TOKYO J. MATH.
VOL. 31, NO. 2, 2008

On Upper Bound Graphs with Edge Operations

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(Communicated by M. Ohta)

Abstract. In the upper bound graph of a poset P with the vertex set $V(P)$, xy is an edge if there exists an $m \in V(P)$ such that $x, y \leq_P m$. We obtain some properties of edge operations of UB-graphs. According to these properties, we consider transformations on split UB-graphs.

1. Introduction

In this paper, we deal with graphs on finite posets. For a poset $P = (X, \leq_P)$ and $x \in X$, $L_P(x) = \{u \in X; u \leq_P x\}$ and $U_P(x) = \{u \in X; x \leq_P u\}$. For a poset $P = (X, \leq_P)$ and $S \subseteq X$, $L_P(S) = \bigcup_{x \in S} L_P(x)$ and $U_P(S) = \bigcup_{x \in S} U_P(x)$. $\mathrm{Max}(P)$ is the set of maximal elements of a poset P and $\mathrm{Min}(P)$ is the set of minimal elements of a poset P.

For a poset $P = (X, \leq_P)$, the *upper bound graph* (UB-graph) of P is the graph $\mathrm{UB}(P) = (X, E_{\mathrm{UB}(P)})$, where $xy \in E_{\mathrm{UB}(P)}$ if and only if $x \neq y$ and there exists $m \in X$ such that $x, y \leq_P m$. McMorris and Zaslavsky [6] introduced this concept and gave a characterization of upper bound graphs. A *clique* in a graph G is the vertex set of a maximal complete subgraph of G. In some cases, we consider that a clique is a maximal complete subgraph. In the same way, we occasionally abuse terms of induced subgraphs and the vertex set of induced subgraphs, especially for complete subgraphs. A family $\mathcal{E}$ of complete subgraphs *edge covers* G if and only if for each edge $uv \in E(G)$, there exists $Q \in \mathcal{E}$ such that $u, v \in Q$.

THEOREM 1 (McMorris and Zaslavsky [6]). *A graph G is a UB-graph if and only if there exists a family $\mathcal{E} = \{Q_1, Q_2, \dots, Q_n\}$ of complete subgraphs of G such that*

(1) *$\mathcal{E}$ edge covers G, and*

(2) *for each Q_i, there is a vertex $v_i \in Q_i - (\bigcup_{j \neq i} Q_j)$.*

Furthermore, such a family $\mathcal{E}$ must consist of cliques of G and is the only such family if G has no isolated vertices.

In the proof of this result, McMorris and Zaslavsky use a fact that principal order ideals of a poset correspond to complete subgraphs of a UB-graph. For a UB-graph G, an edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \dots, Q_n\}$ satisfying the conditions of Theorem 1 is called a *UB edge*

Received July 13, 2007
Key words: Upper bound graphs, edge operations

clique cover. For a UB-graph G and its UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_n\}$, a *kernel* of G is a vertex subset $\{v_1, v_2, \ldots, v_n\}$ such that $v_i \in Q_i - (\bigcup_{j \neq i} Q_j)$ for each $i = 1, 2, \ldots, n$. A kernel is not uniquely defined for a UB-graph. However, we know a fact that for a UB-graph G and a kernel K, there exists a poset P with $UB(P) = G$ and $\mathrm{Max}(P) = K$.

For a graph G and $v \in V(G)$, $N_G(v) = \{u \,;\, uv \in E(G)\}$ and $N_G[v] = N_G(v) \cup \{u\}$. Furthermore, $v \in V(G)$ is called a *simplicial vertex* if $N_G(v)$ is the vertex set of a complete subgraph of G.

Lundgren and Maybee [5] obtained another characterization of upper bound graphs in terms of ordered edge cover. Using their results, Bergstrand and Jones [1] obtained some properties on transformations of upper bound graphs. Iwai et al. [3] and [4] considered construction methods of upper bound graphs and double bound graphs. In these papers, they use contractions of edges and splits of vertices for constructions of bound graphs.

Ogawa et al. [2], [7], [8] and [9] considered properties of transformations between posets with the same UB-graph. In these papers, they use operations on posets, that is, $x < y$-additions and $x < y$-deletions. In this paper, we deal with properties of transformations between UB-graphs in terms of edge operations.

2. Main results

In this section, we deal with properties of edge operations on UB-graphs. First we consider edge additions on UB-graphs. For a graph G and non adjacent vertices u and v, $G + uv$ is a graph where $V(G + uv) = V(G)$ and $E(G + uv) = E(G) \cup \{uv\}$.

PROPOSITION 2. *Let G be a connected UB-graph with non adjacent vertices u and v. If u and v are simplicial vertices of G and $N_G(u) = N_G(v)$, then $G + uv$ is a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_k\}$ of G. Then $N_G[u]$ and $N_G[v]$ are cliques of G, $N_G[u]$, $N_G[v] \in \mathcal{E}(G)$ and $N_G[u] \cup N_G[v]$ is a clique of $G + uv$. Let

$$\mathcal{E}(G + uv) = (\mathcal{E}(G) - \{N_G[u], N_G[v]\}) \cup \{N_G[u] \cup N_G[v]\}.$$

Then $\mathcal{E}(G + uv)$ is an edge clique cover of $G + uv$. A simplicial vertex in each $Q_i \in \mathcal{E}(G) - \{N_G[u], N_G[v]\}$ is also a simplicial vertex in $G + uv$. Moreover, u is a simplicial vertex of $G + uv$ and only belongs to $N_G[u] \cup N_G[v]$. Therefore, $\mathcal{E}(G + uv)$ is a UB edge clique cover of $G + uv$ and $G + uv$ is a UB-graph. □

PROPOSITION 3. *Let G be a connected UB-graph with non adjacent vertices u and v. If u and v are simplicial vertices of G, $N_G(u) \supset N_G(v)$ and there exists a simplicial vertex $w (\neq u, v)$ of G in $N_G(u) - N_G(v)$, then $G + uv$ is a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_k\}$ of G. Then $N_G[v]$, $N_G[u] \in \mathcal{E}(G)$ and $N_G[v] \cup \{u\}$ is a clique of $G + uv$.

Let

$$\mathcal{E}(G+uv) = (\mathcal{E}(G) - \{N_G[v]\}) \cup \{N_G[v] \cup \{u\}\}\,.$$

For each $Q_i \in \mathcal{E}(G) - \{N_G[u], N_G[v]\}$, Q_i is a clique of $G+uv$ and a simplicial vertex of Q_i is also a simplicial vertex of $G+uv$. Moreover, v is a simplicial vertex of $G+uv$ and does not belong to any cliques of $\mathcal{E}(G+uv)$ other than $N_G[v] \cup \{u\}$. Since $w \in N_G(u) - N_G(v)$ is a simplicial vertex of G, w is also a simplicial vertex of $G+uv$ and $N_G[u]$ is also a clique of $G+uv$ with a simplicial vertex w. Thus $\mathcal{E}(G+uv)$ is a UB edge clique cover of $G+uv$ and $G+uv$ is a UB-graph. □

PROPOSITION 4. *Let G be a connected UB-graph with non adjacent vertices u and v. If u is not a simplicial vertex of G, v is a simplicial vertex of G and $N_G(u) \supset N_G(v)$, then $G+uv$ is a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, \ldots, Q_k\}$ of G. Then $N_G[v]$ is a clique of G and $N_G[v] \cup \{u\}$ is a clique of $G+uv$. v is also a simplicial vertex of $G+uv$. Let

$$\mathcal{E}(G+uv) = (\mathcal{E}(G) - \{N_G[v]\}) \cup \{N_G[v] \cup \{u\}\}.$$

Then $\mathcal{E}(G+uv)$ is a UB edge clique cover of $G+uv$ and $G+uv$ is a UB-graph. □

PROPOSITION 5. *Let G be a connected UB-graph with non adjacent vertices u and v. If u and v are not simplicial vertices of G, then $G+uv$ is not a UB-graph.*

PROOF. We assume that $G+uv$ is UB-graph. Then there exists a kernel K^* of $G+uv$ and a simplicial vertex $m \in K^*$ which is adjacent to u and v. Since u and v are non adjacent vertices in G, m is not a simplicial vertex of G. Therefore, there exists a simplicial vertices u' and v' of G such that u' is adjacent to u, m and v' is adjacent to v, m. Since m is a simplicial vertex of $G+uv$, $u'v' \in E(G+uv)$ and $u'v' \in E(G)$. Since u' is a simplicial vertex of G, $uv' \in E(G)$. Thus v' is adjacent to non adjacent vertices u, v in G, which is a contradiction. □

PROPOSITION 6. *Let G be a connected UB-graph with non adjacent vertices u and v, where u is not a simplicial vertex of G and v is a simplicial vertex of G. If $N_G(u) \not\supset N_G(v)$, then $G+uv$ is not a UB-graph.*

PROOF. Since there exists a vertex $w \in N_G(v) - N_G(u)$, $N_G(v) \cup \{u\}$ is not a complete subgraph in both G and $G+uv$. Thus v is not a simplicial vertex of $G+uv$ and each vertex of $N_G(v) \cap N_G(u)$ is not a simplicial vertex of $G+uv$. Since vertices adjacent to u and v are only those in $N_G(v) \cap N_G(u)$, there exists no simplicial vertices of $G+uv$ for a clique containing an edge uv. Thus $G+uv$ is not a UB-graph. □

PROPOSITION 7. *Let G be a connected UB-graph with non adjacent vertices u and v, where u and v are simplicial vertices of G. If $N_G(u) \not\supseteq N_G(v)$ and $N_G(v) \not\supseteq N_G(u)$, then $G+uv$ is not a UB-graph.*

PROOF. First we consider the case $N_G(u) \cap N_G(v) \neq \emptyset$. Then $\{u, v\} \cup (N_G(u) \cap N_G(v))$ is a clique of $G + uv$ containing u and v. Moreover, any vertex of $\{u, v\} \cup (N_G(u) \cap N_G(v))$ is not a simplicial vertex of $G + uv$. Thus $G + uv$ is not a UB-graph.

Next we consider the case $N_G(u) \cap N_G(v) = \emptyset$. Then, the edge uv is a clique of $G + uv$. Nevertheless, both u and v are not simplicial vertices of $G + uv$, since $N_{G+uv}[u]$ and $N_{G+uv}[v]$ do not induce complete subgraphs in $G + uv$. Thus there does not exist the UB edge clique cover of $G + uv$, and $G + uv$ is not a UB-graph. □

PROPOSITION 8. *Let G be a connected UB-graph with non adjacent vertices u and v, where u and v are simplicial vertices of G and $N_G(u) \supset N_G(v)$. If there exist no simplicial vertices in $N_G(u) - N_G(v)$, then $G + uv$ is not a UB-graph.*

PROOF. Let $w \in N_G(u) - N_G(v)$. If $G + uv$ is a UB-graph, then there exists a simplicial vertex z which is adjacent to u and w. Since w and v are non adjacent vertices, $z \in N_G(u) - N_G(v)$, which is a contradiction. □

By these propositions, we obtain the following result.

THEOREM 9. *Let G be a connected UB-graph with non adjacent vertices u and v. A graph $G + uv = (V(G), E(G) \cup \{uv\})$ is a UB-graph if and only if u and v satisfy one of the following conditions*;

(1) *u and v are simplicial vertices of G and $N_G(u) = N_G(v)$,*
(2) *u and v are simplicial vertices of G, $N_G(u) \supset N_G(v)$ and there exists a simplicial vertex $w(\neq u, v)$ of G in $N_G(u) - N_G(v)$, or*
(3) *u is not a simplicial vertex of G, v is a simplicial vertex of G and $N_G(u) \supset N_G(v)$.*

PROOF. By Proposition 2, 3 and 4, if u and v satisfy one of the conditions (1), (2) or (3), then $G + uv$ is a UB-graph.

Conversely, we assume that $G + uv$ is a UB-graph. By Proposition 5, both u and v are simplicial vertices or either u and v is a simplicial vertex. By Proposition 6, if u is not a simplicial vertex and v is a simplicial vertex, then u and v satisfy the condition (3). In the case that both u and v are simplicial vertices, $N_G(u) \supseteq N_G(v)$ or $N_G(u) \subseteq N_G(v)$ by Proposition 7. If $N_G(u) = N_G(v)$, then u and v satisfy the condition (1). If $N_G(u) \neq N_G(v)$, then $N_G(u) \subset N_G(v)$ or $N_G(v) \subset N_G(u)$, and u and v satisfy the condition (2) by Proposition 8. □

Next we consider edge deletions on UB-graphs. For a graph G and an edge uv of G, $G - uv$ is a graph where $V(G - uv) = V(G)$ and $E(G - uv) = E(G) - \{uv\}$.

PROPOSITION 10. *Let G be a connected UB-graph and uv be an edge of G. If u and v are simplicial vertices of G, then $G - uv$ is a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_k\}$ of G. Then u and v belong to the same clique Q of $\mathcal{E}(G)$, $Q - \{u\}$ and $Q - \{v\}$ are cliques of $G - uv$. Since each vertex of $Q - \{u, v\}$ is adjacent to non adjacent vertices u, v in $G - uv$, $Q - \{u\}$ has the unique simplicial vertex v in $G - uv$ and $Q - \{v\}$

has the unique simplicial vertices u in $G - uv$. Let

$$\mathcal{E}(G - uv) = (\mathcal{E}(G) - \{Q\}) \cup \{Q - \{u\}, Q - \{v\}\}.$$

Each $Q_i \in \mathcal{E}(G) - \{Q\}$ is a clique of $G - uv$ containing a simplicial vertex of $G - uv$ which is not included in other cliques of $\mathcal{E}(G - uv)$. Then $\mathcal{E}(G - uv)$ is a UB edge clique cover of $G - uv$ and $G - uv$ is a UB-graph. □

PROPOSITION 11. *Let G be a connected UB-graph with a kernel K and uv be an edge of G. If u is not a simplicial vertex of G, v is a simplicial vertex of G and there exists a subset $\{w_i \in K; w_i \neq v, uw_i \in E(G)\}$ of K such that $N_G(v) \subseteq \bigcup_i N_G(w_i)$, then $G - uv$ is a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_k\}$ whose kernel is K. Assume that $u, v \in Q \in \mathcal{E}(G)$, then $Q - \{u\}$ is a clique of $G - uv$ with v as a simplicial vertex. Each $Q_i \in \mathcal{E}(G) - \{Q\}$ is also a clique in $G - uv$ with a simplicial vertex belonging to no other $Q_j (i \neq j)$. Therefore, a sufficient condition for $G - uv$ being a UB-graph is that each edge incident to u is covered by some clique other than $Q - \{u\}$ in $G - uv$, which means that each edge incident to u is covered by some clique other than Q in G. By Theorem 1 and the definition of a kernel, this condition implies that there exists a subset $\{w_i \in K; w_i \neq v, uw_i \in E(G)\}$ of K such that $N_G(v) \subseteq \bigcup_i N_G(w_i)$. Under such condition

$$\mathcal{E}(G - uv) = (\mathcal{E}(G) - \{Q\}) \cup (Q - \{u\})$$

is a UB edge clique cover of $G - uv$ with a kernel K and $G - uv$ is a UB-graph. □

PROPOSITION 12. *Let G be a connected UB-graph with a kernel K and uv be an edge of G. If u and v are not simplicial vertices of G, then $G - uv$ is not a UB-graph.*

PROOF. Since G is a UB-graph, there exists a vertex w of K which is adjacent to both u and v. Since $N_G[w]$ is a clique of G, $N_G[w] - \{u\}$ and $N_G[w] - \{v\}$ are cliques of $G - uv$.

Since u and v are not simplicial vertices of G, u is adjacent to a vertex of K without w, and v is adjacent to a vertex of K without w. Then u and v are not simplicial vertices of $G - uv$. Since each vertex of $N_G[w] - \{u, v\}$ is adjacent to non adjacent vertices u and v in $G - uv$, all vertices of $N_G[w] - \{u, v\}$ are not simplicial vertices of $G - uv$. Therefore $N_G[w] - \{u\}$ and $N_G[w] - \{v\}$ have no simplicial vertices of $G - uv$. So there does not exist a UB edge clique cover of $G - uv$ and $G - uv$ is not a UB-graph. □

PROPOSITION 13. *Let G be a connected UB-graph with a kernel K and uv be an edge of G, where u is not a simplicial vertex of G and v is a simplicial vertex of G. If there exist no subsets $\{w_i \in K; w_i \neq v, uw_i \in E(G)\}$ of K such that $N_G(v) \subseteq \bigcup_i N_G(w_i)$, then $G - uv$ is not a UB-graph.*

PROOF. By Theorem 1, there exists the UB edge clique cover $\mathcal{E}(G) = \{Q_1, Q_2, \ldots, Q_k\}$ whose kernel is K. By the assumption, there exists $z \in N_G(v)$ such that the edge uz is covered by no cliques in $\mathcal{E}(G)$ other than $N_G[v]$. If there exists a clique Q'

with a simplicial vertex $w \neq v$ in $G - uv$ such that $u, z \in Q'$, then Q' is also a clique with a simplicial vertex w in G and the edge uz is covered by Q', which contradicts the assumption. Thus $G - uv$ has no cliques which contain u, z and a simplicial vertex, that is $G - uv$ is not a UB-graph. □

By these propositions, we obtain the following result.

THEOREM 14. *Let G be a connected UB-graph with a kernel K and uv be an edge of G. A graph $G - uv = (V(G), E(G) - \{uv\})$ is a UB-graph if and only if u and v satisfy one of the following conditions* ;

(1) *u and v are simplicial vertices of G, or*

(2) *u is not a simplicial vertex of G, v is a simplicial vertex of G and there exists a subset $\{w_i \in K;\, w_i \neq v, uw_i \in E(G)\}$ of K such that $N_G(v) \subseteq \bigcup_i N_G(w_i)$.*

PROOF. By Proposition 10 and 11, if u and v satisfy the condition (1) or (2), then $G-uv$ is a UB-graph.

Conversely, we assume that $G - uv$ is a UB-graph. By Proposition 12, both u and v are simplicial vertices or either u and v is a simplicial vertex. In the case that both u and v are simplicial vertices, u and v satisfy the condition (1). In the case that either u and v is a simplicial vertex, u and v satisfy the condition (2) by Proposition 13. □

REMARK. $G+uv$ and $G-uv$ can be considered mutually invertible operations in each correspondent cases, that is, the condition (1) of Theorem 14 corresponds to the condition (1) of Theorem 9. Also, the condition (2) of the former corresponds to the conditions (2) and (3) of the latter.

3. Split UB-graphs

In this section, we consider split UB-graphs. For a graph G, G is a split graph if and only if there exists a partition $V(G) = I \cup Q$ of its vertex set into an independent set I and a vertex set Q of a complete subgraph. In this section, for a split UB-graph G, I is a maximal independent set of G and $Q = V(G) - I$. Then $\mathcal{E}(G) = \{N_G[v_i]\,;\ v_i \in I\}$ is a UB edge clique cover of G, because I is a maximal independent set and each vertex of I is a simplicial vertex.

PROPOSITION 15. *Let G be a split UB-graph with a maximal independent set I. For non adjacent vertices $u \in Q$ and $v \in I$, $G + uv$ is a split UB-graph.*

PROOF. It is trivial that $G + uv$ is a split graph. Since G is a split UB-graph and I is a maximal independent set, there exists the UB edge clique cover of G such that I is a kernel and v is a simplicial vertex. Moreover, $uv \notin E(G)$ implies $N_G(v) \subseteq Q - \{u\} \subseteq N_G(u)$ and $Q - \{u\} \neq N_G(u)$, that is $N_G(v) \neq N_G(u)$.

In the case u is a simplicial vertex, there exists a vertex $w \in I$ adjacent to u by the maximal property of I. Then u and v satisfy the condition (2) of Theorem 9 and $G + uv$ is a UB-graph.

In the case u is not a simplicial vertex, u and v satisfy the condition (3) of Theorem 9 and $G + uv$ is a UB-graph. □

The graph obtained from G by recursively joining pairs of non adjacent vertices $u \in Q$ and $v \in I$ is a split graph. Thus we have the following result. For a graph G and $S \subseteq V(G)$, $\langle S \rangle$ denotes the subgraph induced by S. For two graphs G and H, the sum $G + H$ is the graph with the vertex set $V(G + H) = V(G) \cup V(H)$ and the edge set $E(G + H) = E(G) \cup E(H) \cup \{uv\ ;\ u \in V(G), v \in V(H)\}$.

COROLLARY 16. *Let G be a split UB-graph with a maximal independent set I, and H be the graph obtained from G by recursively joining pairs of non adjacent vertices $u \in Q$ and $v \in I$ until no such pair remains. Then $H = \langle Q \rangle + \langle I \rangle$ and H is a split UB-graph.*

PROOF. It is trivial that $H = \langle Q \rangle + \langle I \rangle$. By Proposition 15, the resulting graph at each stage is a split UB-graph. So H is a split UB-graph. □

Since $\langle Q \rangle + \langle I \rangle$ is a split UB-graph, we have the following result by Theorem 9.

PROPOSITION 17. *Let $H = \langle Q \rangle + \langle I \rangle$ where I is a maximal independent set of H with $|I| \geq 2$ and $Q = V(H) - I$ is a vertex set of a complete subgraph of H. For $u, v \in I$, $H + uv$ is a split UB-graph.*

PROOF. Since $N_H(u) = N_H(v) = Q$, u and v satisfy the condition (1) of Theorem 9. Thus $H + uv$ is a UB-graph. Since $I - \{v\}$ is a maximal independent set of $H + uv$ and $Q \cup \{v\}$ is a vertex set of a complete subgraph of $H + uv$, $H + uv$ is a split graph. □

For a graph G, $\alpha(G)$ is the cardinality of the maximal independent set of G. For a graph H satisfying the conditions of Proposition 17, $H + uv$ is a split UB-graph and $\alpha(H + uv) = \alpha(H) - 1$. By Corollary 16 and Proposition 17, we have the following result.

NOTE. Let G be a split UB-graph with a maximal independent set I, and G^+ be a graph obtained from G by the following operations. Then G^+ is a complete graph.

Step 1: For any non adjacent vertices pair $u \in Q$ and $v \in I$, add the new edge uv recursively until no such pair remains.

Step 2: If $|I| \geq 2$, then for a pair $u, v \in I$, add the new edge uv and go to Step 1. If $|I| = 1$, we obtain the graph G^+.

PROPOSITION 18. *Let G be a split UB-graph with a maximal independent set I, and uv be an edge of G. If a graph $G - uv$ is a UB-graph, then $G - uv = (V(G), E(G) - \{uv\})$ is a split graph.*

PROOF. By definition of split graphs, $\langle Q \rangle$ is a complete subgraph of G. By Theorem 14, u and v are simplicial vertices of G or one of u and v is a simplicial vertex of G. So we consider the following four cases depending on the conditions of u and v.

CASE (a): $u \in Q$, $v \in I$ and u is a simplicial vertex of G.

Since u is a simplicial vertex, u is not adjacent to vertices of I without v. Thus $N_G[u] = N_G[v]$. Therefore $I \cup \{u\}$ is a maximal independent set of $G - uv$ and $\langle Q - \{u\} \rangle$ is a complete subgraph of $G - uv$. So $G - uv$ is a split graph.

CASE (b): $u \in Q, v \in I$ and u is not a simplicial vertex of G.

Since $G - uv$ is a UB-graph, there exists a subset $\{w_i \in I; w_i \neq v, uw_i \in E(G)\}$ such that $N_G(v) \subseteq \bigcup_i N_G(w_i)$ by Theorem 14. Then I is also a maximal independent set of $G - uv$ and $\langle Q \rangle$ is also a complete subgraph of $G - uv$. So $G - uv$ is a split graph.

CASE (c): $u, v \in Q$ and u, v are simplicial vertices of G.

Since G is a UB-graph and I is a maximal independent set of G, there exists a simplicial vertex $w \in I$ such that w is adjacent to u and v. Since u and v are simplicial vertices of G, $N_G[u] = N_G[v] = N_G[w]$. Thus $(I - \{w\}) \cup \{u, v\}$ is a maximal independent set of $G - uv$ and $\langle (Q - \{u, v\}) \cup \{w\} \rangle$ is a complete subgraph of $G - uv$. So $G - uv$ is a split graph.

CASE (d): $u, v \in Q$, u is not a simplicial vertex of G and v is a simplicial vertex of G.

Since G is a UB-graph and I is a maximal independent set of G, there exists a simplicial vertex $w \in I$ such that w is adjacent to $u, v \in Q$ of G. Since v is a simplicial vertex of G, $N_G[v] = N_G[w]$. Let $I_u = \{w_i \in I : uw_i \in E(G)\}$. Since u is not a simplicial vertex, $(N_G(u) - \{w\}) \cap I \neq \emptyset$. Thus $I_u \neq \emptyset$. Then $\bigcup_{w_i \in I_u} N_G(w_i) \subseteq Q$. Since $w \in N_G(v)$, $N_G(v) \not\subseteq \bigcup_{w_i \in I_u} N_G(w_i)$. Therefore $G - uv$ is not a UB-graph by Theorem 14, which is a contradiction. So this case does not satisfy the condition. □

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