{i\leqq r}(l-i)(\alpha_i+\cdots+\alpha_{r-1})+\sum_{i\leqq r}(l-i)(\alpha_r+\cdots+\alpha_l)$$

$$=\sum_{m=1}^{r-1}\left(\sum_{k=1}^{m}(l-k)\right)\alpha_m+\sum_{m=r}^{l}\left(\sum_{k=1}^{r}(l-k)\right)\alpha_m$$

and

(4.12)
$$\sum_{i \leqq r} \sum_{j=i}^{l-1} (j-i+1)\alpha_j = \sum_{i \leqq r} \sum_{j=i}^{r} (j-i+1)\alpha_j + \sum_{i \leqq r} \sum_{j=r+1}^{l-1} (j-i+1)\alpha_j$$
$$= \sum_{m=1}^{r-1} \Big(\sum_{k=1}^{m} k \Big) \alpha_m + \sum_{m=r}^{l-1} \Big(\sum_{k=1}^{r} (m-k+1) \Big) \alpha_m .$$

Then, from (4.10), (4.11) and (4.12), we have

$$\sum_{\alpha \in \Sigma''} \alpha = l \sum_{m=1}^{r-1} m\alpha_m + r \sum_{m=r}^{l-1} (l+m-r)\alpha_m + \frac{1}{2} r(2l-r-1)\alpha_l ,$$

which, combined with (4.9), implies

$$2\delta_{\mathfrak{m}} = \sum_{\alpha \in \Sigma_{\mathfrak{m}}^{+}} \alpha = (2l-r+1) \Big(\sum_{m=1}^{r-1} m\alpha_m + r \sum_{m=r}^{l-1} \alpha_m + \frac{1}{2} r\alpha_l \Big) .$$

The Cartan matrix C of $\mathfrak{g} = C_l$ and its inverse matrix are given by

$$C = \Big(c_{ij} \Big)_{1 \leqq i,j \leqq l} , \quad c_{ij} = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} ,$$
$$C^{-1} = \Big(d_{ij} \Big)_{1 \leqq i,j \leqq l} ,$$
$$d_{ij} = \begin{cases} j & \text{if } 1 \leqq j \leqq l-1 \text{ and } j \leqq i \leqq l , \\ i & \text{if } 1 \leqq j \leqq l-1 \text{ and } 1 \leqq i \leqq j , \\ \frac{i}{2} & \text{if } j = l , \end{cases}$$

so that the following holds

$$\Lambda_{\alpha_r} = \sum_{m=1}^{l} d_{rm}\alpha_m = \sum_{m=1}^{r-1} m\alpha_m + r \sum_{m=r}^{l-1} \alpha_m + \frac{1}{2} r\alpha_l .$$

Therefore, we obtain

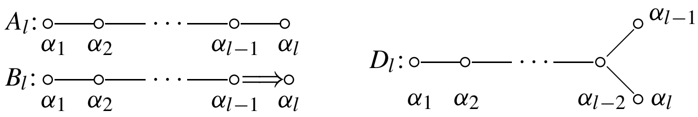
$$2\delta_{\mathfrak{m}} = (2l-r+1)\Lambda_{\alpha_r} .$$

Summing up the above consideration, we obtain following.

THEOREM 4.3. *For the Kähler C-space M of type (C_l, α_r) equipped with the Kähler metric $g(a\Lambda_{\alpha_r})$, the complex dimension, the scalar curvature τ and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given respectively by*

$$\dim_{\mathbf{C}} M = \frac{r(4l-3r+1)}{2} , \quad \tau = \frac{2(2l-r+1)}{a} \dim_{\mathbf{C}} M , \quad \lambda_1(M) = \frac{2(2l-r+1)}{a} .$$

When $\mathfrak{g}$ is a compact simple Lie algebra of the other classical type, suppose that the simple roots α_i are naturally numbered as follows:



By an argument similar to Theorem 4.3, we can obtain the following theorem:

THEOREM 4.4. *Let $\mathfrak{g}$ be a compact simple Lie algebra of classical type. Then, for the Kähler C-space M of type $(\mathfrak{g}, \alpha_r)$ equipped with the Kähler metric $g(a\Lambda_{\alpha_r})$, the complex dimension and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given as follows*:

$\mathfrak{g}$	$\dim_{\mathbf{C}} M$	$\lambda_1(M)$	
A_l	$r(l-r+1)$	$\frac{2(l+1)}{a}$	
B_l	$\frac{r(4l-3r+1)}{2}$	$\frac{2(2l-r)}{a}$	$1 \leqq r \leqq l-1$
	$\frac{l(l+1)}{2}$	$\frac{4l}{a}$	$r = l$
C_l	$\frac{r(4l-3r+1)}{2}$	$\frac{2(2l-r+1)}{a}$	
D_l	$\frac{r(4l-3r-1)}{2}$	$\frac{2(2l-r-1)}{a}$	$1 \leqq r \leqq l-2$
	$\frac{l(l-1)}{2}$	$\frac{4(l-1)}{a}$	$r = l-1,\ l$

5. The homogeneous Kähler hypersurface (C_l, α_2)

In this section, we will consider a Kähler C-space of type (C_l, α_r) as a Kähler submanifold of $G_r(\mathbf{C}^{2l})$.

Let's set

$$\mathfrak{g} = \mathfrak{sp}(l) = \left\{ \begin{pmatrix} A & -\overline{C} \\ C & \overline{A} \end{pmatrix} \;\middle|\; \begin{matrix} A, C \in M_l(\mathbf{C}), \\ A^* = -A,\ {}^tC = C \end{matrix} \right\},$$

then $\mathfrak{g}$ is a compact semisimple Lie algebra of type C_l whose complexification is given by

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{sp}(l, \mathbf{C}) = \left\{ \begin{pmatrix} A & B \\ C & -{}^tA \end{pmatrix} \;\middle|\; \begin{matrix} A, B, C \in M_l(\mathbf{C}), \\ {}^tB = B,\ {}^tC = C \end{matrix} \right\}.$$

Note that the Killing form $B_{\mathfrak{g}}$ is given by

$$B_{\mathfrak{g}}(X, Y) = 2(l+1) tr XY, \quad X, Y \in \mathfrak{g}.$$

For integers i and j with $1 \leq i, j \leq l$, let E_{ij} be the matrix in $M_l(\mathbf{C})$ whose (i, j)-coefficient is 1 and others are zero. and let's set

$$e_{ij} = \begin{pmatrix} E_{ij} & 0 \\ 0 & -E_{ji} \end{pmatrix}, \quad f_{ij} = \begin{pmatrix} 0 & E_{ij} + E_{ji} \\ 0 & 0 \end{pmatrix}, \quad g_{ij} = \begin{pmatrix} 0 & 0 \\ E_{ij} + E_{ji} & 0 \end{pmatrix},$$

$$\theta_i = \frac{\sqrt{-1}}{4(l+1)} e_{ii}$$

for $1 \leq i, j \leq l$. Relative to an abelian subalgebra $\mathfrak{t} = \mathbf{R}\{\theta_i,\ 1 \leqq i \leqq l\}$, the set Σ^+ of all positive roots is given as

$$\Sigma^+ = \{\theta_i - \theta_j\ (i < j)\,, \quad \theta_i + \theta_j\ (i \leqq j)\}\,.$$

The simple roots α_i numbered as the last section is given by

$$\alpha_i = \theta_i - \theta_{i+1}\ (1 \leqq i \leqq l-1)\,, \quad \alpha_l = 2\theta_l\,,$$

so that we have linear combinations

$$\theta_i - \theta_j = \alpha_i + \cdots + \alpha_{j-1} \quad (1 \leqq i < j \leqq l)\,,$$

$$\theta_i + \theta_j = (\alpha_i + \cdots + \alpha_{l-1}) + (\alpha_j + \cdots + \alpha_{l-1}) + \alpha_l \quad (1 \leqq i \leqq j \leqq l-1)\,,$$

$$\theta_i + \theta_l = \alpha_i + \cdots + \alpha_l \quad (1 \leqq i \leqq l-1)\,, \quad 2\theta_l = \alpha_l\,.$$

The root vectors

$$E_{\theta_i - \theta_j} = \frac{1}{2\sqrt{l+1}}\, e_{ij}\,, \quad E_{-\theta_i + \theta_j} = -\frac{1}{2\sqrt{l+1}}\, e_{ji}\,,$$

$$E_{\theta_i + \theta_j} = \frac{1}{2\sqrt{l+1}}\, f_{ij}\,, \quad E_{-\theta_i - \theta_j} = -\frac{1}{2\sqrt{l+1}}\, g_{ij}\,, \quad \text{for } 1 \leqq i < j \leqq l$$

$$E_{2\theta_i} = \frac{1}{2\sqrt{2(l+1)}}\, f_{ii}\,, \quad E_{-2\theta_i} = -\frac{1}{2\sqrt{2(l+1)}}\, g_{ii}\,, \quad \text{for } 1 \leqq i \leqq l$$

satisfy (4.6).

Σ_0 and $\Sigma_{\mathfrak{m}}^+$ are given by, for $1 \leqq r \leqq l-1$,

$$\Sigma_0 = \left\{ \begin{array}{ll} \pm(\theta_i - \theta_j) & (1 \leqq i < j \leqq r \text{ or } r+1 \leqq i < j \leqq l)\,, \\ \pm(\theta_i + \theta_j) & (r+1 \leqq i \leqq j \leqq l) \end{array} \right\},$$

$$\Sigma_{\mathfrak{m}}^+ = \left\{ \begin{array}{ll} \theta_i - \theta_j & (1 \leqq i \leqq r \text{ and } r+1 \leqq j \leqq l)\,, \\ \theta_i + \theta_j & (1 \leqq i \leqq r \text{ and } i \leqq j \leqq l) \end{array} \right\},$$

and, for $r = l$,

$$\Sigma_0 = \{\pm(\theta_i - \theta_j) \quad (1 \leqq i < j \leqq l)\}\,,$$

$$\Sigma_{\mathfrak{m}}^+ = \{\theta_i + \theta_j \quad (1 \leqq i \leqq j \leqq l)\}\,.$$

By a direct computation, (4.2) and (4.3) imply

$$\mathfrak{u} = \left\{ \begin{pmatrix} A & A'' & B & B'' \\ 0 & A' & {}^tB'' & B' \\ 0 & 0 & -{}^tA & 0 \\ 0 & C' & -{}^tA'' & -{}^tA' \end{pmatrix} \;\middle|\; \begin{matrix} A, B \in M_r(\mathbf{C}) , \\ A', B', C' \in M_{l-r}(\mathbf{C}) , \\ A'', B'' \in M_{r,l-r}(\mathbf{C}) , \\ {}^tB = B \,,\ {}^tB' = B' \,,\ {}^tC' = C' \end{matrix} \right\} ,$$

$$\begin{aligned} \mathfrak{k} &= \mathfrak{g} \cap \mathfrak{u} \\ &= \left\{ \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & A' & 0 & -\overline{C'} \\ 0 & 0 & \overline{A} & 0 \\ 0 & C' & 0 & \overline{A'} \end{pmatrix} \;\middle|\; \begin{matrix} A \in M_r(\mathbf{C}) , \\ A', C' \in M_{l-r}(\mathbf{C}) , \\ A^* = -A \,,\ A'^* = -A' \,,\ {}^tC' = C' \end{matrix} \right\} \\ &= \mathfrak{u}(r) + \mathfrak{sp}(l-r) \,. \end{aligned}$$

Therefore, the Kähler C-space M of type (C_l, α_r) is identified with the homogeneous space $G/K = Sp(l)/U(r) \cdot Sp(l-r)$.

For $x, y \in M_{l-r,r}(\mathbf{C})$ and $z \in M_r(\mathbf{C})$ with ${}^tz = z$, define

$$\eta(x, y, z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ z & {}^ty & 0 & -{}^tx \\ y & 0 & 0 & 0 \end{pmatrix} .$$

Note that, if $r = l$, then we ignore x and y, and $\eta(x, y, z)$ and $\eta(0, 0, z)$ denote a matrix $\begin{pmatrix} 0_l & 0_l \\ z & 0_l \end{pmatrix}$, $z \in M_l(\mathbf{C})$, ${}^tz = z$. (4.5) implies

$$\begin{aligned} \mathfrak{m} &= \{\eta(x, y, z) - \eta(x, y, z)^*\} \,, \\ \mathfrak{m}^+ &= \{\eta(x, y, z)\} \,. \end{aligned}$$

If $1 \leqq r \leqq l-1$, then $(\alpha_r, \alpha_r) = \frac{1}{2(l+1)}$. Thus, define subsets of $\Sigma^+_{\mathfrak{m}}$ by

$$\begin{aligned} \Sigma^+_{\mathfrak{m}_1} &= \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; (\alpha, \Lambda_{\alpha_r}) = \frac{1}{4(l+1)} \right\} = \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \frac{2(\alpha, \Lambda_{\alpha_r})}{(\alpha_r, \alpha_r)} = 1 \right\} \\ &= \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \alpha = \alpha_r + (\text{sum of other } \alpha_i) \right\} \\ &= \left\{ \begin{matrix} \theta_i - \theta_j & (1 \leqq i \leqq r \text{ and } r+1 \leqq j \leqq l), \\ \theta_i + \theta_j & (1 \leqq i \leqq r \text{ and } r+1 \leqq j \leqq l) \end{matrix} \right\} , \end{aligned}$$

$$\begin{aligned} \Sigma^+_{\mathfrak{m}_2} &= \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; (\alpha, \Lambda_{\alpha_r}) = \frac{1}{2(l+1)} \right\} = \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \frac{2(\alpha, \Lambda_{\alpha_r})}{(\alpha_r, \alpha_r)} = 2 \right\} \\ &= \left\{ \alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \alpha = 2\alpha_r + (\text{sum of other } \alpha_i) \right\} \\ &= \{ \theta_i + \theta_j \quad (1 \leqq i \leqq r \text{ and } i \leqq j \leqq r) \} , \end{aligned}$$

and we have an orthogonal decomposition $\mathfrak{m}^+ = \mathfrak{m}_1^+ + \mathfrak{m}_2^+$,

$$\mathfrak{m}_1^+ = \sum_{\alpha \in \Sigma^+_{\mathfrak{m}_1}} \mathfrak{g}^{\mathbf{C}}_{-\alpha} = \{\eta(x, y, 0)\} ,$$

$$\mathfrak{m}_2^+ = \sum_{\alpha \in \Sigma^+_{\mathfrak{m}_2}} \mathfrak{g}^{\mathbf{C}}_{-\alpha} = \{\eta(0, 0, z)\} .$$

From (4.7), the G-invariant Kähler metric corresponding to $a\Lambda_{\alpha_r}$ is given by

$$g(a\Lambda_{\alpha_r}) = \frac{a}{4(l+1)}\left\{(\, , \,)_{\mathfrak{m}_1^+ \times \overline{\mathfrak{m}_1^+}} \; + \; 2(\, , \,)_{\mathfrak{m}_2^+ \times \overline{\mathfrak{m}_2^+}}\right\},$$

so that, for $X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}$, we get

$$\begin{aligned} g(a\Lambda_{\alpha_r})(X, X) &= 2g(a\Lambda_{\alpha_r})(X^+, \overline{X^+}) \\ &= \frac{a}{2(l+1)}\left\{(X_1^+, \overline{X_1^+}) + 2(X_2^+, \overline{X_2^+})\right\} = 2a\, tr(x^*x + y^*y + \bar{z}z) , \end{aligned}$$

where $X^+ = \eta(x, y, z) \in \mathfrak{m}^+$, $X_1^+ = \eta(x, y, 0) \in \mathfrak{m}_1^+$ and $X_2^+ = \eta(0, 0, z) \in \mathfrak{m}_2^+$.

If $r = l$, then $(\alpha_l, \alpha_l) = \frac{1}{l+1}$. So, $\Sigma^+_{\mathfrak{m}}$ satisfies the following:

$$\begin{aligned} \Sigma^+_{\mathfrak{m}} &= \left\{\alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; (\alpha, \Lambda_{\alpha_l}) = \frac{1}{2(l+1)}\right\} = \left\{\alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \frac{2(\alpha, \Lambda_{\alpha_l})}{(\alpha_l, \alpha_l)} = 1\right\} \\ &= \left\{\alpha \in \Sigma^+_{\mathfrak{m}} \;\middle|\; \alpha = \alpha_l + (\text{sum of other } \alpha_i)\right\} . \end{aligned}$$

From (4.7), the G-invariant Kähler metric corresponding to $a\Lambda_{\alpha_r}$ is given by

$$g(a\Lambda_{\alpha_l}) = \frac{a}{2(l+1)}(\, , \,)_{\mathfrak{m}^+ \times \overline{\mathfrak{m}^+}} ,$$

so that, for $X = \eta(0, 0, z) - \eta(0, 0, z)^* \in \mathfrak{m}$, we get

$$g(a\Lambda_{\alpha_l})(X, X) = 2g(a\Lambda_{\alpha_l})(X^+, \overline{X^+}) = \frac{a}{l+1}(X^+, \overline{X^+}) = 2a\, tr(\bar{z}z) ,$$

where $X^+ = \eta(0, 0, z) \in \mathfrak{m}^+$.

Consequently, for any r with $1 \leqq r \leqq l$, we see

(5.1) $$g(a\Lambda_{\alpha_r})(X, X) = 2a\, tr(x^*x + y^*y + \bar{z}z) , \quad X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m} .$$

The natural inclusion $Sp(l) \to SU(2l)$ defines an immersion φ of M into $\tilde{M} = G_r(\mathbf{C}^{2l}) = \tilde{G}/\tilde{K} = SU(2l)/S(U(r) \cdot U(2l - r))$ by

$$\varphi(g \cdot K) = g \cdot \tilde{K} , \quad g \in G .$$

Under identification of $T_o\tilde{M}$ with $\tilde{\mathfrak{m}}$, the image of $X = \eta(x, y, z) - \eta(x, y, z)^* \in \mathfrak{m}$ is

$$\varphi_*(X) = \begin{pmatrix} 0 & -x^* & -\bar{z} & -y^* \\ x & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ y & 0 & 0 & 0 \end{pmatrix},$$

so that we have

$$\tilde{g}_c(\varphi_*(X), \varphi_*(X)) = \frac{4}{c}\, tr(x^*x + y^*y + \bar{z}z)\,, \tag{5.2}$$

where c is the maximal holomorphic sectional curvature of $G_r(\mathbf{C}^{2l})$. Therefore, Theorem 4.3, (5.1) and (5.2) imply the following.

THEOREM 5.1. *For the Kähler C-space $M = Sp(l)/U(r)\cdot Sp(l-r)$ of type (C_l, α_r) equipped with the Kähler metric $g(\frac{2}{c}\Lambda_{\alpha_r})$, M is immersed in $G_r(\mathbf{C}^{2l})$ by the Kähler immersion φ. The complex dimension, and the first eigenvalue $\lambda_1(M)$ of the Laplacian are given by*

$$\dim_{\mathbf{C}} M = \frac{r(4l-3r+1)}{2}\,, \quad \lambda_1(M) = c\,(2l-r+1)\,.$$

In particular, if $r = 2$, then $M = Sp(l)/U(2)\cdot Sp(l-2)$ is a Kähler hypersurface of $G_2(\mathbf{C}^{2l})$, whose first eigenvalue $\lambda_1(M)$ of the Laplacian is given by

$$\lambda_1(M) = c\,(2l-1)\,.$$

REMARK 5.1.
(1) (C_l, α_l) is a Hermitian symmetric space $Sp(l)/U(l)$.
(2) (C_l, α_1) is a complex projective space $\mathbf{C}P^{2l-1}$ so it is Hermitian symmetric. But the pair $(Sp(l),\ U(1)\cdot Sp(l-1))$ is not a compact symmetric pair.
(3) Other $(C_l, \alpha_r),\ 2 \leqq r \leqq l-1$ are not symmetric spaces.

For $z \in M_r(\mathbf{C})$, define an unit vector ν at the origin o of $G_2(\mathbf{C}^{2l})$ by

$$\nu(z) = \begin{pmatrix} 0 & 0 & -z^* & 0 \\ 0 & 0 & 0 & 0 \\ z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in \tilde{\mathfrak{m}}\,, \quad \frac{4}{c}\, tr\, z^*z = 1\,.$$

Then $\nu(z)$ is tangent to M if and only if z is symmetric.

The Kähler hypersurface $M = (C_l, \alpha_2)$ satisfies the following property relative to the quaternionic Kähler structure $\mathfrak{J}$ of $G_2(\mathbf{C}^{2l})$.

PROPOSITION 5.2. *The Kähler hypersurface $M = Sp(l)/U(2)\cdot Sp(l-2)$ of $G_2(\mathbf{C}^{2l})$ satisfies*

$$\mathfrak{J}\, T^\perp M \subset TM \quad (\Leftrightarrow J\xi \ \perp\ \mathfrak{J}\xi \ \textit{ for any } \xi \in T^\perp M)\,, \tag{5.3}$$

where TM and $T^{\perp}M$ are the tangent bundle and the normal bundle of M, respectively.

PROOF. Let ν_o be an unit normal vector of M at o defined by

$$\nu_o = \nu(z_o), \quad z_o = \frac{1}{2}\sqrt{\frac{c}{2}}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

so that the normal space $T_o^{\perp}M$ is given by

$$T_o^{\perp}M = \mathbf{R}\{\nu_o,\ J\nu_o = \nu(\sqrt{-1}z_o)\}.$$

Then we see

$$\begin{aligned} \mathfrak{J}_o\, T_o^{\perp}M &= \mathbf{R}\,\{J_i\nu_o,\ J_iJ\nu_o, \quad i = 1, 2, 3\} \\ &= \mathbf{R}\{\nu(z_o\varepsilon_i),\ \nu(\sqrt{-1}z_o\varepsilon_i), \quad i = 1, 2, 3\}, \end{aligned}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. It is easy to check that $z_o\varepsilon_i$ and $\sqrt{-1}z_o\varepsilon_i$ are symmetric, so that we obtain

$$\mathfrak{J}_o\, T_o^{\perp}M \subset T_oM.$$

Since the quaternionic Kähler structure $\mathfrak{J}$ is $\tilde{G}$-invariant, and since the immersion φ is G-equivariant, (5.3) holds at any point of M. □

If the ambient space is $G_2(\mathbf{C}^4)$, then the condition (5.3) determines a Kähler hypersurface as follows:

PROPOSITION 5.3. *Suppose that a Kähler hypersurface M of $Q^4 = G_2(\mathbf{C}^4)$ satisfies the condition*

$$\mathfrak{J}\, T^{\perp}M \subset TM.$$

Then M is totally geodesic. Moreover, if M is compact, then M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$.

PROOF. Denote by $\tilde{\nabla}$ the Riemannian connection of Q^4, and denote by ∇, σ, A and $\nabla^{\perp}$, the Riemannian connection, the second fundamental form, the shape operator, and the normal connection of M, respectively. It is well-known that Gauss' formula and Weingarten's formula hold:

$$\begin{aligned} \tilde{\nabla}_X Y &= \nabla_X Y + \sigma(X, Y), \\ \tilde{\nabla}_X \xi &= -A_\xi X + \nabla^{\perp}_X \xi, \end{aligned} \tag{5.4}$$

for $X, Y \in TM$ and $\xi \in T^{\perp}M$. The metric condition implies

$$\tilde{g}_c(\sigma(X, Y), \xi) = \tilde{g}_c(A_\xi X, Y). \tag{5.5}$$

Relative to the complex structure J, σ and A satisfy

$$\sigma(X, JY) = J\sigma(X, Y), \quad A_\xi \circ J = -J \circ A_\xi = -A_{J\xi}. \tag{5.6}$$

For a local unit normal vector field ξ, we define local vector fields as follow: $e_i = J_i\xi$, $i = 1, 2, 3$, where J_1, J_2 and J_3 are a local canonical basis of $\mathfrak{J}$. Then, under the assumption of this proposition, $\{e_1, e_2, e_3, Je_1, Je_2, Je_3, \xi, J\xi\}$ is a local orthonormal frame field of Q^4 such that $\{e_1, e_2, e_3, Je_1, Je_2, Je_3\}$ is a tangent frame of M. For $X \in TM$, (5.4) implies

$$\begin{aligned}\nabla_X e_i + \sigma(X, e_i) = \tilde{\nabla}_X e_i &= (\tilde{\nabla}_X J_i)\xi + J_i(\tilde{\nabla}_X \xi) \\ &= (\tilde{\nabla}_X J_i)\xi - J_i A_\xi X + J_i(\nabla^{\perp}_X \xi)\,.\end{aligned} \tag{5.7}$$

Since $\mathfrak{J}$ is parallel with respect to the connection $\tilde{\nabla}$, we have $\tilde{\nabla}_X J_i \in \mathfrak{J}$, so that the normal component of (5.7) is

$$\begin{aligned}\sigma(X, e_i) &= -\tilde{g}_c(J_i A_\xi X,\ \xi)\xi - \tilde{g}_c(J_i A_\xi X,\ J\xi)J\xi \\ &= g_c(A_\xi X,\ e_i)\xi + g_c(A_\xi X,\ Je_i)J\xi\,,\end{aligned}$$

where g_c is the induced Kähler metric of M. On the other hand, (5.5) and (5.6) imply

$$\begin{aligned}\sigma(X, e_i) &= \tilde{g}_c(\sigma(X, e_i),\ \xi)\xi + \tilde{g}_c(\sigma(X, e_i),\ J\xi)J\xi \\ &= g_c(A_\xi X,\ e_i)\xi - g_c(A_\xi X,\ Je_i)J\xi\,.\end{aligned}$$

From these two equations, we get

$$g_c(A_\xi X,\ Je_i) = 0\,. \tag{5.8}$$

Instead of X, applying to JX, we have

$$g_c(A_\xi X,\ e_i) = g_c(-A_\xi JX,\ Je_i) = 0\,.$$

Therefore, we have $A_\xi = 0$, or $\sigma = 0$, so that M is totally geodesic. By B. Y. Chen and T. Nagano [5]'s results, if M is compact, M is congruent to a complex quadric $Q^3 = Sp(2)/U(2)$. □

The Kähler submanifold $M = (C_l, \alpha_r)$ satisfies another interesting property as follows:

PROPOSITION 5.4. *The isometric immersion $\Psi \circ \varphi : M = Sp(l)/U(r)\cdot Sp(l-r) \longrightarrow HM(2l, \mathbf{C})$ is a sum of ($HM(2l, \mathbf{C})$-valued) eigenfunctions with eigenvalues 0, $c(2l-r+1)$ and $2cl$. More precisely, $\Psi \circ \varphi$ satisfies*

$$\Psi \circ \varphi = F_0 + F_1 + F_2\,,$$

$$\Delta F_0 = 0\,, \quad \Delta F_1 = c(2l - r + 1)F_1\,, \quad \Delta F_2 = 2clF_2\,,$$

where F_0, F_1 and F_2 are $HM(2l, \mathbf{C})$-valued functions defined by

$$F_0 = \frac{r}{2l}I_{2l}\,, \quad F_1 = \frac{1}{2}(A + S\bar{A}S)\,, \quad F_2 = -\frac{r}{2l}I_{2l} + \frac{1}{2}(A - S\bar{A}S)\,,$$

$A = \Psi \circ \varphi$ is a position vector in $HM(2l, \mathbf{C})$, and

$$S = \begin{pmatrix} 0 & -I_l \\ I_l & 0 \end{pmatrix}.$$

REMARK 5.2. If $r = l$, then F_2 vanishes. If $r = 1$, then two positive eigenvalues coincide with each other, and $\Psi \circ \varphi$ is the first standard imbedding of $\mathbf{C}P^{2l-1}$.

COROLLARY 5.5. *For $l \geqq 3$ and $2 \leqq r \leqq l-1$, $2cl$ is an eigenvalue of the Laplacian of $Sp(l)/U(r) \cdot Sp(l-r)$, which is greater than the first eigenvalue.*

REMARK 5.3. By B. Y. Chen's definition, if $l \geqq 3$ and $2 \leqq r \leqq l-1$, then $Sp(l)/U(r) \cdot Sp(l-r)$ is a mass-symmetric 2-type submanifold of order $\{c(2l-r+1),\ 2cl\}$. On the other hand, for any $l \geqq 1$, $(C_l, \alpha_1) = \mathbf{C}P^{2l-1}$ is a mass-symmetric 1-type submanifold of order $\{2cl\}$, and $(C_l, \alpha_l) = Sp(l)/U(l)$ is a mass-symmetric 1-type submanifold of order $\{c\,(l+1)\}$. (cf. [4])

PROOF OF PROPOSITION 5.4. Notice that $G = Sp(l)$ is a subgroup of $\tilde{G} = SU(2l)$ and satisfies

$$G = Sp(l) = \{g \in SU(2l) \mid {}^t gSg = S\}\,.$$

For $1 \leqq i < j \leqq r$, let's set

$$z_{ij} = \frac{1}{2}\sqrt{\frac{c}{2}}(E_{ij} - E_{ji})\,,$$

so that $\nu(z_{ij}),\ J\nu(z_{ij}) = \nu(\sqrt{-1}z_{ij}),\ 1 \leqq i < j \leqq r$ are an orthonormal basis of $T_o^{\perp}M$. By a simple computation, we get

$$\sum_{i<j} \Psi_*(\nu(z_{ij}))^2 = \sum_{i<j} \Psi_*(J\nu(z_{ij}))^2 = \frac{c}{4}\frac{r-1}{2}\begin{pmatrix} I_r & 0 & 0 & 0 \\ 0 & 0_{l-r} & 0 & 0 \\ 0 & 0 & I_r & 0 \\ 0 & 0 & 0 & 0_{l-r} \end{pmatrix}.$$

From (2.9), at the origin $A_o = \Psi(o) = \begin{pmatrix} I_r & 0 \\ 0 & 0_{2l-r} \end{pmatrix}$,

$$\begin{aligned} &\sum_{i<j}\big(\tilde{\sigma}_{A_o}(\nu(z_{ij}), \nu(z_{ij})) + \tilde{\sigma}_{A_o}(J\nu(z_{ij}), J\nu(z_{ij}))\big) \\ &\qquad = 4\bigg(\sum_{i<j} \Psi_*(\nu(z_{ij}))^2\bigg)(I - 2A_o) = \frac{c(r-1)}{2}(-A_o - SA_oS)\,. \end{aligned}$$

Since M is minimal in $G_r(\mathbf{C}^{2l})$, it follows from (2.10) that, at the origin A_o, the mean curvature vector H_{A_o} of M in $HM(2l, \mathbf{C})$ is given by

$$\begin{aligned} 2\dim_{\mathbf{C}} M\, H_{A_o} &= 2r(2l-r)\tilde{H}_{A_o} - \sum_{i<j}\big(\tilde{\sigma}_{A_o}(\nu(z_{ij}), \nu(z_{ij})) + \tilde{\sigma}_{A_o}(J\nu(z_{ij}), J\nu(z_{ij}))\big) \\ &= \frac{c}{2}\big(2rI - (4l-r+1)A_o + (r-1)SA_oS\big)\,. \end{aligned}$$

Since the immersions φ and Ψ are equivariant under the actions G and $\tilde{G}$, at a point $A = gA_og^*,\ g \in G$, the mean curvature H_A is given by

$$2\dim_{\mathbf{C}} M\ H_A = 2\dim_{\mathbf{C}} M\ g\,H_{A_o}g^* = \frac{c}{2}\big(2rI - (4l - r + 1)A + (r-1)S\bar{A}S\big)\,.$$

Therefore, we obtain

$$\Delta A = -2\dim_{\mathbf{C}} M\ H_A = -\frac{c}{2}\big(2rI - (4l - r + 1)A + (r-1)S\bar{A}S\big)\,.$$

which implies Proposition 5.4. □

REMARK 5.4. A quaternionic projective space $\mathbf{H}P^{l-1}$ admits a totally geodesic embedding $\varphi_{\mathbf{H}P^{l-1}}$ into $G_2(\mathbf{C}^{2l})$. (See [5] and [6].) $\varphi_{\mathbf{H}P^{l-1}}$ is a quaternionic embedding with respect to the quaternionic Kähler structure of $G_2(\mathbf{C}^{2l})$, and is a totally real embedding with respect to the complex structure of $G_2(\mathbf{C}^{2l})$. It is known that the Kähler hypersurface $M = (C_l, \alpha_2)$ is the focal set of $\mathbf{H}P^{l-1}$ in $G_2(\mathbf{C}^{2l})$. (cf. [1])

6. Proof of main theorems

Let M be a compact connected Kähler hypersurface of $G_r(\mathbf{C}^n)$ immersed by a immersion φ. It is well-known that every $HM(n, \mathbf{C})$-valued function F satisfies

$$(\Delta F,\ \Delta F)_{L^2} - \lambda_1(\Delta F,\ F)_{L^2} \geqq 0\,. \tag{6.1}$$

The equality holds if and only if F is a sum of eigenfunctions with respect to eigenvalues 0 and λ_1. It is equivalent to that there exists a constant vector $C \in HM(n, \mathbf{C})$ such that $\Delta(F - C) = \lambda_1(F - C)$.

Denote by H the mean curvature vector of the isometric immersion $\Phi = \Psi \circ \varphi$. Then, since M is minimal in $G_r(\mathbf{C}^n)$, (2.10) implies

$$\begin{aligned} 2(r(n-r)-1)H_A &= 2r(n-r)\tilde{H}_A - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J\xi, J\xi) \\ &= c(rI - nA) - \tilde{\sigma}_A(\xi, \xi) - \tilde{\sigma}_A(J\xi, J\xi)\,, \end{aligned} \tag{6.2}$$

where A is a position vector of $\Phi(M)$ in $HM(n, \mathbf{C})$, and ξ is a local unit normal vector field of φ. Using (2.12) and (6.2), we get

$$(H_A,\ A) = -1\,. \tag{6.3}$$

$HM(n, \mathbf{C})$-valued function Φ satisfies $\Delta\Phi = -2(r(n-r)-1)H$, so that (6.1) and (6.3) imply the following. The equality condition dues to T. Takahashi's theorem in [15].

LEMMA 6.1.

$$2(r(n-r)-1)\int_M (H_A,\ H_A)dv_M - \lambda_1 vol(M) \geqq 0\,. \tag{6.4}$$

The equality holds if and only if Φ is a minimal immersion of M into some round sphere in $HM(n, \mathbf{C})$, more precisely, there exists some positive constant R and some constant vector $C \in HM(n, \mathbf{C})$ such that H_A satisfies

$$H_A = \frac{1}{R^2}(C - A)\,. \tag{6.5}$$

LEMMA 6.2. *If the equality holds in* (6.4), *then M is contained in a totally geodesic submanifold of $G_r(\mathbf{C}^n)$ which is product of Grassmann manifolds, more precisely, there exist integers k_i, r_i, $i = 1, \cdots, m$ such that*

$$0 \leqq r_i \leqq k_i\,, \quad r_1 \geqq r_2 \geqq \cdots \geqq r_m\,,$$

$$\sum_{i=1}^{m} r_i = r\,, \quad \sum_{i=1}^{m} k_i = n\,,$$

$$M \subset G_{r_1}(\mathbf{C}^{k_1}) \times G_{r_2}(\mathbf{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbf{C}^{k_m}) \subset G_r(\mathbf{C}^n)\,. \tag{6.6}$$

Notice that $G_0(\mathbf{C}^{k_i}) = G_{k_i}(\mathbf{C}^{k_i}) = \{$one point$\}$.

PROOF. Assume that the equality holds in (6.4).

Since M is minimal in $G_r(\mathbf{C}^n)$, H is normal to $G_r(\mathbf{C}^n)$. Then, from (2.5) and (6.5), we get

$$CA = AC\,, \tag{6.7}$$

where C is a constant vector in Lemma 6.1. Since $SU(n)$ acts on $G_r(\mathbf{C}^n)$ transitively, without loss of generality, we can assume that C is a diagonal matrix as follows:

$$C = \begin{pmatrix} c_1 I_{k_1} & & & 0 \\ & c_2 I_{k_2} & & \\ & & \ddots & \\ 0 & & & c_m I_{k_m} \end{pmatrix}, \quad k_i > 0\,, \quad c_i \neq c_j \ (i \neq j)\,. \tag{6.8}$$

Notice that

$$n = k_1 + k_2 + \cdots + k_m\,.$$

Define a linear subspace L of $HM(n, \mathbf{C})$ by $L = \{Z \in HM(n, \mathbf{C}) \mid ZC = CZ\}$, so that

$$L = \left\{ \begin{pmatrix} Z_1 & & & 0 \\ & Z_2 & & \\ & & \ddots & \\ 0 & & & Z_m \end{pmatrix} \;\middle|\; Z_i \in M_{k_i}(\mathbf{C}) \right\}.$$

From (6.7), M is contained in $G_r(\mathbf{C}^n) \cap L$.

For each integer r_i with $0 \leqq r_i \leqq k_i$, $\sum_{i=1}^{m} r_i = r$, let's define connected subsets of $G_r(\mathbf{C}^n)$ by

$$W_{r_1,\cdots,r_m} = \left\{ \begin{pmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_m \end{pmatrix} \middle| \begin{array}{c} A_i \in M_{k_i}(\mathbf{C}) , \\ A_i^2 = A_i , \quad tr\, A_i = r_i \end{array} \right\} .$$

So, $G_r(\mathbf{C}^n) \cap L$ is a disjoint union of all $W_{r_1,\cdots,r_m}$'s. Since M is connected, M is contained in suitable one of $W_{r_1,\cdots,r_m}$'s, saying $W_{r_1,\cdots,r_m}$. By the definition, we see

$$W_{r_1,\cdots,r_m} = G_{r_1}(\mathbf{C}^{k_1}) \times G_{r_2}(\mathbf{C}^{k_2}) \times \cdots \times G_{r_m}(\mathbf{C}^{k_m}) .$$

Without loss of generality, we can choose a diagonal matrix C with respect to which the inequalities $r_1 \geqq r_2 \geqq \cdots \geqq r_m$ hold. □

From (2.9), (2.11) and (6.2), we get

$$H_A = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c} (\Psi_* \xi)^2 (I - 2A) \right\} . \tag{6.9}$$

Using (2.3) and (2.4), we see

$$(H_A,\, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ nr(n-r) - 2\, tr\, \frac{4}{c} r\, (\Psi_* \xi)^2 \left(I + \frac{n-2r}{r} A \right) \right. \\ \left. + tr\, \frac{16}{c^2} (\Psi_* \xi)^2 (I - 2A) (\Psi_* \xi)^2 (I - 2A) \right\} . \tag{6.10}$$

Since the immersion Ψ is $\tilde{G}$-equivariant, for any $A \in \Phi(M)$, there exists a element $g_A \in \tilde{G}$ and a matrix $v_A \in M_{n-r,r}(\mathbf{C})$ satisfying $A_o = g_A A\, g_A^*$ and

$$\sqrt{\frac{c}{4}} \begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} = g_A (\Psi_* \xi)\, g_A^* . \tag{6.11}$$

Since the inner product (,) is $\tilde{G}$-equivariant and ξ is unit, we have $tr\, v_A^* v_A = tr\, v_A v_A^* = 1$. After translating by g_A, together with (6.11), (6.10) implies

$$(H_A,\, H_A) = \frac{c}{2(r(n-r)-1)^2} \left\{ n(r(n-r)-2) + 2\, tr\, (v_A^* v_A v_A^* v_A) \right\} . \tag{6.12}$$

LEMMA 6.3. *For* $v \in M_{n-r,r}(\mathbf{C})$ *with* $tr\, v^* v = 1$, *the following inequality holds*

$$tr\, v^* v v^* v \leqq 1 . \tag{6.13}$$

Moreover, the following three conditions are equivalent to each other.

(1) *The equality holds in* (6.13).

(2) *The hermitian* r*-matrix* $v^* v$ *is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix}$.

(3) *The hermitian $(n-r)$-matrix vv^* is similar to* $\begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix}$.

If the equality holds in (6.13), *then there exists* $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} \in S(U(r) \cdot U(n-r))$ *such that* $v' = QvP^*$ *satisfies*

$$v'^* v' = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v' v'^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} .$$

PROOF. Lemma 6.3 follows from that both of Hermitian matrices $v^* v$ and vv^* are similar to diagonal matrices with non-negative eigenvalues. □

Form (6.12) and Lemma 6.3, the following lemma is immediately obtained, which is used to prove Theorem A.

LEMMA 6.4.

$$(H_A, H_A) \leqq \frac{c}{2(r(n-r)-1)} \left\{ n - \frac{n-2}{r(n-r)-1} \right\} . \tag{6.14}$$

The equality holds if and only if, for any $A \in \Phi(M)$, it is possible to choose v_A satisfying

$$v_A^* v_A = \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \quad \text{and} \quad v_A v_A^* = \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} . \tag{6.15}$$

PROOF OF THEOREM A. (6.4) and (6.14) imply

$$\lambda_1 \leqq c \left(n - \frac{n-2}{r(n-r)-1} \right) .$$

Let's assume that this equality holds. Then, the equality conditions of Lemmas 6.1 and 6.4 hold.

Assume $m = 1$. Then, (6.5) and (6.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(r(n-r)-1)} \left\{ (rI - nA) - \frac{4}{c}(\Psi_* \xi)^2 (I - 2A) \right\} .$$

After translating by g_A, together with (6.11) and (6.15), we obtain

$$\frac{1}{R^2}(c_1 - 1) I_r = \frac{c}{2(r(n-r)-1)} \left\{ (r-n) I_r + \begin{pmatrix} 1 & 0 \\ 0 & 0_{r-1} \end{pmatrix} \right\} ,$$

$$\frac{1}{R^2} c_1 I_{n-r} = \frac{c}{2(r(n-r)-1)} \left\{ r I_{n-r} - \begin{pmatrix} 1 & 0 \\ 0 & 0_{n-r-1} \end{pmatrix} \right\} .$$

The first equation implies $r = 1$, and the second one implies $n - r = 1$. So, we have $n = 2$ and $r = 1$. This contradicts that M is a complex hypersurface.

Since $m \geqq 2$, from Lemma 6.2, M is contained in a proper totally geodesic submanifold of $G_r(\mathbf{C}^n)$. On the other hand, M is of complex codimension 1 in $G_r(\mathbf{C}^n)$. Consequently, either $r = 1$ or $r = n-1$ occurs, and M is a totally geodesic complex hypersurface of a complex projective space $\mathbf{C}P^{n-1} \cong G_1(\mathbf{C}^n) \cong G_{n-1}(\mathbf{C}^n)$. □

PROOF OF THEOREM B. Let's assume that M is a compact connected Kähler hypersurface of $G_2(\mathbf{C}^n)$ satisfying the condition $J\xi \perp \mathfrak{J}\xi$. Since both of the complex structure and the quaternionic Kähler structure are $\tilde{G}$-invariant, we obtain, at the origin A_o,

$$J\begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix} \perp J_i\begin{pmatrix} 0 & v_A^* \\ v_A & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{6.16}$$

where J_1, J_2 and J_3 are a canonical basis of $\mathfrak{J}_o$ defined in the section 2. Set

$$v_A = (v'_A \quad v''_A), \quad v'_A,\, v''_A \in M_{n-2,1}(\mathbf{C}) \cong \mathbf{C}^{n-2}.$$

Using (2.7) and (2.8), (6.16) implies that $|v'_A| = |v''_A|$ and $v'_A \perp v''_A$. Combining these with $tr\, v_A^* v_A = 1$, we obtain $|v'_A| = |v''_A| = \frac{1}{\sqrt{2}}$, so that

$$v_A^* v_A = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.17}$$

Together with (6.17), (6.12) implies

$$(H_A,\ H_A) = \frac{c}{2(2n-5)}\left\{n - \frac{n-1}{2n-5}\right\}.$$

Therefore, form Lemma 6.1, we obtain

$$\lambda_1 \leqq c\left(n - \frac{n-1}{2n-5}\right).$$

Let's assume that this equality holds. Then, the equality conditions of Lemma 6.1 holds.

Computing dimensions of manifolds in (6.6), we have

$$2n - 5 \leqq \sum_{i=1}^{m} r_i(k_i - r_i). \tag{6.18}$$

From $\sum_{i=1}^m r_i = 2$ and $r_1 \geqq r_2 \geqq \cdots \geqq r_m$, the following two cases occur:

$$\text{Case I} : \quad r_1 = r_2 = 1, \quad r_3 = \cdots = r_m = 0,$$
$$\text{Case II} : \quad r_1 = 2, \quad r_2 = \cdots = r_m = 0.$$

In Case I, (6.18) implies $2n-5 \leqq k_1 + k_2 - 2 \leqq n-2$, so $n \leqq 3$. This is contradiction.

Therefore, Case II occurs. Then, (6.18) implies $2n - 5 \leqq 2(k_1 - 2)$, so that we have $n = k_1, m = 1, k_2 = \cdots = k_m = 0$. (6.5) and (6.9) imply

$$\frac{1}{R^2}(c_1 I - A) = \frac{c}{2(2n-5)}\left\{(2I - nA) - \frac{4}{c}(\Psi_*\xi)^2(I - 2A)\right\}.$$

After translating by g_A, together with (6.11) and (6.17), we obtain

$$\frac{1}{R^2}(c_1 - 1) = \frac{c}{2(2n-5)}\left\{2 - n + \frac{1}{2}\right\},$$
$$\frac{1}{R^2}c_1 I_{n-2} = \frac{c}{2(2n-5)}\left\{2I_{n-2} - v_A v_A^*\right\}.$$

The second equation implies

$$v_A v_A^* = dI_{n-2}\,, \quad d = 2 - \frac{2(2n-5)}{c}\frac{c_1}{R^2}\,. \tag{6.19}$$

From (6.17), we have

$$d\,v_A = dI_{n-2}v_A = (v_A v_A^*)v_A = v_A(v_A^* v_A) = \frac{1}{2}v_A\,,$$

so that $d = \frac{1}{2}$. Consequently, taking traces of both sides of (6.19), we obtain $n = 4$.

Therefore, from Proposition 5.3, M is congruent to Q^3. □

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