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A Filtration of Algebraic Loops

William LIU

Bowling Green State University, Firelands College

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Abstract. A filtration of the loop group of unitary group by singular complex algebraic varieties defined by S. Mitchell and G. Segal is studied, focusing on determining the singularities of those varieties.

1. Introduction

The (based) loop group ΩU_n is the space of all smooth maps (that we shall call loops) $\gamma : \mathbf{S}^1 \to U_n$ such that $\gamma(1) = I$. Here $\mathbf{S}^1$ is the unit circle in the complex plane. ΩU_n is an infinite dimensional Lie group. Notice that each loop has a Laurent series expansion. Consider those loops that have finite Laurent series expansion, called *algebraic loops*. Denote the set of all algebraic loops by $\Omega_{alg} U_n$ which is a subgroup of ΩU_n. S. Mitchell in [3] and G. Segal in [5] define a filtration by singular complex algebraic varieties of $\Omega_{alg} U_n$. In [3], this filtration is used to study the stable homotopy type of ΩSU_n while, in [5], it is used to study harmonic maps from the two sphere to U_n. The space of holomorphic maps from the two sphere to this filtration is studied in [1] by M. Guest. In [2] Guest and Ohnita use this filtration to study deformations for harmonic maps. In this article, we determine the singularities of each stratum of this filtration. The strategy is to realize each stratum of the filtration as intersection of two smooth varieties and study the intersection of the tangent spaces.

The outline of this article is as follows. In Sect. 2, we gather some facts about ΩU_n and $\Omega_{alg} U_n$. Sect. 3 is devoted to describing the said filtration. In Sect. 4 we determine the singular points of the filtration (Theorem 4.1).

2. Basic facts about ΩU_n and $\Omega_{alg} U_n$

The basic reference for this section is [4].

There are isomorphisms $\pi_0(\Omega U_n) \cong \pi_1 U_n \cong \mathbf{Z}$. Each connected component of ΩU_n is determined by the degree of the determinant of loops. All connected components are diffeomorphic to each other.

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The subgroup $\Omega_{alg}U_n$ is an approximation to ΩU_n in the sense that its natural inclusion in ΩU_n is a homotopy equivalence. Hence $\pi_0(\Omega_{alg}U_n) \cong \mathbf{Z}$ with each connected component determined by the degree of the determinant of loops. Notice that $\Omega_{alg}U_n$ is *not* a smooth manifold. On the other hand, similar to the case of smooth loops, all components of $\Omega_{alg}U_n$ are homeomorphic to each other.

While ΩSU_n is properly included in the identity component of ΩU_n, the subgroup $\Omega_{alg}SU_n$ is equal to the identity component of $\Omega_{alg}U_n$.

There is a "Grassmannian model" for ΩU_n defined as follows.

Let $H = L^2(\mathbf{S}^1, \mathbf{C}^n) = H_+ \oplus H_-$, where

$$H_+ = \overline{\langle z^i e_j : i \geq 0, j = 1, \cdots, n\rangle}$$

and H_- its orthogonal complement. Here $\{e_1, \cdots, e_n\}$ is the standard basis of $\mathbf{C}^n$ and $\{z^i e_j : i \in \mathbf{Z},\ j = 1, 2, \cdots, n\}$ a basis of H and the closure is taken with respect to the L^2 norm topology. The Grassmannian Gr_∞ consists of all *closed* subspaces $W \subset L^2(\mathbf{S}^1, \mathbf{C}^n)$ such that

1. the orthogonal projections $pr_\pm : W \to H_\pm$ are Fredholm and Hilbert-Schmidt operators respectively;
2. $zW \subset W$;
3. the images of both orthogonal projections $W \to H_-$ and $W^\perp \to H_+$ consist of smooth functions.

It is an infinite dimensional smooth manifold with connected components determined by the Fredholm index of $pr_+ : W \to H_+$, called the virtual dimension of W. The map

$$\varphi : \Omega U_n \to Gr_\infty : \gamma \mapsto \gamma H_+$$

is a diffeomorphism such that if deg.det$(\gamma) = -k$, then virt.dim$(\varphi(\gamma)) = k$.

Let Gr_{alg} be the subspace of Gr_∞ consisting of elements W such that for some $k \in \mathbf{N}$,

$$z^k H_+ \subset W \subset z^{-k} H_+ .$$

Notice that this condition implies that W must be closed and that the orthogonal projections $pr_\pm : W \to H_\pm$ are Fredholm and Hilbert-Schmidt respectively. So one has

$$Gr_{alg} = \{W \subset H \mid z^k H_+ \subset W \subset z^{-k} H_+,\ zW \subset W\}.$$

One can check that Gr_{alg} is precisely the image of $\Omega_{alg}U_n \subset \Omega U_n$ under φ.

3. A filtration of algebraic loops

We describe a filtration of $\Omega_{alg}U_n$ defined in [3] and [5]. For each k, denote by $(\Omega_{alg}U_n)_k$ the connected component with deg.det$=$ k. Let M_k be the set of all loops in $(\Omega_{alg}U_n)_{-k}$ that are polynomials in z^{-1}. One can "shift" M_k by multiplying z^m to obtain a set

$$z^m M_k = \{z^m \gamma \mid \gamma \in M_k\}$$

for any integer m. It is easy to see that $z^m M_k \subset (\Omega_{alg}U_n)_{-k+mn}$ and it is homeomorphic to M_k. Moreover, there is a filtration

$$M_0 \subset zM_n \subset z^2 M_{2n} \subset \cdots \subset \bigcup_{k\geq 0} z^k M_{kn} = (\Omega_{alg}U_n)_0 \ = \Omega_{alg}SU_n\,.$$

One can define a Grassmannian analogue: For each k, define

$$\begin{aligned} F_k &= \{W \in Gr_{alg} \mid H_+ \subset W \subset z^{-k}H_+, \dim W/H_+ = k\} \\ &= \{W \subset H \mid H_+ \subset W \subset z^{-k}H_+, \dim W/H_+ = k,\ zW \subset W\}\,. \end{aligned}$$

It is easy to see that F_k is contained in $(Gr_\infty)_k$, the connected component with virt.dim$= k$. Moreover, the homeomorphism $\varphi : \Omega_{alg}U_n \to Gr_{alg}$ restricts to a homeomorphism $\varphi : M_k \to F_k$.

Similar to the situation for M_k, one can "shift" F_k to a set

$$\begin{aligned} z^m F_k &= \{z^m W \mid W \in F_k\} \\ &= \ \{W' \subset H \mid z^m H_+ \subset W' \subset z^{-k+m}H_+, \dim W'/H_+ = k,\ zW' \subset W'\}\,. \end{aligned}$$

This set is contained in $(Gr_\infty)_{k-mn}$ and is homeomorphic to F_k. We have a sequence

$$F_0 \subset zF_n \subset z^2 F_{2n} \subset \cdots \subset \bigcup_{k\geq 0} z^k F_{kn} = (Gr_{alg})_0\,.$$

For each k, F_k can be realized as an algebraic subvariety of a *finite dimensional* Grassmannian. To see that, we first notice that by taking quotient by H_+, we obtain a homeomorphism

$$F_k \cong \{V \subset z^{-k}H_+/H_+ \mid zV \subset V, \dim V = k\}\,.$$

With the identifications

$$z^{-k}H_+/H_+ \cong \langle z^i e_j : j = 1, \cdots, n;\ i = -1, \cdots, -k\rangle \cong \mathbf{C}^{nk}\,,$$

multiplication by z induces a map

$$N : z^{-k}H_+/H_+ \cong \mathbf{C}^{nk} \to z^{-k}H_+/H_+ \cong \mathbf{C}^{nk}\,.$$

This map is nilpotent: $N^k = 0$. So we have

$$F_k \cong \{V \in Gr_k(\mathbf{C}^{nk}) \mid NV \subset V\} \subset Gr_k(\mathbf{C}^{nk})\,.$$

LEMMA 3.1. *F_k embeds into a complex projective space as the intersection of two smooth subvarieties.*

PROOF. Consider the linear map $I + N : \mathbf{C}^{nk} \to \mathbf{C}^{nk}$, where $I : \mathbf{C}^{nk} \to \mathbf{C}^{nk}$ is the identity map. It is an isomorphism, and 1 is the only eigenvalue. It induces an isomorphism

$(I+N)_{Gr} : Gr_k(\mathbf{C}^{kn}) \to Gr_k(\mathbf{C}^{kn})$. Notice that for any $E \in Gr_k(\mathbf{C}^{kn})$, we have

$$NE \subset E \Leftrightarrow (I+N)_{Gr}E = E\,.$$

Embed the Grassmannian $Gr_k(\mathbf{C}^{kn})$ in projective space using Plücker embedding ι : $Gr_k(\mathbf{C}^{kn}) \to \mathbf{CP}^{r-1}$, where $r = \begin{pmatrix} nk \\ k \end{pmatrix}$.

Consider the induced linear map

$$\Lambda^k(I+N) : \Lambda^k\mathbf{C}^{nk} \cong \mathbf{C}^r \to \Lambda^k\mathbf{C}^{nk} \cong \mathbf{C}^r\,.$$

Again, 1 is the only eigenvalue. Consider the 1−eigenspace

$$\mathcal{V} = \{v \in \Lambda^k\mathbf{C}^{nk} \mid \Lambda^k(I+N)v = v\}\,.$$

Its quotient $\mathcal{V}^* = \mathcal{V}/\mathbf{C}^* \subset \mathbf{CP}^{r-1}$ is a smooth subvariety. It is clear that

$$F_k = \mathcal{V}^* \cap \iota(Gr_k(\mathbf{C}^{nk}))\,. \qquad \square$$

The above argument also applies to the "shifted" F_k : There are homeomorphisms

$$\begin{aligned} z^m F_k &\cong \{V \subset z^{-k+m}H_+/z^m H_+ \mid NV \subset V, \dim V = k\} \\ &\cong \{V \in Gr_k(\mathbf{C}^{nk}) \mid NV \subset V\} \end{aligned}$$

with the identifications

$$z^{-k+m}H_+/z^m H_+ \cong \langle z^i e_j : j = 1, \cdots, n;\ i = m-1, \cdots, m-k\rangle \cong \mathbf{C}^{nk}\,.$$

Again, $z^m F_k$ is an algebraic subvariety of the Grassmannian and embeds into a complex projective space as intersection of two smooth subvarieties.

4. Singularities of F_k

In this section, we assume that $n \geq 2$.

Define the subset

$$F'_k = \{E \in F_k \mid \dim(E \cap \mathbf{C}^n) \geq 2\}$$

of F_k, where $\mathbf{C}^n = \langle z^{-1}e_1, \cdots, z^{-1}e_n\rangle$.

LEMMA 4.1.

$$F'_k = \{E \in F_k \mid z^{k-1}E = \{0\}\}\,.$$

PROOF. We need to show that for any $E \in F_k$, the two conditions

(i) $\dim(E \cap \mathbf{C}^n) \geq 2$;

(ii) $z^{k-1}E = \{0\}$

are equivalent. Recall that $\dim zE < \dim E$, $\dim z^2E < \dim zE$ etc. This means the dimension drops at least by one each time we apply z. Assuming (i), the dimension drops by at least two when we apply z to E. Since E is of dimension k, condition (ii) must be satisfied.

Similarly, if $z^{k-1}E = \{0\}$, then there must be a jump

$$\dim z^{i+1}E \leq \dim z^i E - 2\,.$$

That is,

$$\dim(z^{i+1}E \cap \mathbf{C}^n) \geq 2\,.$$

Since $z^{i+1}E \subset E$, condition (i) is satisfied. □

Since condition (ii) in the above proof is an algebraic condition, F_k' is a subvariety of F_k.

THEOREM 4.1. *F_k' is precisely the set of all singular points in F_k.*

PROOF. Recall from the previous section that the space F_k is the intersection of two smooth subvarieties of $\mathbf{CP}^{r-1}$, $r = \binom{nk}{k}$:

$$F_k = \mathcal{V}^* \cap \iota(Gr_k(\mathbf{C}^{nk}))\,,$$

where ι is the Plücker embedding; $\mathcal{V}^* = \mathcal{V}/\mathbf{C}^*$, where

$$\mathcal{V} = \{v \in \mathbf{C}^r = \Lambda^k\mathbf{C}^{nk} \mid \Lambda^k(I+N)v = v\}\,.$$

Since F_k is the intersection of two smooth subvarieties, a point $E \in F_k$ is singular if and only if the tangent space

$$T_E\mathcal{V}^* \cap T_E\,\iota(Gr_k(\mathbf{C}^{nk}))$$

is not of minimal dimension. This gives us a criterion to find all such points. To do this, let us compute the above intersection of tangent spaces in general. Put a metric on $\mathbf{C}^{nk} = \langle z^i e_j : i = -1, \cdots, -k;\ j = 1, \cdots, n\rangle$ such that the basis $\{z^i e_j : i = -1, \cdots, -k;\ j = 1, \cdots, n\}$ is orthonormal. This naturally induces a metric on $\mathbf{C}^r = \Lambda^k\mathbf{C}^{nk}$. Then for any $\eta \in \mathcal{V}^*$, we have

$$T_\eta\mathcal{V}^* \cong \mathrm{Hom}(\eta, \eta^\perp \cap \mathcal{V})\,,$$

where $\eta^\perp$ is the orthogonal complement of η with respect to the metric described above.

On the other hand, we have, for any $E \in Gr_k(\mathbf{C}^{nk})$,

$$T_{\iota E}\,\iota(Gr_k(\mathbf{C}^{nk})) = d\iota\, T_E Gr_k(\mathbf{C}^{nk}) = d\iota\,\mathrm{Hom}(E, E^\perp)\,.$$

To compute this tangent space, let $f_t : E \to E^\perp$ be a family of linear maps such that $f_0 = I$, the identity map. Write $E = \langle u_1, \cdots, u_k\rangle$, we then have a curve $\alpha_t = \langle u_1 + f_t u_1, \cdots, u_k + f_t u_k\rangle$ on $Gr_k(\mathbf{C}^{nk})$ such that $\alpha_0 = E$. Hence

$$
\begin{aligned}
d\iota\left(\frac{d}{dt}\alpha_t\Big|_{t=0}\right) &= \frac{d}{dt}\iota(\alpha_t)\Big|_{t=0} \\
&= \frac{d}{dt}(u_1 + f_t u_1) \wedge \cdots \wedge (u_k + f_t u_k)\Big|_{t=0} \\
&= u_1 \wedge \cdots \wedge u_k \mapsto \sum_{i=1}^{k}(u_1 + f_0 u_1) \wedge \cdots \wedge (u_{i-1} + f_0 u_{i-1}) \\
&\quad \wedge \dot{f}_0 u_i \wedge (u_{i+1} + f_0 u_{i+1}) \wedge \cdots \wedge (u_k + f_0 u_k) \\
&= u_1 \wedge \cdots \wedge u_k \mapsto \sum_{i=1}^{k} u_i \wedge \cdots \wedge u_{i-1} \wedge \dot{f}_0 u_i \wedge u_{i+1} \wedge \cdots \wedge u_k \\
&\in \mathrm{Hom}(\wedge^k E, (\wedge^k E)^{\perp}) .
\end{aligned}
$$

Notice that here $\dot{f}_0 \in \mathrm{Hom}(E, E^{\perp})$. We can now see that $T_{\iota E}\, \iota(Gr_k(\mathbf{C}^{nk}))$ consists of maps

$$
u_1 \wedge \cdots \wedge u_k \mapsto \sum_{i=1}^{k} u_1 \wedge \cdots \wedge u_{i-1} \wedge v_i \wedge u_{i+1} \wedge \cdots \wedge u_k
$$

for $v_1, \cdots, v_k \in E^{\perp}$. It is isomorphic to $\bigoplus_{i=1}^{k} E^{\perp}$ because $v_1, \cdots, v_k$ are arbitrary.

Hence for any $\eta = \Lambda^k E \in \mathcal{V}^* \cap \iota(Gr_k(\mathbf{C}^{nk}))$, where

$$
E = \langle u_1, \cdots, u_k \rangle \in Gr_k(\mathbf{C}^{nk}) ,
$$

we have

$$
\begin{aligned}
&T_\eta \mathcal{V}^* \cap T_\eta\, \iota(Gr_k(\mathbf{C}^{nk})) \\
&\quad = \left\{ f : \eta \to \eta^{\perp} \cap \mathcal{V} \;\middle|\; \begin{array}{l} f(u_1 \wedge \cdots \wedge u_k) = \sum_{i=1}^{k} u_1 \wedge \cdots \wedge u_{i-1} \wedge v_i \wedge \\ \qquad\qquad\qquad\quad u_{i+1} \wedge \cdots \wedge u_k \in \mathcal{V} \end{array} \right\} \\
&\quad \cong \left\{ \begin{array}{c} \sum_{i=1}^{k} u_1 \wedge \cdots \wedge u_{i-1} \wedge \\ v_i \wedge u_{i+1} \wedge \cdots \wedge u_k \end{array} \;\middle|\; \begin{array}{l} (I+N)(\sum_{i=1}^{k} u_1 \wedge \cdots \wedge u_{i-1} \wedge \\ \qquad v_i \wedge u_{i+1} \wedge \cdots u_k) = \\ \sum_{i=1}^{k} u_1 \wedge \cdots \wedge u_{i-1} \wedge v_i \wedge u_{i+1} \wedge \cdots \wedge u_k \end{array} \right\} .
\end{aligned}
$$

We now look at the condition (the equality) defining the above space more closely. First notice that the left hand side of the equality is equal to

$$
\begin{aligned}
&\sum_{i=1}^{k} (I+N)u_1 \wedge \cdots \wedge (I+N)v_i \wedge \cdots \wedge (I+N)u_k \\
&\qquad = \sum_{i=1}^{k} u_1 \wedge \cdots \wedge v_i \wedge \cdots \wedge u_k + \sum_{i=1}^{k} \sum^{\vee} g_1 u_1 \wedge \cdots \wedge g_i v_i \wedge \cdots \wedge g_k u_k ,
\end{aligned}
$$

where $\sum^{\vee}$ sums all terms such that $g_j = N$ for at least one j, and g_j equals the identity for other j's.

Therefore the equality becomes

$$\sum_{i=1}^{k} \sum^{\vee} g_1 u_1 \wedge \cdots \wedge g_i v_i \wedge \cdots \wedge g_k u_k = 0\,. \tag{4.1}$$

We consider the following two cases:

Case 1: $\dim(E \cap \mathbf{C}^n) \leq 1$.

Since N is nilpotent and $NE \subset E$, we must have $\dim(E \cap \mathbf{C}^n) = 1$, and E must have the form $\langle u_1, \cdots, u_k \rangle$ such that $Nu_i = u_{i-1}$ for all i and $Nu_1 = 0$.

PROPOSITION 4.1. *The complex dimension of the above intersection of the two tangent spaces is equal to $k(n-1)$.*

PROOF. First notice that the possibly non-zero terms on the left hand side of equation (4.1) are:

$$v_1 \wedge \sum_{i=2}^{k} J_i\,, \quad -v_2 \wedge \sum_{i=3}^{k} J_i\,, \quad v_3 \wedge \sum_{i=4}^{k} J_i\,, \quad -v_4 \wedge \sum_{i=5}^{k} J_i\,,$$
$$\cdots\cdots,\ (-1)^k v_{k-1} \wedge \sum_{i=k}^{k} J_i$$

and

$$Nv_1 \wedge \sum_{i=1}^{k} J_i\,, \quad -Nv_2 \wedge \sum_{i=2}^{k} J_i\,, \quad Nv_3 \wedge \sum_{i=3}^{k} J_i\,, \quad -Nv_4 \wedge \sum_{i=4}^{k} J_i\,,$$
$$\cdots\cdots,\ (-1)^k N v_{k-1} \wedge \sum_{i=k-1}^{k} J_i\,, \quad (-1)^{k+1} N v_k \wedge \sum_{i=k}^{k} J_i\,,$$

where $J_i = u_1 \wedge \cdots \wedge \hat{u_i} \wedge \cdots \wedge u_k$. Here $\hat{u_i}$ means omitting the factor u_i.

Adding and regrouping all the terms above, equation (4.1) becomes

$$\begin{gathered} Nv_1 \wedge J_1 \\ +(Nv_1 + v_1 - Nv_2) \wedge J_2 \\ +(Nv_1 + v_1 - Nv_2 - v_2 + Nv_3) \wedge J_3 \\ \vdots \\ +\{(I+N)(v_1 - v_2 + v_3 - \cdots + (-1)^k v_{k-1}) + (-1)^{k+1} N v_k\} \wedge J_k = 0\,. \end{gathered}$$

Write

$$\xi_1 = Nv_1\,,$$

$$\xi_2 = Nv_1 + v_1 - Nv_2\,,$$
$$\xi_3 = Nv_1 + v_1 - Nv_2 - v_2 + Nv_3\,,$$
$$\vdots$$
$$\xi_k = (1+N)(v_1 - v_2 + v_3 - \cdots + (-1)^k v_{k-1}) + (-1)^{k+1} N v_k\,.$$

Then (4.1) becomes

$$\sum_{i=1}^{k} \xi_i \wedge J_i = 0\,. \tag{4.2}$$

By wedging both sides of (4.2) with u_i, one can see that $\xi_i \wedge (u_1 \wedge \cdots \wedge u_k) = 0$ for all i. It is easy to see that this is equivalent to the conditions that

$$-Nv_1 \in E\,,$$
$$v_1 - Nv_2 \in E\,,$$
$$v_2 - Nv_3 \in E\,,$$
$$\vdots$$
$$v_{k-1} - Nv_k \in E\,.$$

Moreover, if we write $\xi_i = \sum_{j=1}^{k} \xi_i^j u_j$ for each i, then (4.2) implies that

$$\xi_1^1 - \xi_2^2 + \xi_3^3 - \xi_4^4 + \cdots = 0\,.$$

Hence $T_\eta \mathcal{V}^* \cap T_\eta\, \iota(Gr_k(\mathbf{C}^{nk}))$ is isomorphic to the set of $(v_1, \cdots, v_k) \in E^\perp \oplus \cdots \oplus E^\perp$ which satisfy

(1) $-Nv_1, v_1 - Nv_2, \cdots, v_{k-1} - Nv_k \in E$,

(2) $\xi_1^1 - \xi_2^2 + \xi_3^3 - \xi_4^4 + \cdots = 0$.

LEMMA 4.2. *Condition* (1) *above implies condition* (2).

PROOF. First, modulo $E^\perp$, we have

$$\xi_1 = Nv_1\,,$$
$$\xi_2 = N(v_1 - v_2)\,,$$
$$\xi_3 = N(v_1 - v_2 + v_3)\,,$$
$$\vdots$$
$$\xi_k = N(v_1 - v_2 + v_3 - \cdots + (-1)^{k-1} v_k)\,.$$

Let $Nv_i = \sum_{s=1}^{k} a_i^s u_s$ modulo $E^{\perp}$. We then have

$$\xi_1 = \sum_s a_1^s u_s \,,$$

$$\xi_2 = \sum_s (a_1^s - a_2^s) u_s \,,$$

$$\xi_3 = \sum_s (a_1^s - a_2^s + a_3^s) u_s \,,$$

$$\vdots$$

$$\xi_k = \sum_s (a_1^s - a_2^s + a_3^s - \cdots + (-1)^{k-1} a_k^s) u_s \,.$$

Hence

$$\begin{aligned}
\xi_1^1 - \xi_2^2 + \xi_3^3 - \xi_4^4 + \cdots &= a_1^1 - (a_1^2 - a_2^2) + (a_1^3 - a_2^3 + a_3^3) \\
&\quad - (a_1^4 - a_2^4 + a_3^4 - a_4^4) + \cdots \\
&= (a_1^1 - a_1^2 + a_1^3 - \cdots) + (a_2^2 - a_2^3 + a_2^4 - \cdots) \\
&\quad + (a_3^3 - a_3^4 + \cdots) + \cdots + (a_k^k) \,.
\end{aligned}$$

On the other hand, modulo $E^{\perp}$, we have $(I + N)v_i = Nv_i$ for each i. Therefore

$$\begin{aligned}
0 = v_i &= (I - N + N^2 - N^3 + \cdots) Nv_i \\
&= Nv_i - N^2 v_i + N^3 v_i + \cdots \,.
\end{aligned}$$

For $i = 1$, this equality becomes

$$\begin{aligned}
0 &= Nv_1 - N^2 v_1 + N^3 v_1 + \cdots \\
&= (a_1^1 - a_1^2 + a_1^3 - \cdots) u_1 + \cdots \,,
\end{aligned}$$

which implies that $a_1^1 - a_1^2 + a_1^3 - \cdots = 0$.

Also, for each i,

$$\begin{aligned}
0 &= N(Nv_i - N^2 v_i + N^3 v_i + \cdots) \\
&= N^2 v_i - N^3 v_i + N^4 v_i + \cdots \,.
\end{aligned}$$

For $i = 2$,

$$\begin{aligned}
0 &= N^2 v_2 - N^3 v_2 + N^4 v_2 + \cdots \\
&= (a_2^2 - a_2^3 + a_2^4 - \cdots) u_1 + \cdots \,,
\end{aligned}$$

which yields $a_2^2 - a_2^3 + a_2^4 - \cdots = 0$.

Proceeding in a similar way, one gets $\xi_1^1 - \xi_2^2 + \xi_3^3 - \xi_4^4 + \cdots = 0$. $\square$

Consider the linear map

$$\begin{aligned} T : E^{\perp} \oplus \cdots \oplus E^{\perp} &\to \mathbf{C}^{nk} \oplus \cdots \oplus \mathbf{C}^{nk} \\ (v_1, \cdots, v_k) &\mapsto (-Nv_1, v_1 - Nv_2, \cdots, v_{k-1} - Nv_k)\,. \end{aligned}$$

Write $E^{\perp} =< w_1, w_2, \cdots, w_m >$, where $m = nk - k$. The vectors

$$\{u_1, \cdots, u_k, w_1, \cdots, w_m\}$$

form an ordered basis $\mathcal{B}$ of $\mathbf{C}^{nk}$. This naturally gives an ordered basis $\mathcal{B}^k$ of $\mathbf{C}^{nk} \oplus \cdots \oplus \mathbf{C}^{nk}$. With respect to this basis, the map T has the form

$$\begin{pmatrix} -N' & 0 & & \\ I & -N' & 0 & \\ 0 & \ddots & \ddots & \\ & & I & -N' \end{pmatrix},$$

where N' and $I + N'$ are the restrictions of N and $I + N$ on $E^{\perp}$. Write

$$N' = \begin{pmatrix} A \\ B \end{pmatrix}, \quad I' = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $A : E^{\perp} \to E; \quad 1, B : E^{\perp} \to E^{\perp}$.

We can see that $T(v_1, \cdots, v_k) \in E \oplus \cdots \oplus E$ if and only if

$$\begin{pmatrix} -B & & & \\ 1 & -B & & \\ & \ddots & \ddots & \\ & & 1 & -B \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_k \end{pmatrix} = 0\,.$$

Denote the above matrix by D. We have

$$T_{\eta}\mathcal{V}^* \cap T_{\eta}\, \iota(Gr_k(\mathbf{C}^{nk})) = \ker D\,.$$

We now calculate the rank of D. First notice that since $B : E^{\perp} \to E^{\perp}$ is just the restriction of N followed by a projection, it must be nilpotent. Hence by choosing the basis $\{w_1 \cdots, w_m\}$ appropriately, we can assume that $-B$ has the form

$$\begin{pmatrix} X & & & \\ & X & & \\ & & \ddots & \\ & & & X \end{pmatrix},$$

where

$$X = \begin{pmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Observe that because of the presence of the blocks 1, the row vectors in D with such blocks (that is, from the $(nk-k+1)$-st row to the last row) are linearly independent, while, because of the form of $-B$, the first $nk-k$ rows can be expressed as linear combinations of the rows below them. Hence the rank of D is equal to $(nk-k)(k-1)$, which implies that

$$\dim\ker D = (nk-k)k - \operatorname{rank} D = (nk-k)k - (nk-k)(k-1) = k(n-1)\,.$$

The proof of the proposition is completed. $\square$

Case 2: $\dim(E \cap \mathbf{C}^n) > 1$.

Let us first examine the case $\dim(E \cap \mathbf{C}^n) = 2$.

Then E must be of the form $\langle u_1, u_3, \cdots, u_2, u_4, \cdots\rangle$ such that $Nu_1 = Nu_2 = 0$ and $Nu_i = u_{i-2}$.

The possibly non-zero terms on the left hand side of (4.1) are

$$\begin{array}{rl} Nv_1 \wedge J_1\,, & -Nv_2 \wedge J_2\,, \\ \{(1+N)(-v_1) + Nv_3\} \wedge J_3\,, & \{(1+N)(v_2) - Nv_4\} \wedge J_4\,, \\ \{(1+N)(v_1 - v_3) + Nv_5\} \wedge J_5\,, & \{(1+N)(-v_2 + v_4) - Nv_6\} \wedge J_6\,, \\ \{(1+N)(-v_1 + v_3 - v_5) + Nv_7\} \wedge J_7\,, & \{(1+N)(v_2 - v_4 + v_6) - Nv_8\} \wedge J_8\,, \\ \vdots & \vdots \qquad . \end{array}$$

As in case 1, let ξ_i be such that the above terms become $\xi_i \wedge J_i = 0$ for all i. Then (4.1) implies $\xi_i \in E$.

We claim again that, if we write $\xi_i = \sum_{s=1}^{k} \xi_i^s u_s$, then we automatically have $\xi_1^1 - \xi_2^2 + \xi_3^3 - \cdots = 0$. Writing $Nv_i = \sum a_i^s u_s$ modulo $E^\perp$, we have

$$\begin{aligned} \xi_1 &= Nv_1 = \sum a_1^s u_s\,, \\ \xi_2 &= -Nv_2 = -\sum a_2^s u_s\,, \\ \xi_3 &= -Nv_1 + Nv_3 = \sum (-a_1^s + a_3^s) u_s\,, \\ \xi_4 &= Nv_2 - Nv_4 = \sum (a_2^s - a_4^s) u_s\,, \\ \xi_5 &= Nv_1 - Nv_3 + Nv_5 = \sum (a_1^s - a_3^s + a_5^s) u_s\,, \\ &\vdots \qquad . \end{aligned}$$

Therefore

$$\begin{aligned}\xi_1^1 - \xi_2^2 + \xi_3^3 - \cdots &= a_1^1 + a_2^2 + (-a_1^3 + a_3^3) - (a_2^4 - a_4^4) + (a_1^5 - a_3^5 + a_5^5) - \cdots \\ &= (a_1^1 - a_1^3 + a_1^5 - a_1^7 + \cdots) + (a_2^2 - a_2^4 + a_2^6 - \cdots) \\ &\quad + (a_3^3 - a_3^5 + a_3^7 - \cdots) + \cdots .\end{aligned}$$

Manipulating $\xi_1, \xi_3, \xi_5, \cdots$ as in case 1, one gets

$$\begin{aligned} a_1^1 - a_1^3 + a_1^5 - a_1^7 + \cdots &= 0 , \\ a_3^3 - a_3^5 + a_3^7 - \cdots &= 0 , \\ &\vdots \quad , \end{aligned}$$

while combining $\xi_2, \xi_4, \xi_6, \cdots$ as in case 1 gives

$$\begin{aligned} a_2^2 - a_2^4 + a_2^6 - \cdots &= 0 , \\ &\vdots \quad . \end{aligned}$$

Consider the linear map $T : E^\perp \oplus \cdots \oplus E^\perp \to \mathbf{C}^{nk} \oplus \cdots \oplus \mathbf{C}^{nk}$ which sends $(v_1 \cdots , v_k)$ to $(\xi_1, \cdots , \xi_k)$. In terms of the basis $\mathcal{B}^k$ defined in case 1, we have

$$T = \begin{pmatrix} -N & 0 & & & \\ 0 & -N & 0 & & \\ I & 0 & -N & 0 & \\ & I & 0 & -N & \\ & & \ddots & \ddots & \ddots \end{pmatrix} .$$

Writing $N = \begin{pmatrix} A \\ B \end{pmatrix}$ as in case 1, we again have

$$T_\eta \mathcal{V}^* \cap T_\eta \, \iota(Gr_k(\mathbf{C}^{nk})) = \ker D ,$$

where

$$D = \begin{pmatrix} -B & 0 & & & \\ 0 & -B & 0 & & \\ I & 0 & -B & 0 & \\ & I & 0 & -B & \\ & & \ddots & \ddots & \ddots \end{pmatrix} .$$

Arguing as in case 1, one can see that rank $D = (k-2)(nk-k)$ because of the presence of the submartix

$$\begin{pmatrix} -B & \\ & -B \end{pmatrix} .$$

at the upper left corner of D. Hence

$$\dim \ker D = (nk - k)k - (k - 2)(nk - k) = 2k(n - 1)\,.$$

In general, one can see that if $\dim E \cap \mathbf{C}^n = d$, then

$$\operatorname{rank} D = (k - d)(nk - k)$$

because there will be a submatrix

$$\begin{pmatrix} -B & & \\ & \ddots & \\ & & -B \end{pmatrix}$$

of size $d(nk - k)$ at the upper left corner of D. Hence we have

LEMMA 4.3. *If* $\dim E \cap \mathbf{C}^n = d$, *then* $T_\eta \mathcal{V}^* \cap T_\eta\, \iota(Gr_k(\mathbf{C}^{nk}))$ *has complex dimension* $dk(n - 1)$.

In particular, if $\dim E \cap \mathbf{C}^n > 1$, the dimension of $T_\eta \mathcal{V}^* \cap T_\eta\, \iota(Gr_k(\mathbf{C}^{nk}))$ is greater than the case 1 situation. In other words, in the case 1 situation we have the minimal dimension, while in the case 2 situation the dimension is greater. Therefore we can conclude that $E \in F_k$ is a smooth point if and only if $\dim(E \cap \mathbf{C}^n) = 1$. And the proof of the theorem is now complete. □

To finish, let us make two remarks.

First, for a general segment

$$z^m F_{mn} \to z^{m+1} F_{(m+1)n}$$

of the filtration, we can ask if the first set is precisely the set of singularities of the second one. This is equivalent to asking the same question for the shifted inclusion:

$$z^{-1} F_{k-n} \to F_k\,,$$

where we write k for $(m + 1)n = mn + n$.

PROPOSITION 4.2. *The space* F'_k *contains* $z^{-1} F_{k-n}$.

PROOF. For any $W \in z^{-1} F_{k-n}$,

$$z^{-1} H_+ \subset W \subset z^{n-k-1} H_+\,.$$

So

$$z^{k-1} W \subset z^{n-2} H_+ \subset H_+$$

because $n \geq 2$. But this is equivalent to the condition (ii) that defines F'_k. □

PROPOSITION 4.3. *In the case of* $n = 2$, $F'_k = F_{k-2}$.

PROOF. Notice that when $n = 2$, we have

$$E \in F_k' \Leftrightarrow \dim E \cap \mathbf{C}^2 \geq 2 \Leftrightarrow E \supset \mathbf{C}^2 \Leftrightarrow E \in F_{k-2}\,. \qquad \square$$

When $n > 2$, $z^{-1}F_{k-n} \subset F_k'$ is a proper inclusion. For example, look at the case $n = k = 3$. The set $z^{-1}F_{k-n} = z^{-1}F_0$ is a point whereas F_3' is easily seen to contain more than a point.

The second remark has to do with a "desingularization" of F_k defined in [3]. Assume that $n \geq 2$. For any k, define $\hat{F}_k$ to be the subspace of the flag manifold $F_{1,2,\cdots,k}(\mathbf{C}^{nk})$ consisting of flags of the form

$$\{0\} = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k \subset \mathbf{C}^{nk}$$

that satisfy the conditions:

(1) $\dim E_i/E_{i-1} = 1$ for all i,

(2) $NE_i \subset E_i$ for all i,

where $\mathbf{C}^{nk}$ is identified with $\langle z^{-i}e_j : i = 1, 2, \cdots, k;\ j = 1, 2, \cdots, n\rangle$.

It is proved in [3] p. 358 that $\hat{F}_k$ is a smooth complex manifold and the projection

$$\pi : \hat{F}_k \to F_k : E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_k \mapsto E_k$$

is surjective and restricts to a biholomorphism

$$\pi : \pi^{-1}(F_k - F_k') \to F_k - F_k'\,.$$

(In [3], $\hat{F}_k$ is denoted $X_{n,k}$ and $F_k - F_k'$ is denoted $V_{n,k}$.)

Since F_k' is precisely the singular set of F_k, we see that the above projection is a genuine desingularization of F_k.

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Present Address:
DEPARTMENT OF NATURAL AND SOCIAL SCIENCES,
BOWLING GREEN STATE UNIVERSITY–FIRELANDS COLLEGE,
ONE UNIVERSITY DRIVE, HURON, OH 44839, U.S.A.
e-mail: nwliu@bgnet.bgsu.edu