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Recurrence-Transience Criteria for Storage Processes and Their Applications

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Abstract. We obtain sufficient conditions for recurrence and sufficient conditions for transience of storage processes. Applying these results, we give a necessary and sufficient condition for transience in case the input process is a stable process and the release rate is a power function. This result is extended using Abelian theorem for Stieltjes transforms. As another application, we refine conditions for recurrence-transience in case the release rate is non-decreasing and bounded. One of these results corresponds to a recurrence-transience criterion for Bessel processes. As a by-product, the necessary and sufficient condition for transience of processes of Ornstein-Uhlenbeck type is simplified in diagonal drift coefficient case.

1. Introduction

In this paper, we deal with storage processes (in some papers these are called dam processes). Roughly speaking, storage process is a stochastic process governed by the following stochastic integral equation:

$$X(t) = x - \int_0^t r(X(s))ds + A(t) \tag{1}$$

where r is a nonnegative function called release rate and $\{A(t)\}$ is a subordinator called input process.

Although many papers are devoted to the investigation of storage processes, there remain some interesting problems. One of such problems is to give necessary and sufficient conditions for positive recurrence, null recurrence and transience, respectively in terms of r and $\{A(t)\}$ (or equivalently, its Lévy measure). Storage processes are classified into three types: positive recurrent, null recurrent and transient processes (Theorem 1. A necessary and sufficient condition for positive recurrence is given by Brockwell, Resnick and Tweedie ([3]) when $r(x)$ is non-decreasing. However, it seems that a nice criterion for transience-recurrence is not known so far. One exceptional case is the following:

$$r(x) = ax\,, \quad a > 0\,. \tag{2}$$

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Stochastic process with such a special r is the simplest 1-dimensional case of so-called processes of Ornstein-Uhlenbeck type on R^d, $d \geq 1$ ([8]). A necessary and sufficient condition for positive recurrence for processes of Ornstein-Uhlenbeck type is given in [8], a necessary and sufficient condition for transience is obtained by Shiga [9] in one dimension and by Watanabe [11] in higher dimensions.

Our aim is to get sufficient conditions for recurrence and sufficient conditions for transience (Theorems 3–6). Applying these results, we give, in Section 6, a necessary and sufficient condition for transience in case the input process A is a stable process and the release rate r is a power function (Theorem 8). This result is extended using Abelian theorem for Stieltjes transforms (Theorem 9). As another application, we refine conditions for recurrence-transience in case r is non-decreasing and bounded. One of these results, Theorem 10, corresponds to a recurrence-transience criterion for Bessel processes. It should be remarked that the necessary and sufficient condition for transience of processes of Ornstein-Uhlenbeck type can be simplified in at least 1-dimensional case (Theorem 7).

We describe the construction of the storage process in Section 2 which gives the precise definition and is necessary in the successive arguments. This is due to Brockwell, Resnick and Tweedie ([3]). In section 3, we classify storage processes into three classes: positive recurrent, null recurrent and transient processes. In Sections 3 and 4, we give preliminary results.

2. Construction of storage processes

Let us define a storage process following [3] . Let r be a nonnegative function defined on $[0, \infty)$ such that $r(0) = 0$, $r(x) > 0$ for $x > 0$, left continuous on $(0, \infty)$ and has right limits on $[0, \infty)$ which are positive on $(0, \infty)$. Let $\{A(t)\}$ be an increasing Lévy process such that $A(0) = 0$ and

$$Ee^{-\theta A(t)} = \exp \int_0^\infty t(e^{-\theta y} - 1)\nu(dy) \quad \text{for all } \theta \geq 0,$$

where ν is a measure on $(0, \infty)$ satisfying

$$0 < \int_0^\infty (x \wedge 1)\nu(dx) < \infty .$$

This measure ν is called the Lévy measure of $\{A(t)\}$. Consider first a simple case $\lambda = \nu((0, \infty)) < \infty$. Then the number of jumps of A is finite on any finite time intervals. Denote the number of jumps and jump sizes by T_n and Y_n, $n = 1, 2, \cdots$, respectively and set $T_0 = 0$ for convenience. The inter-arrival times are denoted by $\sigma_n = T_n - T_{n-1}$ for $n \geq 1$. Set

$$R(x, y) = \int_x^y \frac{1}{r(z)} dz$$

for $y > 0$ and $0 < x \leq y$. Define $R(0, y) := \lim_{x \downarrow 0} R(x, y) \leq \infty$. Since the function $R(x, y)$ is continuous and strictly decreasing in $x \in (0, y]$, it has a continuous inverse $R_y^{-1}(t)$

on $[0, R(0, y))$. Define $q(t, y)$ for $y > 0$ by

$$q(t, y) = \begin{cases} R_y^{-1}(t) & \text{for } 0 \le t < R(0, y)\,, \\ 0 & \text{for } R(0, y) \le t \quad (\text{if } R(0, y) < \infty) \end{cases}$$

and set $q(t, 0) = 0$ for $t \ge 0$. Then, the function q has the following properties:

1. $q(R(x, y), y) = x$ for $0 < x \le y < \infty$ and $R(q(t, y), y) = t$ for $y > 0$ and $0 \le t < R(0, y)$.
2. $q(t, y)$ is continuous and decreasing in t and increasing in y.
3. Since $R(x, y)$ is decreasing and left differentiable in x, $q(t, y)$ is right differentiable in t and it satisfies

$$\begin{cases} \dfrac{d^+}{dt} q(t, y) = -r(q(t, y))\,, \\ q(0, y) = y\,, \end{cases}$$

for $y > 0$ and $0 \le t < R(0, y)$, where $\frac{d^+}{dt} q(x, t)$ is the right derivative of $q(x, t)$ for fixed x. Unique solution of (1) is given by

$$\begin{aligned} X(0) &= x\,, \\ X(T_m) &= q(\sigma_m, X(T_{m-1})) + Y_m \quad (m \ge 1)\,, \\ X(t) &= q(t - T_m, X(T_m)) \quad \text{for} \quad T_m < t < T_{m+1}\,,\ m \ge 0\,. \end{aligned}$$

In case $\lambda = \infty$, we define $\{A_n(t)\}$ by

$$A_n(t) = \sum_{s \le t} (A(s) - A(s-)) 1_{\{A(s) - A(s-) > \frac{1}{n}\}}\,.$$

Then, $\{A_n(t)\}$ is an increasing Lévy process with Lévy measure $\nu_n(\cdot) = \nu(\cdot \cap (\frac{1}{n}, \infty))$. Clearly, $\nu_n((0, \infty)) < \infty$. For this $\{A_n(t)\}$, we can construct unique process $\{X_n(t)\}$ satisfying (1) as above. Since $A_n(t)$ is nondecreasing in n, $\{X_n(t)\}$ is also nondecreasing in n. Hence, $X(t) = \lim_{n\to\infty} X_n(t)$ exists. The convergence is uniform on every finite time interval for each fixed $x > 0$ since A_n converges to A uniformly on every finite interval and r is left continuous ([3]). Hence $\{X(t)\}$ has cad lag sample paths. The process $\{X(t)\}$ satisfies (1) and is a Hunt process ([3]). This process is called storage process starting at x corresponding to (r, ν). The storage process $X(t)$ starting at x is denoted as $X(t, x)$ if necessary.

3. Recurrence classification

Let τ_y be the hitting time of $\{y\}$ and let τ_y^* be the hitting time of $[y, \infty]$ by $\{X(t)\}$.

THEOREM 1. *Storage process $\{X(t)\}$ can be classified into the following three types*:

(a) *For all $x > y > 0$, $0 < P_x(\tau_y < \infty) < 1$,*

(b) *for all $x > y > 0$, $P_x(\tau_y < \infty) = 1$ and $E_x(\tau_y) = \infty$,*

(c) *for all* $x > y > 0$, $P_x(\tau_y < \infty) = 1$ *and* $E_x(\tau_y) < \infty$.
In case (c), *there is unique stationary distribution* π *satisfying*

$$\lim_{t\to\infty} P_x(X(t) \in B) = \frac{1}{E_x(\tau_x)} E_x[\int_0^{\tau_x} 1_B(X(t))dt] = \pi(B)$$

for every $x > 0$ *and every Borel set* B *on* $[0, \infty)$.

Since storage process has no downward jump, we have:

LEMMA 1. *Let* $0 \le z < y < x$ *and* $\alpha \ge 0$. *Then,*

$$E_x[e^{-\alpha\tau_z}] = E_x[e^{-\alpha\tau_y}]E_y[e^{-\alpha\tau_z}]\,. \tag{3}$$

Letting $\alpha = 0$ in (3), we have

$$P_x(\tau_z < \infty) = P_x(\tau_y < \infty)P_y(\tau_z < \infty) \quad \text{for } 0 \le z < y < x\,. \tag{4}$$

LEMMA 2. *The following hold.*
(a) $P_x(\lim_{y\to\infty} \tau^*_y = \infty) = 1$,
(b) $P_x(\tau^*_y < \infty) = 1$ *and* $E_x(\tau^*_y) < \infty$ *for all* $y > x \ge 0$.

PROOF. (a) Since τ^*_y is non-decreasing in y, $\lim_{y\to\infty} \tau^*_y = \tau$ exists. On $\{\tau < \infty\}$, $X(\tau^*_y) \to X(\tau)$ a.s. as $y \to \infty$ by quasi-left continuity. On the other hand, $y \le X(\tau^*_y)$. Hence $X(\tau) = \infty$. This is a contradiction. Hence $\tau = \infty$ a.s.

(b) Fix $y > 0$ arbitrarily. Since $X(t, x) \le X(t, z)$ for $0 \le x < z$,

$$P_x\left(\sup_{0\le t\le a} X(t) < y\right) \ge P_z\left(\sup_{0\le t\le a} X(t) < y\right)$$

for $0 \le x < z < y$. We show that for some $a > 0$,

$$p := P_0\left(\sup_{0\le t\le a} X(t) < y\right) < 1. \tag{5}$$

Then, by the Markov property, we have

$$P_0(\tau^*_y > ma) \le p^m$$

for every integer $m \ge 1$. This yields (b). Let $\nu_n = \nu|_{(1/n,\infty)}$ and let $\lambda_n = \nu_n(R_+)$. Choose n so that $\lambda_n > 0$. Define $A_n(t)$ as in Section 2. Let $\{\sigma_k\}$ and $\{Y_k\}$ be sequences of inter arrival times and jump sizes defined for $A_n(t)$ in Section 2. Let $X_n(t)$ be a storage process constructed by $r(x)$ and $A_n(t)$. Choose c and d so that $\nu([d, \infty)) > 0$ and $0 < c < d$. Let $t_0 = (d - c)\inf\{\frac{1}{r(u)} : c \le u \le y\}$. Choosing an integer k so that $d + kc > y$, we have

$$\begin{aligned} P_0(X(t) \ge y \quad &\text{for some } t \text{ in } [0, (k+1)t_0]) \\ &\ge P_0(X_n(t) \ge y \quad \text{for some } t \text{ in } [0, (k+1)t_0]) \\ &\ge P_0(\sigma_1 \le t_0, \cdots, \sigma_{k+1} \le t_0, Y_1 \ge d, \cdots, Y_{k+1} \ge d) \end{aligned}$$

$$\geq (1 - e^{-\lambda_n t_0})^{k+1} \{\nu([d, \infty))/\lambda_n\}^{k+1} > 0 .$$

We have shown (5). □

Let $f(x)$ be a nonnegative and non-increasing measurable function on $(0, \infty)$ and integrable near 0 and let $F(x) = \int_0^x f(y)dy$. Let

$$f^{1*}(x) = f(x) \quad \text{for } x > 0,$$

$$f^{m*}(x) = \int_0^x f(z) f^{(m-1)*}(x - z)dz \quad \text{for } x > 0 \text{ and } m \geq 2 .$$

Then $f^{m*}(x)$ is finite for almost every $x \in [0, \infty)$. Since

$$\int_0^x f^{m*}(y)dy \leq e^{\lambda x} \int_0^x e^{-\lambda y} f^{m*}(y)dy \leq e^{\lambda x} \left\{ \int_0^\infty e^{-\lambda y} f(y)dy) \right\}^m$$

for $\lambda > 0$, choosing $\lambda_0 > 0$ so that $\hat{f}(\lambda_0) = \int_0^\infty e^{-\lambda_0 y} f(y)dy < 1$, we have that

$$\sum_{m=1}^\infty \int_0^x f^{m*}(y)dy < e^{\lambda_0 x} \hat{f}(\lambda_0)/\{1 - \hat{f}(\lambda_0)\} < \infty$$

and $f^*(x) = \sum_{m=1}^\infty f^{m*}(x)$ is finite for almost every x in $(0, \infty)$.

LEMMA 3. *Suppose that $r(x) = r > 0$ for all $x > 0$. Let $f(x) = \nu((x, \infty))/r$. Then*

$$P_x(\tau_b^* < \tau_a) = \int_{b-x}^{b-a} f^*(y)dy \Big/ \left\{ 1 + \int_0^{b-a} f^*(y)dy \right\}$$

for $a < x < b$ and hence $0 < P_x(\tau_b^* < \tau_a) < 1$.

PROOF. In case $0 < f(0) < \infty$, the result is obtained by Harrison and Resnick ([6]). Assume that $f(0) = \infty$. Define $A_n(t)$ as in Section 2 and let $X_n(t)$ be the storage process corresponding to (r, A_n) starting at x. Let $\tau_b^*(n)$ and $\tau_a(n)$ be the hitting times of $[b, \infty)$ and $\{a\}$ by $X_n(t)$, respectively. We have $\tau_b^*(n) \downarrow \tau_b^*$ and $\tau_a(n) \uparrow \tau_a$ as $n \to \infty$ a.s. since $X(\tau_b^*) > b$ a.s., $X_n(t) \uparrow X(t)$ as $n \to \infty$ uniformly on any bounded time interval a.s. and both $\{X_n(t)\}$ and $\{X(t)\}$ are quasi-left continuous. Hence

$$P_x(\tau_b^*(n) < \tau_a(n)) \uparrow P_x(\tau_b^* < \tau_a)$$

as $n \to \infty$. Let

$$f_n(x) = \begin{cases} f(x) & \text{for } x > 1/n , \\ 0 & \text{otherwise} \end{cases}$$

and let $f_n^*(x) = \sum_{m=1}^\infty f_n^{m*}(x)$. Note that

$$P_x(\tau_b^*(n) < \tau_a(n)) = \int_{b-x}^{b-a} f_n^*(y)dy \Big/ \left\{ 1 + \int_0^{b-a} f_n^*(y)dy \right\} .$$

Hence, $P_x(\tau_b^* < \tau_a) > 0$. Since $\{f_n^{m*}(x)\}$ is non-decreasing in n and $\lim_{n\to\infty} f_n^{m*}(x) = f^{m*}(x)$ for $x > 0$,

$$\lim_{n\to\infty} \int_{b-x}^{b-a} f_n^{m*}(y)dy = \int_{b-x}^{b-a} f^{m*}(y)dy\,.$$

Hence, by bounded convergence theorem,

$$\int_{b-x}^{b-a} f_n^{*}(y)dy = \sum_{m=1}^{\infty} \int_{b-x}^{b-a} f_n^{m*}(y)dy \to \sum_{m=1}^{\infty} \int_{b-x}^{b-a} f^{m*}(y)dy = \int_{b-x}^{b-a} f^{*}(y)dy$$

as $n \to \infty$. □

REMARK 1. Lemma 3 is stated in [1] in another form.

LEMMA 4. *For every* $0 < a < x < b$,

$$0 < P_x(\tau_b^* < \tau_a) < 1$$

PROOF. Let $r_0 = \inf_{a\le z\le b} r(z)$. Then $r_0 > 0$. Define $r_1(x) = r_0$ for all $x > 0$ and

$$r_2(x) = \begin{cases} r_0 & \text{if } a < x \le b\,, \\ r(x) & \text{otherwise}\,. \end{cases}$$

Let $\{X_j(t)\}$ be the storage processes corresponding to (r_j, A) starting at x for $j = 1, 2$, respectively. Let $\tau_a(j)$ and $\tau_b^*(j)$ be the hitting times of $\{a\}$ and $[b, \infty)$ for the storage process $\{X_j(t)\}$ for $j = 1, 2$, respectively. Then, since $X_2(t) \ge X(t)$ and $X_1(t) = X_2(t)$ up to the first exit time from the interval (a, b),

$$P_x(\tau_b^* < \tau_a) \le P_x(\tau_b^*(2) < \tau_a(2)) = P_x(\tau_b^*(1) < \tau_a(1)) < 1$$

by Lemma 3. Similarly, we have $P_x(\tau_b^* < \tau_a) > 0$. □

LEMMA 5. *For every* $0 < y < x$, $P_x(\tau_y < \infty) > 0$.

PROOF. Choose b so that $b > x$. By Lemma 4, we have

$$P_x(\tau_y < \infty) \ge P_x(\tau_y < \tau_b^*) > 0\,.$$ □

PROOF OF THEOREM 1. Assume that there are x_0 and y_0 such that $0 < y_0 < x_0$ and $P_{x_0}(\tau_{y_0} < \infty) < 1$. Then, by Lemma 1, it holds that $P_x(\tau_y < \infty) < 1$ for all y in $(0, y_0)$ and for all $x > x_0$. For $y \in (0, y_0)$ and $x \in (y, x_0)$, choose b so that $b > x_0$. Note that $0 < P_x(\tau_b^* < \tau_y)$ by Lemma 4. We have

$$\begin{aligned} P_x(\tau_y < \infty) &= P_x(\tau_b^* < \tau_y < \infty) + P_x(\tau_y \le \tau_b^* < \infty) \\ &\le P_b(\tau_y < \infty) P_x(\tau_b^* < \tau_y) + 1 - P_x(\tau_b^* < \tau_y) \\ &< 1 \end{aligned}$$

by strong Markov property and nonincreasingness of $P_x(\tau_y < \infty)$ in x. Hence $P_x(\tau_y < \infty) < 1$ for $y \in (0, y_0)$ and $x > y$. Now suppose that there is some $y_1 \in [y_0, \infty)$ such that

$$P_x(\tau_y < \infty) = 1 \quad \text{for all } x > y > y_1$$

and

$$P_x(\tau_y < \infty) < 1 \quad \text{for all } y \in (0, y_1) \text{ and for all } x > y\,.$$

Choose y_2, y_3, x so that $y_2 < y_1 < y_3 < x$. Then $P_{y_3}(\tau_x^* \geq \tau_{y_2}) > 0$. Note that

$$P_{X(\tau_x^*)}(\tau_{y_2} < \infty) = P_{X(\tau_x^*)}(\tau_{y_3} < \infty)P_{y_3}(\tau_{y_2} < \infty) = P_{y_3}(\tau_{y_2} < \infty)\,.$$

Hence, by the strong Markov property, we have

$$\begin{aligned} P_{y_3}(\tau_{y_2} < \infty) &= E_{y_3}(P_{X(\tau_x^*)}(\tau_{y_2} < \infty) : \tau_x^* < \tau_{y_2}) + P_{y_3}(\tau_x^* \geq \tau_{y_2}) \\ &= P_{y_3}(\tau_{y_2} < \infty)P_{y_3}(\tau_x^* < \tau_{y_2}) + P_{y_3}(\tau_x^* \geq \tau_{y_2}) \\ &= P_{y_3}(\tau_{y_2} < \infty)\{1 - P_{y_3}(\tau_x^* \geq \tau_{y_2})\} + P_{y_3}(\tau_x^* \geq \tau_{y_2})\,. \end{aligned}$$

This yields

$$\{1 - P_{y_3}(\tau_{y_2} < \infty)\}P_{y_3}(\tau_x^* \geq \tau_{y_2}) = 0\,.$$

This is a contradiction. Hence $y_1 = \infty$. Now assume that

$$P_x(\tau_y < \infty) = 1 \quad \text{for all } x > y > 0\,. \tag{6}$$

We have the following equality.

$$P_x(X(t) \in B) = P_x(X(t) \in B, \tau_y > t) + \int_0^t P_y(X(t-s) \in B)P_x(\tau_y \in ds) \tag{7}$$

for $x, y > 0$. This is a renewal equation for $x = y$. It is easy to see that $P_x(\tau_x \in ds)$ is not arithmetic for any $x > 0$. If $E_x(\tau_x) < \infty$, then $P_t(X(t) \in B, T_x > t)$ is directly Riemann integrable in t since it is nondecreasing in t. By the renewal theorem (e.g. [5]), we have

$$P_x(X(t) \in B) \to \frac{1}{E_x(\tau_x)} \int_0^\infty P_x(X(t) \in B, \tau_x > t)dt \tag{8}$$

as $t \to \infty$ for x satisfying $E_x(\tau_x) < \infty$ and

$$P_x(X(t) \in B) \to 0 \tag{9}$$

as $t \to \infty$ for x satisfying $E_x(\tau_x) = \infty$. Denote the right hand sides of both (8) and (9) by the same symbol $\pi_x(B)$. Note that $P_x(\tau_y < \infty) = 1$ for all $x, y > 0$ by Lemma 2 and the assumption (6). Hence, by (7), we have

$$\pi_x(B) = \pi_y(B)$$

for $x, y > 0$. This shows the dichotomy (b) and (c). □

We say that a storage process $\{X(t)\}$ is transient if (a) of Theorem 1 holds, null recurrent if (b) holds, positive recurrent if (c) holds and recurrent if (b) or (c) holds, respectively.

Necessary and sufficient condition for positive recurrence in case that $r(x)$ is nondecreasing is known as follows ([3]): there is $c > 0$ such that

$$\int_0^\infty \frac{\nu([x,\infty))}{r(x+c)}dx < 1\,. \tag{10}$$

The following is the restatement of the above theorem of [3].

THEOREM 2. *Assume that r is nondecreasing.*

(a) *In case*

$$\int_0^\infty \nu([x,\infty))dx = \int_0^\infty x\nu(dx) = m < \infty\,,$$

$\{X(t)\}$ *is positive recurrent iff* $\frac{m}{r(\infty)} < 1$.

(b) *In case $m = \infty$, $\{X(t)\}$ is positive recurrent iff*

$$\int_c^\infty \frac{\nu([x,\infty))}{r(x)}dx < \infty \quad \textit{for some } c > 0\,. \tag{11}$$

Now we are interested in finding a criterion for recurrence and transience.

4. Lemmas

Let $\varphi(x) = E_x[e^{-\alpha\tau_a}]$ for $x \geq a \geq 0$ and $\alpha \geq 0$.

REMARK 2. Let $a \geq 0$. Proposition (2.31) in [4] is valid under our assumption on r and ν. By this proposition, only one of the following two cases can occur:

(a) $\varphi(x) > 0$ for all $x > a$ and $\varphi(a) = 1$,

(b) $\varphi(x) = 0$ for all $x \geq a$.

If $a > 0$, then (b) does not occur.

LEMMA 6. *Suppose that $\nu(R_+) < \infty$ or $\inf_{x\geq 1} r(x) > 0$. Let $a \geq 0$. Then $\varphi(x)$ is uniformly continuous on $[a,\infty)$, $\varphi(x)$ is left differentiable on (a,∞), the left derivative $\frac{d^-}{dx}\varphi(x)$ is left continuous on (a,∞) and satisfies*

$$-r(x)\frac{d^-}{dx}\varphi(x) + \int_0^\infty \{\varphi(x+z) - \varphi(x)\}\nu(dz) = \alpha\varphi(x) \tag{12}$$

for $x \in (a,\infty)$ and $\alpha \geq 0$.

PROOF. If $\varphi(a) < 1$, then the assertion is obvious by Remark 2. So we consider the case $\varphi(a) = 1$. First, assume that $\lambda = \nu(R_+) < \infty$. Then $\varphi(a) = 1$ is equivalent to $R(a,x) < \infty$ for some $x > a$ (hence for all $x > a$) and then $\varphi(x)$ satisfies the following equality:

$$\varphi(x) = \exp\{-(\alpha+\lambda)R(a,x)\} + \int_0^\infty \left[\int_a^x \varphi(u+z)r(u)^{-1}\exp\{-(\alpha+\lambda)R(u,x)\}du\right]\nu(dz) \tag{13}$$

for $x \geq a$ and $\alpha \geq 0$. The right hand side of (13) is uniformly continuous on $[a, \infty)$ and left differentiable for $x > a$ and $\alpha \geq 0$ and the left derivative $\frac{d^-}{dx}\varphi(x)$ satisfies (12) for $x > a$ and $\alpha \geq 0$. By the identity (12), $r(x)\frac{d^-}{dx}\varphi(x)$ is continuous for $x \geq a$. Hence, $\frac{d^-}{dx}\varphi(x)$ is left continuous for $x > a$. Next, suppose that $\nu(R_+) = \infty$. Define $A_n(t)$ as in Section 2. Let $\{X_n\}$ be the storage process corresponding to (r, A_n) starting at x and let $\tau_a(n)$ be the hitting time of $\{a\}$ by $\{X_n\}$. As in the proof of Lemma 3, we have $\tau_a(n) \uparrow \tau_a$ as $n \to \infty$ for $x > a$. Since $\varphi(a) = 1$, $\varphi_n(x) = E_x e^{-\alpha\tau_a(n)} \downarrow \varphi(x)$ as $n \to \infty$ for $x \geq a$. Since $\varphi_n(x)$ is nonincreasing in x for each $n \geq 1$, the convergence is uniform on $[a, \infty)$. Noting that $\varphi(x)$ is nonincreasing and nonnegative, we have the uniform continuity of $\varphi(x)$ on $[a, \infty)$. Let

$$\psi(\theta) = \int_0^\infty (e^{-\theta z} - 1)\nu(dz) + r\theta\,,$$

with $r > 0$. Let $\psi^{-1}(\alpha)$ be the unique solution of $\alpha = \psi(\theta)$ for $\alpha > 0$ and define $\psi^{-1}(0) = \lim_{\alpha\downarrow 0}\psi^{-1}(\alpha)$. Let T_x be the hitting time of $\{-x\}$ by $A(t) - rt$. Then, $E[e^{-\alpha T_x}] = \exp(-x\psi^{-1}(\alpha))$. If $a = 0$, then define b arbitrarily in $(0, x)$. If $a > 0$, then set $b = a$. Set $r = \inf_{z\geq b} r(z)$. Then, by the assumption, $r > 0$. Since $x + A(t) - rt \geq X(t)$ for $t \leq \tau_b$,

$$\tau_y \leq T_{x-y} \quad \text{for } x > y \geq b\,.$$

Hence $E_x[e^{-\alpha\tau_y}] \geq E[e^{-\alpha T_{x-y}}]$ for $x > y \geq b$. We have, for $x > y \geq b$,

$$\begin{aligned} 1 - E_x[e^{-\alpha\tau_y}] &\leq 1 - E[e^{-\alpha T_{x-y}}] \\ &= 1 - \exp\{-(x-y)\psi^{-1}(\alpha)\}\,. \end{aligned}$$

By Lemma 1, we have, for $z > 0$ and $x \geq b$,

$$\begin{aligned} |\varphi(x+z) - \varphi(x)| &= |E_{x+z}[e^{-\alpha\tau_x}]E_x[e^{-\alpha\tau_a}] - E_x[e^{-\alpha\tau_a}]| \\ &\leq |E_{x+z}[e^{-\alpha\tau_x}] - 1| \\ &\leq 1 - e^{-z\psi^{-1}(\alpha)} \leq 1 \wedge \psi^{-1}(\alpha)z\,. \end{aligned} \tag{14}$$

This shows that $\varphi(x)$ is Lipschitz continuous and absolutely continuous in $x \geq b$. Since $\psi_n(\theta) := \int_0^\infty (e^{-\theta z} - 1)\nu_n(dz) + r\theta \geq \psi(\theta)$, $\psi_n^{-1}(\alpha) \leq \psi^{-1}(\alpha)$. Hence

$$|\varphi_n(x+z) - \varphi_n(x)| \leq 1 \wedge \psi^{-1}(\alpha)z \tag{15}$$

for $x \geq b$. Since (14) and (15) hold, $\varphi_n(x)$ converges to φ on $[a, \infty)$ and $\nu([1, \infty)) < \infty$, we have, by bounded convergence theorem,

$$\int_0^\infty 1_{[1/n,\infty)}(z)[\varphi_n(x+z) - \varphi_n(x)]\nu(dz) \to \int_0^\infty [\varphi(x+z) - \varphi(x)]\nu(dz)$$

for $x \in [b, \infty)$. Since $b > 0$ is arbitrary in case $a = 0$, $r(x)\frac{d^-}{dx}\varphi_n(x)$ converges to some function $g(x)$ on (a, ∞). Since $\int_0^\infty \{\varphi(x+z) - \varphi(x)\}\nu(dz)$ is continuous by (14) and bounded convergence theorem, g is continuous on (a,∞) and gr^{-1} is left continuous on (a,∞). Since $\left\{r(x)\frac{d^-}{dx}\varphi_n(x)\right\}$ are bounded, $\left\{\frac{d^-}{dx}\varphi_n(x)\right\}$ are bounded on compact sets in (a,∞). Hence

$$\int_c^d \frac{d^-}{dx}\varphi_n(x)dx \to \int_c^d g(x)r(x)^{-1}dx$$

as $n\to\infty$ for $a<c<d$ by bounded convergence theorem. On the other hand,

$$\int_c^d \frac{d^-}{dx}\varphi_n(x)dx = \varphi_n(d)-\varphi_n(c) \to \varphi(d)-\varphi(c)\,.$$

Hence, gr^{-1} is a density of φ. Since gr^{-1} is left continuous, φ is left differentiable and gr^{-1} is the left derivative. □

REMARK 3. Lemma 6 is proved in [4] under the assumption that r is nondecreasing and continuous on $(0,\infty)$. Although our proof is essentially the same as [4] , we gave the proof for completeness.

Let $C([0,\infty])$ be the class of real valued continuous functions f on $[0,\infty)$ having $\lim_{x\to\infty} f(x)$. Let $C_0 = C([0,\infty])\cap\{f : \lim_{x\to\infty} f(x)=0\}$ and let $C_{00}=C_0\cap\{f : f(0)=0\}$. If $\nu(R_+)<\infty$ or r is nondecreasing, then semigroup on $C([0,\infty])$ associated with the storage process is strongly continuous ([14]). We denote its generator by $\mathcal{G}$ and domain of $\mathcal{G}$ by $\mathcal{D}$.

LEMMA 7 ([13]), ([14]). *Let f be a function in C_0 which has a derivative f' such that $rf'\in C_{00}$. If (a) $\nu(0,\infty)<\infty$ or (b) r is nondecreasing and f' is bounded, then f is contained in the domain $\mathcal{D}$ of the generator $\mathcal{G}$. The generator has the following form:*

$$\mathcal{G}f(x) = -r(x)f'(x)+\int_0^\infty \{f(x+y)-f(x)\}\nu(dy)$$

for $x\ge 0$.

LEMMA 8. *Assume that $\nu(R_+)<\infty$ or r is nondecreasing. Let $x_0>0$ and let u be a nonnegative function on $[x_0,\infty)$ such that $r(x)u(x)$ is continuous.*

(a) *If*

$$\int_{x_0}^\infty u(x)dx=\infty \tag{16}$$

and

$$r(x)u(x)\ge\int_0^\infty \nu(y,\infty)u(x+y)dy \quad \text{for all } x\ge x_0\,, \tag{17}$$

then the storage process $\{X(t)\}$ is recurrent.

(b) *If*

$$0<\int_{x_0}^\infty u(x)dx<\infty \tag{18}$$

and

$$r(x)u(x) \le \int_0^\infty \nu(y,\infty)u(x+y)dy \quad \text{for all } x \ge x_0\,, \tag{19}$$

then the storage process $\{X(t)\}$ *is transient and*

$$P_x(\tau_y < \infty) \le \int_x^\infty u(z)dz \Big/ \int_y^\infty u(z)dz\,. \tag{20}$$

Moreover, *if equality holds in* (19), *then equality holds in* (20).

PROOF. Let $x > x_0$. Choose a, b, M so that $x_0 \le a < x < b < M$. Let $U(x) = \int_{x_0}^x u(y)dy$. Under the assumption (a), there is $V \in \mathcal{D}$ such that $U(x) = V(x)$ for $x \in [x_0, M]$ and $v(x) := V'(x) \le u(x)$ for $x > M$ by Lemma 7. Under the assumption (b), $r(x)u(x) \to 0$ as $x \to \infty$ and hence $u(x)$ is bounded on $x > x_0$ if r is nondecreasing. Hence, under the assumption (b), there is $V \in \mathcal{D}$ such that $U(x) - U(\infty) = V(x)$ for $x \ge x_0$ by Lemma 7. We have

$$\begin{aligned}\mathcal{G}V(x) &= -r(x)V'(x) + \int_0^\infty \{V(x+y) - V(x)\}\nu(dy)\\ &= -r(x)v(x) + \int_0^\infty \nu(y,\infty)v(x+y)dy \quad \text{for } x \ge 0\,.\end{aligned}$$

Hence,

$$\mathcal{G}V(x)\begin{cases}\le 0 & \text{for } M \ge x \ge x_0 \text{ under } (a)\,,\\ \ge 0 & \text{for } x \ge x_0 \text{ under } (b)\,.\end{cases} \tag{21}$$

Since, for $0 \le s \le t \wedge \tau_a \wedge \tau_b^*$,

$$V(X(s)) = \begin{cases} U(X(s)) & \text{under (a) and}\\ U(X(s)) - U(\infty) & \text{under (b)}\,,\end{cases}$$

we have, by Dynkin formula,

$$E_x(U(X(t\wedge\tau_a\wedge\tau_b^*))) = U(x) + E_x\left[\int_0^{t\wedge\tau_a\wedge\tau_b^*} \mathcal{G}V(X(s))ds\right].$$

Divide the left hand side of the above equality as follows:

$$E_x(U(X(t\wedge\tau_a\wedge\tau_b^*))) = J_1 + J_2 + J_3$$

where

$$\begin{aligned}J_1 &= E_x(U(X(t)); t < \tau_a \wedge \tau_b^*)\,,\\ J_2 &= E_x(U(X(\tau_a)); \tau_a < \tau_b^*, \tau_a \le t)\,,\\ J_3 &= E_x(U(X(\tau_b^*)); \tau_b^* \le \tau_a, \tau_b^* \le t)\,.\end{aligned}$$

By $P_x(\tau_b^* < \infty) = 1$ and $X(\tau_a) = a$,

$$J_2 \to U(a)P_x(\tau_a < \tau_b^*) \quad \text{as} \quad t \to \infty . \tag{22}$$

We have

$$J_3 \to E_x(U(X(\tau_b^*)); \tau_b^* \le \tau_a) \tag{23}$$

as $t \to \infty$. Assume (a). By $X(\tau_b^*) \ge b$ and the nondecreasingness of U

$$E_x(U(X(\tau_b^*)); \tau_b^* \le \tau_a) \ge U(b)P_x(\tau_b^* \le \tau_a) . \tag{24}$$

By (21), we have

$$J_2 + J_3 \le U(x) . \tag{25}$$

Hence, by (22)–(25),

$$\{U(b) - U(a)\}P_x(\tau_b^* \le \tau_a) \le U(x) - U(a) . \tag{26}$$

By letting $b \to \infty$, we have

$$P_x(\tau_b^* \le \tau_a) \to P_x(\tau_a = \infty) ,$$
$$U(b) - U(a) \to \infty$$

by $P_x(\lim_{b\to\infty} \tau_b^* = \infty) = 1$ (Lemma 2) and the assumption. Hence

$$P_x(\tau_a = \infty) \le \frac{U(x) - U(a)}{U(\infty) - U(a)} = 0$$

and $\{X(t)\}$ is recurrent. Now assume (b). By (21), we have

$$J_1 + J_2 + J_3 \ge U(x) . \tag{27}$$

Since $P_x(\lim_{b\to\infty} \tau_b^* = \infty) = 1$, $X(\tau_b^*) \ge b$, $U(\infty) < \infty$ and U is nondecreasing,

$$\lim_{b\to\infty} \lim_{t\to\infty} J_3 = U(\infty)P_x(\tau_a = \infty) \tag{28}$$

and

$$J_1 \le U(\infty)P(t < \tau_a \wedge \tau_b^*) \to 0 \quad as \quad t \to \infty . \tag{29}$$

Hence, by (22), (27)–(29),

$$U(a)P_x(\tau_a < \tau_b^*) + U(\infty)P_x(\tau_a = \infty) \ge U(x) .$$

That is,

$$U(\infty)P_x(\tau_a = \infty) - U(a)P_x(\tau_b^* \le \tau_a) \ge U(x) - U(a) .$$

Letting $b \to \infty$, we have

$$P_x(\tau_b^* \le \tau_a) \to P_x(\tau_a = \infty) \ge \frac{U(x) - U(a)}{U(\infty) - U(a)} > 0 \tag{30}$$

by $P_x(\lim_{b\to\infty} \tau_b^* = \infty) = 1$. Hence $P_x(\tau_a < \infty) < 1$ and the process $\{X(t)\}$ is transient. If equality holds in (19), then equality holds in (27). This yields the equality in (30). $\square$

5. Recurrence-transience criteria

THEOREM 3. *Assume that the release rate r is continuous and nondecreasing. If*

$$r(x) \geq \int_0^\infty \nu(y,\infty)\frac{x}{x+y}dy$$

for all large x, then $\{X(t)\}$ is recurrent.

PROOF. Let $u(x) = x^{-1}$. Then $u(x)$ satisfies the assumptions in Lemma 8 (a). Hence $\{X(t)\}$ is recurrent.

THEOREM 4. *Assume that* $\inf_{x\geq 1} r(x) > 0$. *If there is $c > 0$ such that*

$$\int_c^x \frac{\nu((x-y,\infty))}{r(y)}dy \leq 1 \tag{31}$$

for all $x \geq c$, then $\{X(t)\}$ is recurrent.

PROOF. By Lemma 6, $u(x) = -\frac{\partial}{\partial x}P_x(\tau_c < \infty) \geq 0$ satisfies

$$r(x)u(x) = \int_x^\infty \nu((z-x,\infty))u(z)dz \tag{32}$$

for $x > c$. If $\int_c^\infty u(x)dx = \infty$, then, by Lemma 7, $X(t)$ is recurrent. Assume that $\int_c^\infty u(x)dx < \infty$. Divide the both sides of (32) by $r(x)$ and then integrate the both sides. We have

$$\int_c^\infty \left(\int_c^z \frac{\nu((z-y,\infty))}{r(y)}dy - 1\right)u(z)dz = 0\,.$$

The left hand side of the above equality is non-positive by the assumption (31). Since $\int_0^1 \nu(z,\infty)dz < \infty$, $\int_c^z \frac{\nu((z-y,\infty))}{r(y)}dy \to 0$ as $z \downarrow c$. Hence $u(x) = 0$ for $x > c$ sufficiently close to c. Since $P_x(\tau_c < \infty)$ tends to 1 as $x \downarrow c$, $P_x(\tau_c < \infty) = 1$ for all $x > c$ sufficiently close to c. By Theorem 1, the process is recurrent. □.

THEOREM 5. *Assume that the release rate r is continuous and nondecreasing. If there is $\gamma > 1$ such that*

$$r(x) \leq \int_0^\infty \nu(y,\infty)\left(\frac{x}{x+y}\right)^\gamma dy$$

for all large x, then $\{X(t)\}$ is transient.

PROOF. Let $u(x) = x^{-\gamma}$. Then $u(x)$ satisfies the assumptions in Lemma 8 (b). Hence $\{X(t)\}$ is transient.

THEOREM 6. *Assume that* $\nu(0,\infty) < \infty$ *or* r *is nondecreasing. If, for some* $c > 0$,

$$\int_c^\infty \frac{\nu((x,\infty))}{r(x)}dx = \infty \tag{33}$$

and

$$\int_c^\infty \frac{1}{r(x)} \exp\{-\int_c^x \frac{\nu((y,\infty))}{r(y)}dy\}dx < \infty\,, \tag{34}$$

then $\{X(t)\}$ *is transient.*

PROOF. Choose n so that $c > \frac{1}{n}$ and let $\nu_n = \nu|_{(1/n,\infty)}$. Then $\nu_n((x,\infty)) = \nu((x,\infty))$ for $x > c$. Put $u(x) = \frac{1}{r(x)}\exp\{-\int_c^x \frac{\nu((z,\infty))}{r(z)}dz\}$. By the assumption (34), $0 < \int_0^\infty u(x)dx < \infty$. Obviously, $r(x)u(x)$ is continuous. We have

$$\begin{aligned}
\int_x^\infty \nu_n((y-x,\infty))u(y)dy &= \int_x^\infty \frac{\nu_n((y-x,\infty))}{r(y)} \exp\left\{-\int_c^y \frac{\nu((z,\infty))}{r(z)}dz\right\}dy \\
&\geq \int_x^\infty \frac{\nu_n((y,\infty))}{r(y)} \exp\left\{-\int_c^y \frac{\nu((z,\infty))}{r(z)}dz\right\}dy \\
&= \int_x^\infty \frac{\nu((y,\infty))}{r(y)} \exp\left\{-\int_c^y \frac{\nu((z,\infty))}{r(z)}dz\right\}dy \\
&= -\left[\exp\{-\int_c^y \frac{\nu((z,\infty))}{r(z)}dz\}\right]_{y=x}^\infty \\
&= \exp\left\{-\int_c^x \frac{\nu((z,\infty))}{r(z)}dz\right\} = r(x)u(x)
\end{aligned}$$

for $x > c$ by (33). Hence u satisfies the assumptions of Lemma 8 (b) for r and ν_n. Define $A_n(t)$ as in Section 2 and let $\{X_n(t)\}$ be the storage process corresponding to (r, A_n). Then $\{X_n(t)\}$ is transient by Lemma 8 (b). Since $X_n(t) \leq X(t)$, $\{X(t)\}$ is also transient. □

THEOREM 7. *Assume* (2). *Then* (34) *is a necessary and sufficient condition for transience of* $\{X(t)\}$ (*without* (33)) *of* $\{X(t)\}$.

REMARK 4. If r is of the form (2), the necessary and sufficient condition for transience of $\{X(t)\}$ is the following (Shiga [9]).

$$\int_0^1 z^{-1} \exp\left\{-\int_z^1 \frac{\tilde{\nu}(y)}{r(y)}dy\right\}dz < \infty\,, \tag{35}$$

where $\tilde{\nu}(y) = \int_1^\infty (1-e^{-yx})\nu(dx)$. The condition (34) is simpler than (35). Theorem 7 can be extended to higher dimension with $r(x) = Ix$ and general Lévy processes $\{A(t)\}$, where $x \in R^d$ ($d \geq 1$) and I is the identity matrix.

PROOF OF THEOREM 7. We show that (35) is equivalent to (34). Let $\tilde{\nu}$ be the quantity defined in Remark 4. Integration by parts yields that for $0 < z \le 1$,

$$\begin{aligned}\int_z^1 \frac{\tilde{\nu}(y)}{r(y)}dy &= \int_z^1 \left(\int_1^\infty \frac{1-e^{-yx}}{ay}\nu(dx)\right)dy \\ &= \nu((1,\infty))\int_z^1 \frac{1-e^{-y}}{ay}dy + \int_1^\infty \frac{\nu((x,\infty))}{ax}(e^{-zx}-e^{-x})dx \\ &= O(1) + \int_0^\infty G(x)e^{-zx}dx\,,\end{aligned}$$

where

$$G(x) = \begin{cases} 0 & \text{for } 0 \le x < 1\,, \\ \nu((x,\infty))/ax & \text{for } x \ge 1\,. \end{cases}$$

We have that

$$\int_0^1 z^{-1}\exp\left\{-\int_z^1 \frac{\tilde{\nu}(y)}{r(y)}dy\right\}dz < \infty$$

is equivalent to

$$\int_1^\infty z^{-1}\exp\left\{-\int_0^\infty G(x)e^{-x/z}dx\right\}dz < \infty\,.$$

Integration by parts yields

$$\begin{aligned}\int_0^\infty G(x)e^{-x/z}dx &= \left[\left\{\int_0^x G(u)du\right\}e^{-x/z}\right]_{x=0}^\infty + z^{-1}\int_0^\infty \left\{\int_0^x G(u)du\right\}e^{-x/z}dx \\ &= \int_0^\infty \left\{\int_0^{zv} G(u)du\right\}e^{-v}dv \quad \textit{for } z > 0\,.\end{aligned}$$

Since $|\int_{zv}^z G(u)du| \le \frac{1}{a}|\log v|\nu((z\wedge zv)\vee 1,\infty)$ for $z > 1$, we have

$$\begin{aligned}\left|\int_0^z G(x)dx - \int_0^\infty G(x)e^{-x/z}dx\right| &= \left|\int_0^z G(u)du - \int_0^\infty \left\{\int_0^{zv} G(u)du\right\}e^{-v}dv\right| \\ &\le \int_0^\infty \left|\int_{zv}^z G(u)du\right|e^{-v}dv \\ &\to 0 \quad \text{as } z \to \infty\,.\end{aligned}$$

Hence

$$\int_0^1 z^{-1}\exp\left\{-\int_z^1 \frac{\tilde{\nu}(y)}{r(y)}dy\right\}dz < \infty$$

is equivalent to

$$\int_1^\infty z^{-1} \exp\left\{ - \int_1^z G(x)dx \right\} dz < \infty . \qquad \square$$

6. Applications

6.1. Stable inputs and power function release rates. Let us consider the case $\nu((x,\infty)) = x^{-\alpha}$ and $r(x) = ax^\beta$ for $0 < \alpha, \beta < 1$. Set $a_\alpha = \frac{\pi}{\sin \pi\alpha} = \Gamma(\alpha)\Gamma(1-\alpha)$.

THEOREM 8. (a) *If $\alpha + \beta > 1$ then $\{X(t)\}$ is positive recurrent.*
(b) *If $\alpha + \beta = 1$ and $a \geq a_\alpha$, then $\{X(t)\}$ is null recurrent.*
(c) *If $\alpha + \beta = 1$ and $a < a_\alpha$, then $\{X(t)\}$ is transient.*
(d) *If $\alpha + \beta < 1$, then $\{X(t)\}$ is transient.*

PROOF. (a) is proved by Theorem 2. (b): By Theorem 2, $\{X(t)\}$ is not positive recurrent. By Theorem 3 or Theorem 4, $\{X(t)\}$ is recurrent. (d) is proved by Theorem 5 or Theorem 6. We prove (c). We have

$$\begin{aligned}\int_0^\infty \nu((y,\infty))\left(\frac{x}{x+y}\right)^\gamma dy &= \int_0^\infty y^{-\alpha}\left(\frac{x}{x+y}\right)^\gamma dy \\ &= x^{1-\alpha}\int_0^\infty \frac{1}{(1+y)^\gamma y^\alpha} dy \\ &= \frac{\Gamma(1-\alpha)\Gamma(\gamma-1+\alpha)}{\Gamma(\gamma)} x^{1-\alpha}\end{aligned} \tag{36}$$

for $\gamma > 0$. The integral $\int_0^\infty (1+y)^{-\gamma} y^{-\alpha} dy$ is decreasing in $\gamma > 0$ and is equal to a_α at $\gamma = 1$. If $a_\alpha > a$, then the assumption of Theorem 5 is satisfied with $\gamma > 1$ sufficiently close to 1. Hence if $a_\alpha > a$, then $\{X(t)\}$ is transient by Theorem 5. $\square$

This theorem completes the classification of recurrence-transience in this stable input and power function release rate case. This classification is slightly different from the classification of existence-non existence of a continuous local time at the origin obtained by M. Takano [10] and Zakushilo [15]. We illustrate the situation:

$$\begin{array}{llll}\alpha+\beta<1 & \Rightarrow\ 0<P_x(\tau_0<\infty)<1\,, & 0<P_x(\tau_y<\infty)<1\,, & \\ \alpha+\beta=1, a>a_\alpha & \Rightarrow\ P_x(\tau_0<\infty)=1\,, & P_x(\tau_y<\infty)=1\,, & E_x(\tau_y)=\infty\,, \\ \alpha+\beta=1, a=a_\alpha & \Rightarrow\ P_x(\tau_0<\infty)=0\,, & P_x(\tau_y<\infty)=1\,, & E_x(\tau_y)=\infty\,, \\ \alpha+\beta=1, a<a_\alpha & \Rightarrow\ P_x(\tau_0<\infty)=0\,, & 0<P_x(\tau_y<\infty)<1\,, & \\ \alpha+\beta>1 & \Rightarrow\ P_x(\tau_0<\infty)=0\,, & P_x(\tau_y<\infty)=1\,, & E_x(\tau_y)<\infty\,.\end{array}$$

Here, $0 < y < x$ are arbitrary.

In the first or the second case, 0 is regular for itself ([10], [15]). Hence there exists a continuous local time at 0. In the second case, $P_x(\tau_0 < \infty) = 1$ is not quoted in any literature. This fact can be proved in the following way: Assume that $P_x(\tau_0 < \infty) < 1$ for some $x > 0$. Then, since $P_x(\tau_y < \infty) = 1$ for $0 < y < x$, $P_x(\tau_0 < \infty) = c < 1$ independent of x. Since $P_x(\tau_0 < \infty)$ is uniformly continuous in $x \in [0, \infty)$ by Lemma 6, we have

$$P_0(\tau_0 < \infty) = c < 1\,.$$

This contradicts to the regularity of 0 for itself, i.e. $P_0(\tau_0 < \infty) = 1$. Hence $P_x(\tau_0 < \infty) = 1$.

In case $\alpha + \beta = 1$ and $a < a_\alpha$, we have

$$P_x(\tau_y < \infty) = \left(\frac{y}{x}\right)^{\gamma-1} \quad \text{for} \quad x > y > 0$$

with $\gamma > 1$ satisfying $a = \frac{\Gamma(1-\alpha)\Gamma(\gamma-1+\alpha)}{\Gamma(\gamma)}$ by Lemma 8.

REMARK 5. Note that if $r(x) = ax^{1-\alpha}$ $(a \geq \frac{1}{\alpha})$ and $\nu((x,\infty)) = x^{-\alpha}$ $(0 < \alpha < 1)$, then $a_\alpha > \frac{1}{\alpha}$ and $\int_1^\infty \frac{1}{r(x)} \exp\{-\int_1^x \frac{\nu((y,\infty))}{r(y)} dy\} dx = \infty$ hold. However, if $a_\alpha > a \geq \frac{1}{\alpha}$, then by Theorem 8, the process is transient. Hence the sufficient condition (33) in Theorem 6 for transience is not a necessary condition while it is a necessary and sufficient condition for the processes of Ornstein-Uhlenbeck type (Theorem 7).

Noting that $\int_0^\infty \frac{\nu((y,\infty))}{(x+y)^\gamma} dy$ is the Stieltjes transform of $\nu((y,\infty))dy$ of order γ, we can extend the above Theorem 8 as follows.

THEOREM 9. *Assume that*

$$\nu((x,\infty)) = x^{-\alpha} L(x)$$

and

$$r(x) \sim ax^\beta$$

as $x \to \infty$, *where* $a > 0$, $0 < \alpha, \beta < 1$ *and* $L(x)$ *is a function slowly varying at infinity. Then the following hold*:

(a) *If either* $\alpha + \beta > 1$ *or* $\alpha + \beta = 1$ *and* $\int^\infty \frac{L(x)}{x} dx < \infty$, *then* $\{X(t)\}$ *is positive recurrent.*

(b) *If* $\alpha + \beta = 1$ *and there is* $\varepsilon > 0$ *such that*

$$\frac{(1-\varepsilon)a}{\Gamma(\alpha)\Gamma(1-\alpha)} > L(x)$$

for all large x, *then* $\{X(t)\}$ *is recurrent.*

(c) *If* $\alpha + \beta = 1$ *and there is* $\varepsilon > 0$ *such that*

$$\frac{(1+\varepsilon)a}{\Gamma(\alpha)\Gamma(1-\alpha)} < L(x)$$

for all large x, *then* $\{X(t)\}$ *is transient.*

(d) *If* $\alpha+\beta<1$, *then* $\{X(t)\}$ *is transient.*

PROOF. Let

$$S_\gamma(x)=\int_0^\infty \nu((y,\infty))(x+y)^{-\gamma}dy$$

be the Stieltjes transform of $\int_0^x \nu((y,\infty))dy$ of order $\gamma \quad (\gamma>0)$. Since $\nu((y,\infty))$ is non-increasing and $\nu((y,\infty))=y^{-\alpha}L(y)$,

$$S_\gamma(x)\sim\frac{\Gamma(\gamma-1+\alpha)\Gamma(1-\alpha)}{\Gamma(\gamma)}x^{1-\gamma-\alpha}L(x)$$

by Theorem 1.7.4 in [2]. Hence

$$\frac{x^\gamma}{r(x)}S_\gamma(x)\sim\frac{\Gamma(\gamma-1+\alpha)\Gamma(1-\alpha)}{\Gamma(\gamma)}\frac{x^{1-(\alpha+\beta)}}{a}L(x)$$

as $x\to\infty$. If (b) holds, then letting $\gamma=1$,

$$\frac{x}{r(x)}S_1(x)\le 1 \quad \text{for large } x\,.$$

By Theorem 3, $\{X(t)\}$ is recurrent. We see that the process is positive recurrent if either $\alpha+\beta>1$ or $\alpha+\beta=1$ and $\int^\infty \frac{L(x)}{x}dx<\infty$ by Theorem 2 (b). If (d) holds, then letting $\gamma>1$, we have

$$\frac{x^\gamma}{r(x)}S_\gamma(x)\ge 1 \quad \text{for large } x\,.$$

If (c) holds, then letting $\gamma>1$ close to 1,

$$1+\frac{\varepsilon}{2}\le\frac{\Gamma(\gamma-1+\alpha)\Gamma(1-\alpha)}{\Gamma(\gamma)}L(x)$$

for large x. Hence $\{X(t)\}$ is transient in cases (c) and (d) by Theorem 5. □

REMARK 6. We remark that if we assume $\int^\infty \frac{L(x)}{x}dx=\infty$ in addition to the assumption of Theorem 9 (b), then $\{X(t)\}$ is null recurrent.

6.2. Case $r(\infty)<\infty$**.** Next, we consider the case $r(\infty)=\lim_{x\to\infty}r(x)<\infty$. In case that there is $x_0>0$ such that $r(x)$ is identically equal to r_0 for $x>x_0$, it is known that

if $\int_0^\infty \nu((x,\infty))dx<r_0$, then $\{X(t)\}$ is positive recurrent,
if $\int_0^\infty \nu((x,\infty))dx=r_0$, then $\{X(t)\}$ is null recurrent and
if $\int_0^\infty \nu((x,\infty))dx>r_0$, then $\{X(t)\}$ is transient

In the case r is nondecreasing and $r(x)\ne r(\infty)$ $(0<x<\infty)$, the same conclusions hold if

$$\int_0^\infty \nu((x,\infty))dx\ne r(\infty)\,.$$

However, in the case that r is nondecreasing and $\int_0^\infty \nu((x,\infty))dx = r(\infty)$, it has been unknown whether the process is recurrent or not. We give sufficient conditions for transience and recurrence, respectively, under the assumption that r is nondecreasing.

THEOREM 10. *Assume that r is nondecreasing,* $r(\infty) = \int_0^\infty \nu((y,\infty))dy < \infty$ *and*

$$r(\infty) - r(x) = Cx^{-\beta}(1 + o(1)) \quad as \quad x \to \infty\,,$$

where C is a positive constant. Then the following holds:

(a) *If* $\beta > 1$, *then* $\{X(t)\}$ *is null recurrent.*

(b) *If* $\beta = 1$ *and* $\int_0^\infty y\nu((y,\infty))dy > C$ *then* $\{X(t)\}$ *is null recurrent.*

(c) *If* $\beta = 1$ *and* $\int_0^\infty y\nu((y,\infty))dy < C$, *then* $\{X(t)\}$ *is transient.*

(d) *If* $0 < \beta < 1$ *and* $\int_0^\infty y^\beta \nu((y,\infty))dy < \infty$, *then* $\{X(t)\}$ *is transient.*

PROOF. Note that in all cases, the process is not positive recurrent by Theorem 2 (a). We have

$$\begin{aligned}\int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy &= r(x) + (r(\infty)-r(x)) + \left\{\int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy - r(\infty)\right\}\\ &= r(x) + x^{-\beta}\left\{C - \int_0^\infty \nu((y,\infty))\frac{x^\beta y}{x+y}dy + o(1)\right\}.\end{aligned} \tag{37}$$

Case (a). If $\beta > 1$, then we have by Fatou's lemma,

$$\liminf_{x\to\infty}\int_0^\infty \nu((y,\infty))\frac{x^\beta y}{x+y}dy \geq \int_0^\infty \nu((y,\infty))\liminf_{x\to\infty}\frac{x^\beta y}{x+y}dy = \infty\,.$$

Hence we have, for every large enough x,

$$r(x) \geq \int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy\,.$$

By Theorem 3, $\{X(t)\}$ is recurrent.

Cases (b) and (c). Let $\beta = 1$. Since $\frac{xy}{x+y}$ is increasing in $x > 0$ and $0 \leq \frac{xy}{x+y} \leq y$, we have

$$\lim_{x\to\infty}\int_0^\infty \nu((y,\infty))\frac{xy}{x+y}dy = \int_0^\infty y\nu((y,\infty))dy \leq \infty\,.$$

Hence if $\int_0^\infty y\nu((y,\infty))dy > C$ (resp. $< C$), then

$$C - \int_0^\infty \nu((y,\infty))\frac{xy}{x+y}dy < 0 \quad (\text{resp.} > 0)$$

for all large x. Hence by Theorem 3 (resp. 5), $\{X(t)\}$ is recurrent (resp. transient).

Case (d). Let

$$g(t) = t^{-\beta}\{1-(1+t)^{-\gamma}\}$$

for $0 < \beta < 1$ and $\gamma > 0$. Then $g(t)$ is bounded on $[0, \infty)$ and

$$g(t) \to 0 \tag{38}$$

as $t \to 0$. As (37), we have

$$\begin{aligned}\int_0^\infty \nu((y,\infty))\left(\frac{x}{x+y}\right)^\gamma dy \\ = r(x) + x^{-\beta}\left[C - \int_0^\infty y^\beta \nu((y,\infty))\left(\frac{y}{x}\right)^{-\beta}\left\{1 - \left(1+\frac{y}{x}\right)^{-\gamma}\right\}dy + o(1)\right]\end{aligned}$$

By (38) and the dominated convergence theorem, we have

$$\int_0^\infty y^\beta \nu((y,\infty))\left(\frac{y}{x}\right)^{-\beta}\left\{1 - \left(1+\frac{y}{x}\right)^{-\gamma}\right\}dy \to 0$$

as $x \to \infty$. We have

$$r(x) \le \int_0^\infty \nu((y,\infty))\left(\frac{x}{x+y}\right)^\gamma dy$$

for all large x with arbitrary $\gamma > 1$. Hence $\{X(t)\}$ is transient by Theorem 5 in case (d). □

Theorem 10 asserts an interesting but curious phenomenon. Let us consider the simplest case

$$\nu(dx) = b\delta_a(dx)$$

with $a, b > 0$ and

$$r(x) = ab - Cx^{-1}(1 + o(1)) \quad \text{as } x \to \infty .$$

Then

$$\int_0^\infty \nu((y,\infty))dy = ab \quad \text{and} \quad \int_0^\infty y\nu((y,\infty))dy = \frac{a^2 b}{2} .$$

By Theorem 10, $\{X(t)\}$ is recurrent if

$$\frac{a^2 b}{2} > C$$

and transient if

$$\frac{a^2 b}{2} < C .$$

Fix ab and C. Then the above fact shows that $\{X(t)\}$ is recurrent if the jump size a is big and the common mean time interval $1/b$ between successive jumps is big.

Yamada [12] showed that if r is nondecreasing, $\nu((0,\infty))$, $\int_0^\infty \nu((y,\infty))dy = r(\infty) > 0$ and

$$r(\infty) - r(x) \sim \frac{C}{x} \quad \text{as } x \to \infty$$

with $C > 0$, then

$$\left\{ \frac{X(nt)}{v\sqrt{n}} \right\}$$

converges in law on $D([0,T] \to \mathbf{R})$ to Bessel process with index $d = 1 + C/v^2$ where $v^2 = \int_0^\infty y\nu((y,\infty))dy > 0$ and $T > 0$ is arbitrary. Bessel process with index d is recurrent if and only if $0 < d \leq 2$. Our results (b) and (c) of Theorem 10 correspond to this fact for Bessel process except the case $d = 2$. In the case $d = 2$, it is unknown whether the storage process is recurrent or transient. However, (b) of Theorem 10 can be extended to the following:

THEOREM 11. *Assume that r is nondecreasing and*

$$r(x) = \sum_{n=0}^{m} C_n x^{-n} + o(x^{-m}) \quad as \ x \to \infty .$$

If

$$(-1)^n \int_0^\infty y^n \nu((y,\infty))dy = C_n , \quad n = 0, 1, \cdots, m-1$$

and

$$(-1)^m \int_0^\infty y^m \nu((y,\infty))dy < C_m .$$

Then $\{X(t)\}$ is null recurrent.

PROOF. We have

$$\begin{aligned}
\int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy &= r(x) - \sum_{n=0}^{m} C_n x^{-n} - o(x^{-m}) + \int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy \\
&= r(x) - \sum_{n=0}^{m} C_n x^{-n} - o(x^{-m}) \\
&\quad + \int_0^\infty \nu((y,\infty))\left\{ \sum_{n=0}^{m-1} (-1)^n \left(\frac{y}{x}\right)^n + (-1)^m \frac{x}{x+y}\left(\frac{y}{x}\right)^m \right\} dy \\
&= r(x) - x^{-m}\left\{ C_m - (-1)^m \int_0^\infty y^m \frac{x}{x+y}\nu((y,\infty))dy + o(1) \right\} .
\end{aligned}$$

Hence

$$r(x) \geq \int_0^\infty \nu((y,\infty))\frac{x}{x+y}dy$$

for large x. By Theorem 3, $\{X(t)\}$ is recurrent. □

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