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On the Mixed Multiplicities of Multi-graded Fiber Cones

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Abstract. Let $(A, \mathfrak{m})$ denote a Noetherian local ring with maximal ideal $\mathfrak{m}$, J an $\mathfrak{m}$-primary ideal, $I_1, \ldots, I_s$ ideals of A; M a finitely generated A-module. This paper will answer when mixed multiplicities of the multi-graded fiber cone

$$F_M(J, I_1, \ldots, I_s) = \bigoplus_{n_1,\ldots,n_s \geqslant 0} \frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}$$

are positive and characterize them in terms of the length of modules.

1. Introduction

Throughout this paper, $(A, \mathfrak{m})$ denotes a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$; M a finitely generated A-module with Krull dimension $\dim M = \dim A = d > 0$.

Let J be $\mathfrak{m}$-primary and $I_1, \ldots, I_s$ ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann} M}$. Define

$$F_M(J, I_1, \ldots, I_s) = \bigoplus_{n_1,\ldots,n_s \geqslant 0} \frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}; \ \ell = \dim\left(\bigoplus_{n \geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\right)$$

to be the *multi-graded fiber cone of M with respect to* $J, I_1, \ldots, I_s$ and the *analytic spread of I with respect to M*, respectively. The multi-graded fiber cone $F_M(J, I_1, \ldots, I_s)$ is an important object of Commutative Algebra and Algebraic Geometry.

Set $f(n_1, \ldots, n_s) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}\right)$. Then by [HHRT], $f(n_1, \ldots, n_s)$ is a polynomial for all large $n_1, \ldots, n_s$, and the degree of this polynomial is $(\ell - 1)$(see Proposition 3.1, Section 3). The terms of total degree $\ell - 1$ in this polynomial have the form

$$\sum_{d_1+\cdots+d_s = \ell-1} E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \frac{n_1^{d_1} \cdots n_s^{d_s}}{d_1! \cdots d_s!}.$$

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Here $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M)$ are non-negative integers not all zero, called the *mixed multiplicity of the multi-graded fiber cone* $F_M(J, I_1, \dots, I_s)$ of the *type* $(d_1, \dots, d_s)$ [HHRT].

We emphasize that $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M)$ is not only an important object in computing the multiplicity of multi-graded fiber cones(see [HHRT, Theorem 4.3]) but also a generalized object of mixed multiplicities in [Ve2] (see Remark 3.2, Section 3).

The purpose of this paper is to answer to the question when

$$E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M)$$

is positive and characterize it in terms of the length of modules (see Theorem 3.5, Section 3).

This paper is divided into three sections. In Section 2, we give some results on weak-(FC)-sequences of modules. Section 3 investigates mixed multiplicities of multi-graded fiber cones.

2. On Weak-(FC)-Sequences of Modules

In this section, we present some results on weak-(FC)-sequences of modules and reductions of ideals with respect to modules which will be used in the paper. Set

$$a : b^{\infty} = \bigcup_{n \geqslant 0} (a : b^n)\,.$$

The notion of weak-(FC)-sequences in [Vi1] is extended to modules as follows.

DEFINITION 2.1 (see [MV, Definition 2.1]). Let $U = (I_1, \dots, I_s)$ be a set of ideals of A such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $M^* = \frac{M}{0_M : I^{\infty}}$. We say that an element $x \in A$ is a weak-(FC)-element of M with respect to U if there exist an ideal I_i of U and a positive integer n_i' such that

$(\mathrm{FC}_1):$ $x \in I_i \setminus \mathfrak{m} I_i$ and

$$I_1^{n_1} \cdots I_s^{n_s} M^* \cap x M^* = x I_1^{n_1} \cdots I_{i-1}^{n_{i-1}} I_i^{n_i - 1} I_{i+1}^{n_{i+1}} \cdots I_s^{n_s} M^*$$

for all $n_i \geqslant n_i'$ and all non-negative integers $n_1, \dots, n_{i-1}, n_{i+1}, \dots, n_s$.

$(\mathrm{FC}_2):$ $0_M : x \subseteq 0_M : I^{\infty}$.

Let $x_1, \dots, x_t$ be a sequence in $\bigcup_{i=1}^s I_i$. For each $i = 0, 1, \dots, t-1$, set $\bar{A} = \frac{A}{(x_1, \dots, x_i)}$, $\bar{I}_1 = I_1 \bar{A}, \dots, \bar{I}_s = I_s \bar{A}$, $\overline{M} = \frac{M}{(x_1, \dots, x_i)M}$. Let $\bar{x}_{i+1}$ denote the image of x_{i+1} in $\bar{A}$. Then $x_1, \dots, x_t$ is called a weak-(FC)-sequence of M with respect to U if $\bar{x}_{i+1}$ is a weak-(FC)-element of $\overline{M}$ with respect to $(\bar{I}_1, \dots, \bar{I}_s)$ for $i = 0, 1, \dots, t-1$.

REMARK 2.2. If I is contained in $\sqrt{\mathrm{Ann}M}$, then the conditions (FC_1) and (FC_2) are usually true for all $x \in \bigcup_{i=1}^s I_i$. This only obstructs and does not carry useful. That is why in Definition 2.1, one has to exclude the case that I is contained in $\sqrt{\mathrm{Ann}M}$.

In [MV], the authors showed the existence of weak-(FC)-sequences of modules. The important key to the proof of this result is the following lemma.

LEMMA 2.3 (see [MV, Lemma 2.2]). *Let $(A, \mathfrak{m})$ be a Noetherian local ring with maximal ideal $\mathfrak{m}$, infinite residue field $k = A/\mathfrak{m}$ and M a finitely generated A-module. Let $U = (I_1, \dots, I_s)$ be a set of ideals of A and $\sum$ a finite set of prime ideals non containing $I_1 \cdots I_s$. Then for each $1 \leqslant i \leqslant s$, there exists an element x_i of I_i not contained in any prime ideal in $\sum$, and a positive integer k_i such that*

$$I_1^{r_1} \cdots I_i^{r_i} \cdots I_s^{r_s} M \cap x_i M = x_i I_1^{r_1} \cdots I_{i-1}^{r_{i-1}} I_i^{r_i - 1} I_{i+1}^{r_{i+1}} \cdots I_s^{r_s} M$$

for any $r_i \geqslant k_i$ and all non-negative integers $r_1, \dots, r_{i-1}, r_{i+1}, \dots, r_s$.

Using the same argument as in Remark 1 [Vi1], the paper [MV] proved the following.

LEMMA 2.4 (see [MV, Proposition 2.3]). *Let $(I_1, \dots, I_s)$ be a set of ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann} M}$. Then for any $1 \leqslant i \leqslant s$, there exists a weak-(FC)-element $x \in I_i$ of M with respect to $(I_1, \dots, I_s)$.*

So the existence of weak-(FC)-sequences is universal.

In the case where $I_1, \dots, I_s$ are $\mathfrak{m}$-primary ideals, by an argument analogous to that used for Remark 3 [Vi1], we have the following result.

LEMMA 2.5 (see [MV, Proposition 2.5]). *Let $U = (I_1, \dots, I_s)$ be a set of $\mathfrak{m}$-primary ideals. Then for all non-negative integers $k_1, \dots, k_s$ such that $k_1 + \cdots + k_s = d$, there exists a weak-(FC)-sequence $x_1, \dots, x_d$ in $\bigcup_{i=1}^{s} I_i$ of M with respect to U consisting of k_1 elements of $I_1, \dots, k_s$ elements of I_s.*

Set $R_A(I) = \bigoplus_{n \geqslant 0} I^n t^n$, $R_M(I) = \bigoplus_{n \geqslant 0} I^n M t^n$. $R_A(I)$ and $R_M(I)$ are called the Rees algebra and the Rees module of I, respectively.

Note that if $I = I_1 \cdots I_s$ is contained in $\sqrt{AnnM}$ then weak-(FC)-element of M with respect to $U = (I_1, \dots, I_s)$ does not exist, and the analytic spread of I with respect to M $\big(\ell = \dim\big(\bigoplus_{n \geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\big)\big)$ is zero. Then Theorem 3.4 [Vi3] is stated in terms of modules as follows.

LEMMA 2.6 (see[Vi3, Theorem 3.4]). *Let $J_1, \dots, J_t$ be $\mathfrak{m}$-primary ideals and $I_1, \dots, I_s$ arbitrary ideals. Set $I = I_1 \cdots I_s, U = (J_1, \dots, J_t, I_1, \dots, I_s), \ell = \dim\big(\bigoplus_{n \geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\big)$. Let $L_U(I_1, \dots, I_s; M)$ denote the set of lengths of maximal weak-(FC)-sequences in $\bigcup_{i=1}^{s} I_i$ of M with respect to U. For any $1 \leqslant j \leqslant s$, set $\hat{I}_j = I_1 \cdots I_{j-1} I_{j+1} \cdots I_s$ if $s > 1$ and $\hat{I}_j = A$ if $s = 1$; $R_j = R_A(I_j)$, $R_j(M) = R_M(I_j)$. Then the following statements hold.*

(i) *For any $1 \leqslant j \leqslant s$, the length of maximal weak-(FC)-sequences in I_j of M with respect to U is an invariant and this invariant does not depend on t and $J_1, \dots, J_t$.*

(ii) *If I is not contained in $\sqrt{\mathrm{Ann}M}$ and p is the length of maximal weak-(FC)-sequences in I_j of M with respect to U, then*

$$p = \dim\left(\frac{R_j}{\bigcup_{k\geq 0}[\mathfrak{m}(\mathfrak{m}\hat{I}_j)^k R_j(M) : (\mathfrak{m}\hat{I}_j)^k R_j(M)]}\right) \leqslant \ell_j\,,$$

where $\ell_j = \dim\left(\bigoplus_{n\geqslant 0} \frac{I_j^n M}{\mathfrak{m} I_j^n M}\right)$.

(iii) *If $x_1, \dots, x_p$ is a maximal weak-(FC)-sequence in I_j of M with respect to U, then*

$$I_1^{n_1} \cdots I_j^{n_j} \cdots I_s^{n_s} M = (x_1, \dots, x_p) I_1^{n_1} \cdots I_j^{n_j - 1} \cdots I_s^{n_s} M$$

for all large $n_1, \dots, n_s$.

(iv) $\max L_U(I_1, \dots, I_s; M) = \ell$.

Let I be an ideal of A. An ideal $\Im$ of A is called a reduction of I with respect to M if $\Im \subseteq I$ and $I^{n+1}M = \Im I^n M$ for all sufficiently large n [NR].

As an immediate consequence of Lemma 2.6, we have the following result.

LEMMA 2.7 (see [Vi3, Theorem 3.5]). *Let J be an $\mathfrak{m}$-primary ideal and I an ideal is not contained in $\sqrt{\mathrm{Ann}M}$. Set $\ell = \dim\left(\bigoplus_{n\geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\right)$. Suppose that p is the length of maximal weak-(FC)-sequences in I of M with respect to (J, I) and $x_1, \dots, x_p$ is a maximal weak-(FC)-sequence in I of M with respect to (J, I). Then*

(i) $p = \ell$.

(ii) $(x_1, \dots, x_p)$ *is a reduction of I with respect to M.*

Let $\Im$ be a reduction of I with respect to M i.e., $\Im \subseteq I$ and

$$I^{n+1}M = \Im I^n M$$

for all sufficiently large n. A reduction $\Im$ of I is called a minimal reduction if it does not properly contain any other reduction of I [NR]. The least integer n such that $I^{n+1}M = \Im I^n M$ is called *the reduction number of I with respect to $\Im$ and M*, and we denote it by $r_\Im(I; M)$. *The reduction number of I with respect to M* is defined by

$$r(I; M) = \min\{r_\Im(I; M) \mid \Im \text{ is a minimal reduction of } I \text{ with respect to } M\}\,.$$

Let J be an $\mathfrak{m}$-primary ideal and I an ideal is not contained in $\sqrt{\mathrm{Ann}M}$. Set $\ell = \dim\left(\bigoplus_{n\geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\right)$. By Lemma 2.7(i), the length of maximal weak-(FC)-sequences in I of M with respect to (J, I) is ℓ. Assume that $\Im$ is a minimal reduction of I with respect to M. It can be verified that if $x_1, \dots, x_t$ is a maximal weak-(FC)-sequence in $\Im$ of M with respect to $(J, I, \Im)$, then $x_1, \dots, x_t$ is also a maximal weak-(FC)-sequence in I of M with respect to (J, I). By Lemma 2.7, $t = \ell$ and $(x_1, \dots, x_\ell) \subseteq \Im$ is a reduction of I with respect to M. Since $\Im$ is a minimal reduction of I with respect to M, we get $\Im = (x_1, \dots, x_\ell)$. So we have:

LEMMA 2.8. *Let J be $\mathfrak{m}$-primary and I an ideal is not contained in $\sqrt{\mathrm{Ann}M}$. Then $\mathfrak{I}$ is a minimal reduction of I with respect to M if and only if $\mathfrak{I}$ is generated by a maximal weak-(FC)-sequence in I of M with respect to (J, I).*

Denote by $L_U(I_1, \ldots, I_s; M)$ the set of the lengths of maximal weak-(FC)-sequences in $\bigcup_{i=1}^s I_i$ of M with respect to $U = (J, I_1, \ldots, I_s)$. We have the following proposition.

PROPOSITION 2.9. *Let J be an $\mathfrak{m}$-primary ideal and $I_1, \ldots, I_s$ ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $U = (J, I_1, \ldots, I_s)$ and $\ell = \dim\big(\bigoplus_{n\geqslant 0} \frac{I^nM}{\mathfrak{m}I^nM}\big)$. Suppose that x_1 is a weak-(FC)-element of M with respect to U. Set $\bar{A} = A/(x_1)$, $\overline{M} = M/x_1M$, $\overline{\mathfrak{m}} = \mathfrak{m}\bar{A}$, $\bar{J} = J\bar{A}$, $\bar{I} = I\bar{A}$, $\bar{I}_i = I_i\bar{A}$ $(i = 1, \ldots, s)$, $\overline{U} = (\bar{J}, \bar{I}_1, \ldots, \bar{I}_s)$, $\ell' = \dim\big(\bigoplus_{n\geqslant 0} \frac{\bar{I}^n\overline{M}}{\overline{\mathfrak{m}}\bar{I}^n\overline{M}}\big)$. Then the following statements hold.*

(i) $\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) \leqslant \max L_U(I_1, \ldots, I_s; M) - 1$, *the equality holds iff there exist $x_2, \ldots, x_\ell$ such that $x_1, \ldots, x_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U.*

(ii) *The length of maximal weak-(FC)-sequences in I of M with respect to (J, I) is* $\max L_U(I_1, \ldots, I_s; M)$.

(iii) *If* $\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \max L_U(I_1, \ldots, I_s; M) - 1$, *then* $\ell' = \ell - 1$.

PROOF. The proof of (i): Set $p = \max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M})$. Let $y_1, \ldots, y_p$ be a sequence in $\bigcup_{i=1}^s I_i$ such that $y_1', \ldots, y_p'$ is a maximal weak-(FC)-sequence in $\bigcup_{i=1}^s \bar{I}_i$ of $\overline{M}$ with respect to $\overline{U}$, where y_k' the initial form of y_k in $\bar{A}$ $(k = 1, \ldots, p)$. By Definition 2.1, $x_1, y_1, \ldots, y_p$ is a maximal weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U. Thus

$$p + 1 \leqslant \max L_U(I_1, \ldots, I_s; M)$$

or $p \leqslant \max L_U(I_1, \ldots, I_s; M) - 1$. By Lemma 2.6 (iv),

$$\ell = \max L_U(I_1, \ldots, I_s; M)\,.$$

Hence

$$\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \max L_U(I_1, \ldots, I_s; M) - 1$$

iff $\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \ell - 1$. This condition is equivalent to the existence of elements $x_2, \ldots, x_\ell \in \bigcup_{i=1}^s I_i$ such that $\bar{x}_2, \ldots, \bar{x}_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s \bar{I}_i$ of $\overline{M}$ with respect to $\overline{U}$, where $\bar{x}_2, \ldots, \bar{x}_\ell$ are the initial forms of $x_2, \ldots, x_\ell$ in $\bar{A}$, respectively. It is clear that $x_1, x_2, \ldots, x_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U.

The proof of (ii): Denote by p the length of maximal weak-(FC)-sequences in I of M with respect to (J, I). By Lemma 2.7, $p = \ell$. On the other hand, we have $\ell = \max L_U(I_1, \ldots, I_s; M)$ by Lemma 2.6 (iv). Hence

$$p = \max L_U(I_1, \ldots, I_s; M)\,.$$

The proof of (iii): By Lemma 2.6 (iv), we have

$$\ell = \max L_U(I_1, \ldots, I_s; M) \text{ and } \ell' = \max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M})\,.$$

Since

$$\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \max L_U(I_1, \ldots, I_s; M) - 1\,,$$

we get $\ell' = \ell - 1$. ∎

3. Mixed Multiplicities of Multi-graded Fiber cones

Using the results on weak-(FC)-sequences of modules in Section 2, this section answers to the question when mixed multiplicities of multi-graded fiber cones are positive and characterizes them in terms of the length of modules.

If we assign the degree -1 to the zero polynomial then we have the following proposition.

PROPOSITION 3.1. *Let J be an $\mathfrak{m}$-primary ideal and $I_1, \ldots, I_s$ arbitrary ideals. Set $I = I_1 \cdots I_s$,*

$$\ell = \dim\Big(\bigoplus_{n \geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\Big),\ f(n_1, \ldots, n_s) = l_A\Big(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}\Big)\,.$$

Then $f(n_1, \ldots, n_s)$ is a polynomial of degree $\ell - 1$ for all large $n_1, \ldots, n_s$.

PROOF. If I is contained in $\sqrt{AnnM}$ then the analytic spread of I is zero and $f(n_1, n_2, \ldots, n_s)$ is the zero polynomial for all large $n_1, n_2, \ldots, n_s$. Hence, the degree of this polynomial is $-1 = 0 - 1$. So the proposition is true. The case that I is not contained in $\sqrt{AnnM}$. By Theorem 4.1 [HHRT], $f(n_1, \ldots, n_s)$ is a polynomial for all sufficiently large $n_1, \ldots, n_s$. Moreover, all monomials of highest degree in this polynomial have non-negative coefficients. Denote this polynomial by $P(n_1, \ldots, n_s)$. We will prove that $\deg P(n_1, \ldots, n_s) = \ell - 1$. Set $Q(n) = P(n, \ldots, n)$. It is clear that $\deg P(n_1, \ldots, n_s) = \deg Q(n)$. We have

$$Q(n) = P(n, \ldots, n) = l_A\Big(\frac{I_1^n \cdots I_s^n M}{J I_1^n \cdots I_s^n M}\Big) = l_A\Big(\frac{I^n M}{J I^n M}\Big)$$

for all sufficiently large n. Thus

$$\deg Q(n) = \dim\Big(\bigoplus_{n \geqslant 0} \frac{I^n M}{J I^n M}\Big) - 1 = \dim\Big(\bigoplus_{n \geqslant 0} \frac{I^n M}{\mathfrak{m} I^n M}\Big) - 1 = \ell - 1\,.$$

Hence $\deg P(n_1, \ldots, n_s) = \ell - 1$. ∎

REMARK 3.2. If set $q = \dim\Big(\bigoplus_{n \geqslant 0} \frac{(JI)^n M}{\mathfrak{m}(JI)^n M}\Big)$ then by Proposition 3.1, $l_A\Big(\frac{J^n I_1^{n_1} \cdots I_s^{n_s} M}{J^{n+1} I_1^{n_1} \cdots I_s^{n_s} M}\Big)$ is a polynomial of degree $q - 1$ for all large $n, n_1, \ldots, n_s$. One writes the terms of total degree $q-1$ in this polynomial in the form

$$\sum_{k_0+k_1+\cdots+k_s=q-1} e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M)\frac{n^{k_0}n_1^{k_1}\cdots n_s^{k_s}}{k_0!k_1!\cdots k_s!},$$

then $e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M)$ is called *the mixed multiplicity* of M with respect to $(J, I_1, \ldots, I_s)$ [Ve2] and [HHRT]. Since

$$l_A\left(\frac{J^nI_1^{n_1}\cdots I_s^{n_s}M}{J^{n+1}I_1^{n_1}\cdots I_s^{n_s}M}\right)=l_A\left(\frac{J^nI_1^{n_1}\cdots I_s^{n_s}M}{JJ^nI_1^{n_1}\cdots I_s^{n_s}M}\right),$$

it follows that $E_J(J^{[k_0]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M) = e_A(J^{[k_0+1]}, I_1^{[k_1]}, \ldots, I_s^{[k_s]}; M)$ for all non-negative integers $k_0, k_1, \ldots, k_s$ such that $k_0+k_1+\cdots+k_s = q-1$. This equality proves that the mixed multiplicity is a particular case of the mixed multiplicity of multi-graded fiber cones. So $E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M)$ is not only an important object in computing the multiplicity of multi-graded fiber cones but also a generalized object of mixed multiplicities.

Next, we need the following proposition.

PROPOSITION 3.3. *Let J be an $\mathfrak{m}$-primary ideal and $I_1, \ldots, I_s$ ideals such that $I = I_1\cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $U = (J, I_1, \ldots, I_s)$ and $\ell = \dim\big(\bigoplus_{n\geqslant 0}\frac{I^nM}{\mathfrak{m}I^nM}\big)$. Assume that $x_1 \in I_j$ is a weak-(FC)-element of M with respect to U for some j. Set $M^* = \frac{M}{0_M:I^\infty}$, $\overline{M} = M/x_1M$. Denote by $P(n_1, \ldots, n_s)$ and $Q(n_1, \ldots, n_s)$ the polynomials such that*

$$P(n_1,\ldots,n_s)=l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M}{JI_1^{n_1}\cdots I_s^{n_s}M}\right),\quad Q(n_1,\ldots,n_s)=l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}}{JI_1^{n_1}\cdots I_s^{n_s}\overline{M}}\right)$$

for all large $n_1, \ldots, n_s$. Then the following statements hold.

(i) $Q(n_1, \ldots, n_s) = P(n_1, \ldots, n_j, \ldots, n_s) - P(n_1, \ldots, n_j - 1, \ldots, n_s)$.

(ii) *If there exist $x_2, \ldots, x_\ell \in A$ such that $x_1, \ldots, x_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U, then*

$$\deg Q(n_1,\ldots,n_s)=\deg P(n_1,\ldots,n_s)-1.$$

(iii) *If $E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \neq 0$ and $d_j > 0$, then*

$$\deg Q(n_1,\ldots,n_s)=\deg P(n_1,\ldots,n_s)-1.$$

PROOF. Set $N = 0_M : I^\infty$ and $\overline{M}^* = \frac{M}{x_1M:I^\infty}$.

The proof of (i): It is clear that $\overline{M}^* \simeq \frac{\overline{M}}{0_{\overline{M}}:I^\infty}$. We have

$$l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{JI_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right)=l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M+(x_1M:I^\infty)}{JI_1^{n_1}\cdots I_s^{n_s}M+(x_1M:I^\infty)}\right)$$

$$= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + x_1M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + (I_1^{n_1}\cdots I_s^{n_s}M + x_1M + N)\bigcap(x_1M : I^{\infty})}\right)$$

$$= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + x_1M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + x_1M + N + I_1^{n_1}\cdots I_s^{n_s}M\bigcap(x_1M : I^{\infty})}\right).$$

By Artin-Rees Lemma, $I_1^{n_1}\cdots I_s^{n_s}M\bigcap(x_1M : I^{\infty}) \subseteq x_1M$ for all sufficiently large $n_1,\ldots,n_s$. Thus

$$\begin{aligned}
l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{JI_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right) &= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + x_1M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + x_1M + N}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + (x_1M + N)\bigcap(I_1^{n_1}\cdots I_s^{n_s}M + N)}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + N}\right)\\
&\quad - l_A\left(\frac{JI_1^{n_1}\cdots I_s^{n_s}M + x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}{JI_1^{n_1}\cdots I_s^{n_s}M + N}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{JI_1^{n_1}\cdots I_s^{n_s}M^*}\right)\\
&\quad - l_A\left(\frac{x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}{(JI_1^{n_1}\cdots I_s^{n_s}M + N)\bigcap(x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N)}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{JI_1^{n_1}\cdots I_s^{n_s}M^*}\right)\\
&\quad - l_A\left(\frac{x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}{(JI_1^{n_1}\cdots I_s^{n_s}M + N)\bigcap(I_1^{n_1}\cdots I_j^{n_j}\cdots I_s^{n_s}M + N)\bigcap(x_1M + N)}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{JI_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}{x_1JI_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M + N}\right)\\
&= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{JI_1^{n_1}\cdots I_s^{n_s}M^*}\right) - l_A\left(\frac{x_1I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{x_1JI_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)
\end{aligned}$$

for all large $n_1, \ldots, n_s$. Since x_1 is a weak-(FC)-element, x_1 is a non-zero-divisor in M^*. This follows that

$$l_A\left(\frac{x_1 I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{x_1 J I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)= l_A\left(\frac{I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{J I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right).$$

From these facts, we get

$$l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right)= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{J I_1^{n_1}\cdots I_s^{n_s}M^*}\right)-l_A\left(\frac{I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}{J I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M^*}\right)$$

for all large $n_1, \ldots, n_s$. It is clear that

$$\begin{aligned} l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{J I_1^{n_1}\cdots I_s^{n_s}M^*}\right) &= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M+N}{J I_1^{n_1}\cdots I_s^{n_s}M+N}\right) \\ &= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M}{J I_1^{n_1}\cdots I_s^{n_s}M + I_1^{n_1}\cdots I_s^{n_s}M\bigcap N}\right). \end{aligned}$$

By Artin-Rees Lemma, $I_1^{n_1}\cdots I_s^{n_s}M\bigcap N = 0$ for all large $n_1, \ldots, n_s$. Thus

$$l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M^*}{J I_1^{n}\cdots I_s^{n_s}M^*}\right)= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M}{J I_1^{n}\cdots I_s^{n_s}M}\right)$$

for all large $n_1, \ldots, n_s$. Using the result just obtained, we also have

$$l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}}{J I_1^{n_1}\cdots I_s^{n_s}\overline{M}}\right)= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}{J I_1^{n_1}\cdots I_s^{n_s}\overline{M}^*}\right)$$

for all sufficiently large $n_1, \ldots, n_s$. Therefore

$$l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}}{J I_1^{n_1}\cdots I_s^{n_s}\overline{M}}\right)= l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M}{J I_1^{n_1}\cdots I_s^{n_s}M}\right)-l_A\left(\frac{I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M}{J I_1^{n_1}\cdots I_j^{n_j-1}\cdots I_s^{n_s}M}\right)$$

for all large $n_1, \ldots, n_s$. Hence

$$Q(n_1, \ldots, n_j, \ldots, n_s) = P(n_1, \ldots, n_j, \ldots, n_s) - P(n_1, \ldots, n_j - 1, \ldots, n_s)\,.$$

The proof of (ii): Set $\bar{A} = A/(x_1)$, $\bar{J} = J\bar{A}, \overline{\mathfrak{m}} = \mathfrak{m}\bar{A}$, $\bar{I}_i = I_i\bar{A}$ for all $i = 1, \ldots, s$ and $\overline{U} = (\bar{J}, \bar{I}_1, \ldots, \bar{I}_s)$. Denote by $L_U(I_1, \ldots, I_s; M)$ the set of lengths of maximal weak-(FC)-sequences in $\bigcup_{i=1}^s I_i$ of M with respect to U and $L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M})$ the set of lengths of maximal weak-(FC)-sequences in $\bigcup_{i=1}^s \bar{I}_i$ of $\overline{M}$ with respect to $\overline{U}$. By Proposition 2.9 (i),

$$\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \max L_U(I_1, \ldots, I_s; M) - 1\,.$$

Set $\ell' = \dim\big(\bigoplus_{n\geqslant 0} \frac{\bar{I}^n\overline{M}}{\mathfrak{m}\bar{I}^n\overline{M}}\big)$. By Lemma 2.6(iv), we have

$$\max L_{\overline{U}}(\bar{I}_1, \ldots, \bar{I}_s; \overline{M}) = \ell' \text{ and } \max L_U(I_1, \ldots, I_s; M) = \ell.$$

Thus $\ell' = \ell - 1$. By Proposition 3.1,

$$\deg P(n_1, \ldots, n_s) = \ell - 1 \text{ and } \deg Q(n_1, \ldots, n_s) = \ell' - 1\,.$$

Hence

$$\deg Q(n_1, \ldots, n_s) = \deg P(n_1, \ldots, n_s) - 1\,.$$

The proof of (iii): By (i),

$$Q(n_1, \ldots, n_j, \ldots, n_s) = P(n_1, \ldots, n_j, \ldots, n_s) - P(n_1, \ldots, n_j - 1, \ldots, n_s)\,.$$

Since $E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \neq 0$ and $d_j > 0$,

$$\deg Q(n_1, \ldots, n_s) = \deg P(n_1, \ldots, n_s) - 1\,. \qquad \blacksquare$$

The following lemma makes up an important role in the proof of our second main result.

LEMMA 3.4. *Let J be an $\mathfrak{m}$-primary ideal and $I_1, \ldots, I_s$ ideals such that $I = I_1 \cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $U = (J, I_1, \ldots, I_s)$ and $\ell = \dim\big(\bigoplus_{n\geqslant 0} \dfrac{I^nM}{\mathfrak{m}I^nM}\big)$. Then the following statements hold.*

(i) *Assume that $E_J(I_1^{[d_1]}, \ldots, I_j^{[d_j]}, \ldots, I_s^{[d_s]}; M) \neq 0$ and $d_j > 0$ for some j. Then for any weak-(FC)-element $x \in I_j$ of M with respect to U, we have*

$$E_J(I_1^{[d_1]}, \ldots, I_j^{[d_j]}, \ldots, I_s^{[d_s]}; M) = E_J(I_1^{[d_1]}, \ldots, I_j^{[d_j-1]}, \ldots, I_s^{[d_s]}; M/xM)\,.$$

(ii) *If $E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \neq 0$ and $x_1, \ldots, x_{\ell-1}$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U consisting of d_1 elements of $I_1, \ldots, d_s$ elements of I_s, then*

$$E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = l_A\left(\frac{I^n\overline{M}}{JI^n\overline{M}}\right)$$

for large n, where $\overline{M} = \frac{M}{(x_1, \ldots, x_{\ell-1})M}$.

PROOF. The proof of (i): Set $M' = M/xM$. Denote by $P(n_1, \ldots, n_s)$ and $Q(n_1, \ldots, n_s)$ the polynomials such that

$$P(n_1, \ldots, n_s) = l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M}{JI_1^{n_1}\cdots I_s^{n_s}M}\right), \quad Q(n_1, \ldots, n_s) = l_A\left(\frac{I_1^{n_1}\cdots I_s^{n_s}M'}{JI_1^{n_1}\cdots I_s^{n_s}M'}\right)$$

for all large $n_1, \ldots, n_s$. By Proposition 3.3(i) and (iii), we have

$$Q(n_1, \ldots, n_j, \ldots, n_s) = P(n_1, \ldots, n_j, \ldots, n_s) - P(n_1, \ldots, n_j - 1, \ldots, n_s)$$

and $\deg Q(n_1,\ldots,n_s) = \deg P(n_1,\ldots,n_s) - 1$. Consequently,

$$E_J(I_1^{[d_1]},\ldots,I_j^{[d_j]},\ldots,I_s^{[d_s]};M) = E_J(I_1^{[d_1]},\ldots,I_j^{[d_j-1]},\ldots,I_s^{[d_s]};M')\,.$$

The proof of (ii): By (i) and the proof of (i), we have

$$E_J(I_1^{[0]},\ldots,I_j^{[0]},\ldots,I_s^{[0]};\overline{M}) = E_J(I_1^{[d_1]},\ldots,I_j^{[d_j]},\ldots,I_s^{[d_s]};M) \neq 0$$

and $l_A\Big(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}}{JI_1^{n_1}\cdots I_s^{n_s}\overline{M}}\Big)$ is a polynomial of degree 0 for all large $n_1,\ldots,n_s$. Consequently, $E_J(I_1^{[0]},\ldots,I_j^{[0]},\ldots,I_s^{[0]};\overline{M}) = l_A\Big(\frac{I_1^{n_1}\cdots I_s^{n_s}\overline{M}}{JI_1^{n_1}\cdots I_s^{n_s}\overline{M}}\Big)$ for all large $n_1,\ldots,n_s$. By taking $n_1 = \cdots = n_s = n$, where n is a sufficiently large integer, we obtain $E_J(I_1^{[0]},\ldots,I_j^{[0]},\ldots,I_s^{[0]};\overline{M}) = l_A\Big(\frac{I^n\overline{M}}{JI^n\overline{M}}\Big)$. Hence

$$E_J(I_1^{[d_1]},\ldots,I_s^{[d_s]};M) = l_A\left(\frac{I^n\overline{M}}{JI^n\overline{M}}\right)$$

for all large n. ■

Let I be an ideal of A. Denote by $r = r(I;M)$ the reduction number of I with respect to M. The vanishing and non-vanishing of mixed multiplicities of multi-graded fiber cones and determining them in terms of the length of modules are showed by the following theorem.

THEOREM 3.5. *Let J be an $\mathfrak{m}$-primary ideal and $I_1,\ldots,I_s$ ideals such that $I = I_1\cdots I_s$ is not contained in $\sqrt{\mathrm{Ann}M}$. Set $U = (J, I_1,\ldots,I_s)$ and $\ell = \dim\Big(\bigoplus_{n\geqslant 0}\frac{I^nM}{\mathfrak{m}I^nM}\Big)$. Then the following statements hold.*

(i) *$E_J(I_1^{[d_1]},\ldots,I_s^{[d_s]};M) \neq 0$ if and only if there exists a weak-(FC)-sequence $x_1,\ldots,x_{\ell-1}$ in $\bigcup_{i=1}^s I_i$ of M with respect to U consisting of d_1 elements of $I_1,\ldots,d_s$ elements of I_s and $\ell_{\overline{M}}(I) = 1$, where $\overline{M} = \frac{M}{(x_1,\ldots,x_{\ell-1})M}$, $\ell_{\overline{M}}(I) = \dim\Big(\bigoplus_{n\geqslant 0}\frac{I^n\overline{M}}{\mathfrak{m}I^n\overline{M}}\Big)$.*

(ii) *Suppose that $E_J(I_1^{[d_1]},\ldots,I_s^{[d_s]};M) \neq 0$. Let $x_1,\ldots,x_{\ell-1}$ be a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U consisting of d_1 elements of $I_1,\ldots,d_s$ elements of I_s. Set $\overline{M} = \frac{M}{(x_1,\ldots,x_{\ell-1})M}$, $\bar{A} = \frac{A}{(x_1,\ldots,x_{\ell-1})}$, $\bar{I} = I\bar{A}$ and $r = r(\bar{I};\overline{M})$. Then*

$$E_J(I_1^{[d_1]},\ldots,I_s^{[d_s]};M) = l_A\left(\frac{I^nM + (x_1,\ldots,x_{\ell-1})M : I^\infty}{JI^nM + (x_1,\ldots,x_{\ell-1})M : I^\infty}\right)$$

for all $n \geqslant r$.

PROOF. The proof of (i): We first prove the necessity. The proof is by induction on $\ell \geqslant 1$. For $\ell = 1$, the result is trivial. Suppose that the result has been proved for $\ell - 1 \geqslant 1$. As the next step, we claim that the result is true for ℓ. Since $d_1 + \cdots + d_s = \ell - 1 > 0$, there exists $1 \leqslant j \leqslant s$ such that $d_j > 0$. By Lemma 2.4, there exists a weak-(FC)-element $x_1 \in I_j$ of M with respect to U. Set

$$A' = A/(x_1)\,, \quad M' = M/x_1M\,, \quad J' = JA'\,, \quad \mathfrak{m}' = \mathfrak{m}A'\,, \quad I_i' = I_iA'\ (i = 1, \dots, s)\,,$$
$$I' = IA'\,, \quad U' = (J', I_1', \dots, I_s')\,.$$

By Lemma 3.4(i), we have

$$E_J(I_1^{[d_1]}, \dots, I_j^{[d_j-1]}, \dots, I_s^{[d_s]}; M') = E_J(I_1^{[d_1]}, \dots, I_j^{[d_j]}, \dots, I_s^{[d_s]}; M) \neq 0\,.$$

It is clear that

$$E_{J'}(I_1'^{[d_1]}, \dots, I_j'^{[d_j-1]}, \dots, I_s'^{[d_s]}; M') = E_J(I_1^{[d_1]}, \dots, I_j^{[d_j-1]}, \dots, I_s^{[d_s]}; M')\,.$$

So $E_{J'}(I_1'^{[d_1]}, \dots, I_j'^{[d_j-1]}, \dots, I_s'^{[d_s]}; M') \neq 0$. Denote by $P(n_1, \dots, n_s)$ and $Q(n_1, \dots, n_s)$ the polynomials such that

$$P(n_1, \dots, n_s) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{JI_1^{n_1} \cdots I_s^{n_s} M}\right), \quad Q(n_1, \dots, n_s) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M'}{JI_1^{n_1} \cdots I_s^{n_s} M'}\right)$$

for all large $n_1, \dots, n_s$. By Proposition 3.1, $\deg P(n_1, \dots, n_s) = \ell - 1$. Since $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) \neq 0$ and $d_j > 0$, applying Proposition 3.3(iii) we have

$$\deg Q(n_1, \dots, n_s) = \deg P(n_1, \dots, n_s) - 1 = (\ell - 1) - 1 = \ell - 2\,.$$

On the other hand, if set $\ell' = \dim\Big(\bigoplus_{n \geqslant 0} \frac{I'^n M'}{\mathfrak{m}' I'^n M'}\Big)$ then by Proposition 3.1, $\deg Q(n_1, \dots, n_s) = \ell' - 1$. Thus

$$\ell' = \deg Q(n_1, \dots, n_s) + 1 = (\ell - 2) + 1 = \ell - 1\,.$$

By the inductive assumption applied to $\ell' = \ell - 1 \geqslant 1$, there exist $x_2, \dots, x_{\ell-1}$ consisting of d_1 elements of $I_1, \dots, d_j - 1$ elements of $I_j, \dots, d_s$ elements of I_s such that $x_2', \dots, x_{\ell-1}'$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i'$ of M' with respect to U' and $\ell_{M''}(I') = 1$, where x_k' the image of x_k in A' $(k = 2, \dots, \ell - 1)$, $M'' = \frac{M'}{(x_2', \dots, x_{\ell-1}')M'}$ and $\ell_{M''}(I') = \dim\Big(\bigoplus_{n \geqslant 0} \frac{I'^n M''}{\mathfrak{m}' I'^n M''}\Big)$. Set

$$\overline{M} = \frac{M}{(x_1, \dots, x_{\ell-1})M}\,.$$

It is clear that $M'' \simeq \overline{M}$ and $\ell_{\overline{M}}(I) = \ell_{M''}(I')$. Hence $\ell_{\overline{M}}(I) = 1$ and $x_1, x_2, \dots, x_{\ell-1}$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U consisting of d_1 elements of $I_1, \dots, d_j$ elements of $I_j, \dots, d_s$ elements of I_s.

We turn to the proof of sufficiency. The result is proved by induction on $\ell \geqslant 1$. For $\ell = 1$, it implies that $d_1 = \cdots = d_s = 0$ and $l_A\Big(\frac{I_1^{n_1} \dots I_s^{n_s} M}{JI_1^{n_1} \dots I_s^{n_s} M}\Big)$ is a polynomial of degree 0 for all large $n_1, \dots, n_s$. Thus

$$E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) = E_J(I_1^{[0]}, \dots, I_s^{[0]}; M) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}\right)$$

for all large $n_1, \dots, n_s$. Since $I \nsubseteq \sqrt{\mathrm{Ann} M}$, $l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}\right) \neq 0$ for all $n_1, \dots, n_s$. Hence $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) \neq 0$. The result is true for $\ell = 1$. Assume that the result is true for $\ell - 1 \geqslant 1$. As the next step, we show that the result is true for ℓ. Let $x_1, \dots, x_{\ell-1}$ be a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U consisting of d_1 elements of $I_1, \dots, d_s$ elements of I_s and $\ell_{\overline{M}}(I) = 1$. Then $\bar{I} \nsubseteq \sqrt{\mathrm{Ann}_{\bar{A}}(\overline{M})}$, where $\bar{A} = A/(x_1, \dots, x_{\ell-1})$, $\bar{I} = I\bar{A}$. By Lemma 2.4, there exists $x_\ell \in \bigcup_{i=1}^s I_i$ such that $\bar{x}_\ell$ is a weak-(FC)-element in $\bigcup_{i=1}^s \bar{I}_i$ of $\overline{M}$ with respect to $(\bar{J}, \bar{I}_1, \dots, \bar{I}_s)$, where $\bar{x}_\ell$ the image of x_ℓ in $\bar{A}$, $\bar{J} = J\bar{A}$, $\bar{I}_i = I_i\bar{A}$ for $i = 1, \dots, s$. Since $\ell - 1 \geqslant 1$, there exists $1 \leqslant j \leqslant s$ such that $d_j > 0$ and $x_1 \in I_j$. Set

$$A' = A/(x_1)\,, \quad M' = M/x_1M\,, \quad J' = JA'\,, \quad \mathfrak{m}' = \mathfrak{m}A'\,, \quad I_i' = I_iA'(i = 1, \dots, s)\,,$$
$$I' = IA'\,, \quad \ell' = \dim\left(\bigoplus_{n \geqslant 0} \frac{I'^n M'}{\mathfrak{m}' I'^n M'}\right).$$

Denote by $P(n_1, \dots, n_s)$ and $Q(n_1, \dots, n_s)$ the polynomials such that

$$P(n_1, \dots, n_s) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M}{J I_1^{n_1} \cdots I_s^{n_s} M}\right), \quad Q(n_1, \dots, n_s) = l_A\left(\frac{I_1^{n_1} \cdots I_s^{n_s} M'}{J I_1^{n_1} \cdots I_s^{n_s} M'}\right)$$

for all large $n_1, \dots, n_s$. By Proposition 3.1, we have

$$\deg Q(n_1, \dots, n_s) = \ell' - 1 \text{ and } \deg P(n_1, \dots, n_s) = \ell - 1\,.$$

Note that $x_1, \dots, x_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U. By Proposition 3.3,

$$Q(n_1, \dots, n_j, \dots, n_s) = P(n_1, \dots, n_j, \dots, n_s) - P(n_1, \dots, n_j - 1, \dots, n_s)$$

and $\deg Q(n_1, \dots, n_s) = \deg P(n_1, \dots, n_s) - 1$. So that $\ell' = \ell - 1$ and

$$\begin{aligned} E_J(I_1^{[d_1]}, \dots, I_j^{[d_j]}, \dots, I_s^{[d_s]}; M) &= E_J(I_1^{[d_1]}, \dots, I_j^{[d_j-1]}, \dots, I_s^{[d_s]}; M') \\ &= E_{J'}(I_1'^{[d_1]}, \dots, I_j'^{[d_j-1]}, \dots, I_s'^{[d_s]}; M')\,. \end{aligned}$$

By the inductive assumption applied to $\ell' = \ell - 1 \geqslant 1$, it follows that

$$E_{J'}(I_1'^{[d_1]}, \dots, I_j'^{[d_j-1]}, \dots, I_s'^{[d_s]}; M') \neq 0\,.$$

Hence $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) \neq 0$.

The proof of (ii): Set $\ell^* = \dim\Big(\bigoplus_{n\geqslant 0} \frac{\bar{I}^n\overline{M}}{\bar{\mathfrak{m}}\bar{I}^n\overline{M}}\Big)$, $\bar{J} = J\bar{A}$, $\bar{I}_i = I_i\bar{A}$ $(i = 1, \ldots, s)$, $\overline{M}^* = \frac{M}{(x_1,\ldots,x_{\ell-1})M:I^\infty}$. By Lemma 3.4 (ii),

$$l_{\bar{A}}\Big(\frac{\bar{I}^n\overline{M}}{\bar{J}\bar{I}^n\overline{M}}\Big) = l_A\Big(\frac{I^n\overline{M}}{JI^n\overline{M}}\Big) = E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) \neq 0$$

for large n. Therefore, $\bar{I} = \bar{I}_1 \cdots \bar{I}_s \nsubseteq \sqrt{\mathrm{Ann}_{\bar{A}}(\overline{M})}$. By Lemma 2.4, for any $1 \leqslant j \leqslant s$ there exists an element $x_\ell \in I_j$ such that $\bar{x}_\ell$ (the initial form of x_ℓ in $\bar{A}$) is a weak-(FC)-element in $\bar{I}_j$ of $\overline{M}$ with respect to $(\bar{J}, \bar{I}_1, \ldots, \bar{I}_s)$. Then $x_1, \ldots, x_\ell$ is a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to U. Applying Proposition 2.9, we have

$$\ell^* = \ell - (\ell - 1) = 1\,.$$

By Lemma 2.8, there exists a weak-(FC)-element $x \in \bar{I}$ of $\overline{M}$ with respect to $(\bar{J}, \bar{I})$ such that (x) is a minimal reduction of $\bar{I}$ with respect to $\overline{M}$ and $r = r_{(x)}(\bar{I}; \overline{M})$. Therefore $\bar{I}^n\overline{M} = x^{n-r}\bar{I}^r\overline{M}$ for all $n \geqslant r$. It follows that

$$\bar{I}^n\overline{M}^* = x^{n-r}\bar{I}^r\overline{M}^*$$

for all $n \geqslant r$. It is easy to see that

$$l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big) = l_A\Big(\frac{\bar{I}^n\overline{M}}{\bar{J}\bar{I}^n\overline{M} + \bar{I}^n\overline{M}\bigcap(0_{\overline{M}} : \bar{I}^\infty)}\Big).$$

By Artin-Rees Lemma, $\bar{I}^n\overline{M}\bigcap(0_{\overline{M}} : \bar{I}^\infty) = \{0_{\overline{M}}\}$ for all large n. Thus

$$l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big) = l_A\Big(\frac{\bar{I}^n\overline{M}}{\bar{J}\bar{I}^n\overline{M}}\Big) = l_A\Big(\frac{I^n\overline{M}}{JI^n\overline{M}}\Big)$$

for all large n. From these facts and by Lemma 3.4(ii), we have

$$E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = l_A\Big(\frac{I^n\overline{M}}{JI^n\overline{M}}\Big) = l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big)$$

for all large n. It is clear that $l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big) = l_A\Big(\frac{x^{n-r}\bar{I}^r\overline{M}^*}{x^{n-r}\bar{J}\bar{I}^r\overline{M}^*}\Big)$ for all $n \geqslant r$. Since x is a non-zero-divisor in $\overline{M}^*$,

$$l_A\Big(\frac{x^{n-r}\bar{I}^r\overline{M}^*}{x^{n-r}\bar{J}\bar{I}^r\overline{M}^*}\Big) = l_A\Big(\frac{\bar{I}^r\overline{M}^*}{\bar{J}\bar{I}^r\overline{M}^*}\Big)$$

for all $n \geqslant r$. Thus $l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big) = l_A\Big(\frac{\bar{I}^r\overline{M}^*}{\bar{J}\bar{I}^r\overline{M}^*}\Big)$ for all $n \geqslant r$. Hence

$$E_J(I_1^{[d_1]}, \ldots, I_s^{[d_s]}; M) = l_A\Big(\frac{\bar{I}^n\overline{M}^*}{\bar{J}\bar{I}^n\overline{M}^*}\Big) = l_A\Big(\frac{I^nM + (x_1, \ldots, x_{\ell-1})M : I^\infty}{JI^nM + (x_1, \ldots, x_{\ell-1})M : I^\infty}\Big)$$

for all $n \geqslant r$. The proof of Theorem 3.5 is complete. ∎

In the case where $I_1, \dots, I_s$ are $\mathfrak{m}$-primary ideals, it is easy to see that $\ell = d$. By combining Lemma 2.5 and Theorem 3.5(i), we get the following result.

COROLLARY 3.6. *Let $J, I_1, \dots, I_s$ be $\mathfrak{m}$-primary ideals and $d_1, \dots, d_s$ non-negative integers such that $d_1 + \cdots + d_s = d - 1$. Then the following statements hold.*

(i) $E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) \neq 0$.

(ii) *Let $x_1, \dots, x_{d-1}$ be a weak-(FC)-sequence in $\bigcup_{i=1}^s I_i$ of M with respect to $(J, I_1, \dots, I_s)$ consisting of d_1 elements of $I_1, \dots, d_s$ elements of I_s. Set $\bar{A} = \frac{A}{(x_1,\dots,x_{d-1})}$, $\bar{I} = I\bar{A}$, $\overline{M} = \frac{M}{(x_1,\dots,x_{d-1})M}$ and $r = r(\bar{I}; \overline{M})$. Then*

$$E_J(I_1^{[d_1]}, \dots, I_s^{[d_s]}; M) = l_A\left(\frac{I^n M + (x_1, \dots, x_{d-1})M : I^\infty}{JI^n M + (x_1, \dots, x_{d-1})M : I^\infty}\right)$$

for all $n \geqslant r$.

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