Global Classical Solutions of IBVP to Nonlinear Equation of a Suspended String

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Abstract. We are concerned with the existence and uniqueness of the *classical* solution to IBVP for a nonlinear equation of a suspended string with uniform density to which a monotonous nonlinear time-independent outer force works. For this purpose we derive the higher order energy estimates, and employ the Galerkin method combining with the compactness argument along the refined method of [Sath]. We need the regularity theory of Nirenberg type for the suspended string operator L.

1. Introduction

Let Ω be a cylindrical domain $(0, a) \times (0, T)$. Consider IBVP for a nonlinear equation of a flexible and heavy suspended string of finite length a with a uniform density

(P)
$$\begin{cases} \partial_t^2 u(x, t) + L u(x, t) + u(x, t)^3 = 0, & (x, t) \in \Omega, \\ u(a, t) = 0, & t \in (0, T), \\ u(x, 0) = \phi(x), \quad \partial_t u(x, 0) = \psi(x), & x \in (0, a), \end{cases}$$

where L is a second order differential operator of the form

$$L = L_0(x, \partial_x) = -(x \, \partial_x^2 + \partial_x).$$

The operator L is a particular form of suspended string equations introduced by [K-G-S] and [Ko]. As for IBVP to the linear equation of the suspended string with the quasi-periodic forcing term, the existence of almost periodic C^2-solutions was proved in [Ya1], and for several periodic problems of nonlinear equation of suspended string, see [Ya2] and [Ya-Na-Ma]. Quite recently, the present authors proved the existence of time-global weak solutions to these equations with the blowing-up term like $|u|^{p-1}u$, $p > 1$, by using the potential well method (see [Wo-Ya]).

The purpose of this paper is *to show the existence and uniqueness of a classical solution of* (P) *with a monotonous cubic nonlinear term u^3*. It will be shown that the unique classical solution exists for any large initial data (see Main Theorem in §2), while in [Wo-Ya] some boundedness condition related to the potential well on the initial data is assumed. To prove our result we employ the Galerkin method combining with the higher order energy estimates of approximate solutions and compactness argument in the refined way of [Sath], which proved the existence and uniqueness of the classical solution to a nonlinear 3D wave equation. Contrary to [Sath], the difficulty of our problem comes from the singularity of solutions at $x = 0$. We note that L is the special case of the differential operator L_μ (see §3), and degenerates at the origin. Fortunately, the second author made some appropriate function spaces suitable for suspended string operators L_μ (see [Ya1], [Ya2]). As will be seen, our argument will be quite effective in Lebesgue and Sobolev type spaces with some power weight of x. In §2 we review the definitions of these function spaces, and in §3 we will provide the properties of the operator L on these spaces and some Poincaré–Sobolev type inequalities suited for these spaces (see [Ya1], [Ya2], [Ya-Na-Ma], [Wo-Ya]). It will be seen that these spaces have an algebraic property for an appropriate regularity (see Lemma 4.3), which plays important roles in our argument. In §4 we derive the higher order energy estimates of the approximate solutions, and §5 will be devoted to prove our main theorem. The strategy is to use these higher order energy estimates and the compactness method in the standard way. We need the elliptic regularity theory of Nirenberg type for our operator L which enables us to obtain the appropriate regularity of solutions. In the last section this elliptic regularity theorem will be proved as an appendix.

2. Function spaces and Main Theorem

Let R_+^1 and Z_+ be the set of nonnegative numbers and the set of nonnegative integers, respectively. Let $1 \leqq p \leqq +\infty$, $s \in Z_+$ and $\mu > -1$. In the following we assume that all functions are real-valued. For any open set O in R^n, $L^p(O)$ and $H^s(O)$ are the usual Lebesgue and Sobolev spaces, respectively.

We denote by $L^p(0, a; x^\mu)$ a Banach space whose elements $f(x)$ are measurable in $(0, a)$ and satisfy $x^{\mu/p} f(x) \in L^p(0, a)$, and the norm is defined by

$$|f|_{L^p(0,a;x^\mu)} = \left(\int_0^a x^\mu |f(x)|^p dx \right)^{1/p}.$$

In particular, $L^2(0, a; x^\mu)$ is a Hilbert space with inner product

$$(f, g)_{L^2(0,a;x^\mu)} = \int_0^a x^\mu f(x) g(x) dx.$$

We denote by $W^{s,p}(0, a; x^\mu)$ a Banach space whose elements $f(x)$ and their weighted derivatives $x^{j/p} f^{(j)}(x)$, $j = 1, \ldots, s$, belong to $L^p(0, a; x^\mu)$, where $f^{(j)}(x)$ means the j-th derivative of $f(x)$. Its norm is defined by

$$|f|_{W^{s,p}(0,a;x^\mu)} = \left(\sum_{j=0}^{s} \int_0^a x^{\mu+j} |f^{(j)}(x)|^p dx \right)^{1/p}.$$

We set $H^s(0, a; x^\mu) = W^{s,2}(0, a; x^\mu)$ that is a Hilbert space with naturally induced inner product.

Let $T > 0$, and put $\Omega = (0, a) \times (0, T)$. We denote by $L^p(\Omega; x^\mu)$ a Banach space whose elements $f(x, t)$ are measurable in Ω and satisfy $x^{\mu/p} f(x, t) \in L^p(\Omega)$, where the norm is defined by

$$|f|_{L^p(\Omega;x^\mu)} = \left(\int_\Omega x^\mu |f(x, t)|^p dx dt \right)^{1/p}.$$

We denote by $H^s(\Omega; x^\mu)$ a Hilbert space whose elements $f(x, t)$ and their weighted derivatives $x^{j/2} \partial_x^j \partial_t^k f(x, t)$, $0 \leq j + k \leq s$, belong to $L^2(\Omega; x^\mu)$. Its norm is defined by

$$|f|_{H^s(\Omega;x^\mu)} = \left(\sum_{j+k \leq s} \int_\Omega x^{\mu+j} (\partial_x^j \partial_t^k f(x, t))^2 dx dt \right)^{1/2}.$$

$H_0^1(0, a; x^\mu)$ is a subspace of $H^1(0, a; x^\mu)$ whose elements $f(x)$ satisfy $f(a) = 0$. Similarly, $H_0^1(\Omega; x^\mu)$ is a subspace of $H^1(\Omega; x^\mu)$ whose elements $f(x, t)$ satisfy $f(a, t) = 0$ for a. e. $t \in I$. $K^s(0, a; x^\mu)$ is a subspace of $H^s(0, a; x^\mu)$ whose elements $f = f(x)$ satisfy $L_\mu^j f \in H_0^1(0, a; x^\mu)$ for $j = 0, \ldots, [(s - 1)/2]$. Note that

$$K^0(0, a; x^\mu) = L^2(0, a; x^\mu), \quad K^1(0, a; x^\mu) = H_0^1(0, a; x^\mu),$$

$$K^2(0, a; x^\mu) = H^2(0, a; x^\mu) \cap H_0^1(0, a; x^\mu).$$

Let X be a Banach space and J an interval of R^1. We denote by $C^j(J; X)$ the set of j-times continuously differentiable mappings of J into X.

We are now in a position to state our main theorem. We will always assume the following condition on the initial data:

(A) ϕ belongs to $K^4(0, a; x^0)$ and ψ belongs to $K^3(0, a; x^0)$.

We set $I = [0, T]$ for any fixed $0 < T < +\infty$. Then our main result reads as follows.

MAIN THEOREM.   Assume (A). Then IBVP (P) has a unique global solution u in $C^2((0, a] \times I)$.

3. Some properties of the operator L_μ

3.1. The Basic Properties of L_μ. We recall the more general suspended string operator L_μ for $\mu > -1$:

$$(3.1) \qquad L_\mu = -\left(\frac{x}{\mu+1}\partial_x^2 + \partial_x\right).$$

Needless to say, L_μ coincides with our differential operator L for $\mu = 0$. Then we have

PROPOSITION 3.1 ([Ya1]). *Let L_μ be as in (3.1) for $\mu > -1$. Then we have the following assertions*:

(i) *For $f \in K^2(0,a;x^\mu)$ and $g \in K^1(0,a;x^\mu)$, we have*

$$(L_\mu f, g)_{L^2(0,a;x^\mu)} = \int_0^a \frac{x^{\mu+1}}{\mu+1}\partial_x f(x)\partial_x g(x)dx.$$

(ii) *L_μ is a positive definite and self-adjoint elliptic operator on $L^2(0,a;x^\mu)$ with domain $D(L) = K^2(0,a;x^\mu)$, i.e.,*

$$(L_\mu f, f)_{L^2(0,a;x^\mu)} \ge 0, \quad (L_\mu f, g)_{L^2(0,a;x^\mu)} = (f, L_\mu g)_{L^2(0,a;x^\mu)}$$

for $f, g \in K^2(0,a;x^\mu)$.

(iii) *For $k, \ell \in Z_+$, there exist constants C_i, $i = 1, 2$, depending on k, ℓ, μ such that*

$$C_1|f|_{H^{2k+\ell}(0,a;x^\mu)} \le |L^k f|_{H^\ell(0,a;x^\mu)} \le C_2|f|_{H^{2k+\ell}(0,a;x^\mu)}$$

for $f \in K^{2k+\ell}(0,a;x^\mu)$.

3.2. Imbedding theorems and Rellich type theorem

PROPOSITION 3.2 ([Wo-Ya]). *Let $s \ge 1$. Then we have the following assertions*:

(i) *$H^s(0,a;x^\mu)$ is continuously embedded in $L^2(0,a;x^{\mu-\delta})$ for any δ with $\delta \le 1$ and $\delta < \mu + 1$ for $\mu > -1$.*

(ii) *$H^s(0,a;x^0)$ is continuously embedded in $L^r(0,a;x^k) \cap C^{s-1}((0,a])$ for $k \in R_+$ and $1 \le r \le +\infty$. Moreover, we have*

$$|u|_{L^r(0,a;x^k)} \le C\,|u|_{H^s(0,a;x^0)}, \quad u \in H^s(0,a;x^0),$$

where C depends on r, a and is independent of s.

As a consequence of Proposition 3.2, we have

COROLLARY 3.3 ([Wo-Ya]). *Let $s \ge 1, k \ge 0, 1 \le r < +\infty$ and $\mu > -1$. Then, for $u \in H^s(\Omega;x^\mu)$ we have*

$$|u|_{L^r(\Omega;x^{\mu+k})} \le C\,|u|_{H^s(\Omega;x^\mu)},$$

where C depends on r, μ, a and is independent of s, T.

The next lemma is the Poincaré type inequality in $H_0^1(0, a; x^\mu)$.

LEMMA 3.4 ([Ya1]).   *Let $\mu > -1$. Then, for $u \in H_0^1(0, a; x^\mu)$, we have*

$$|u|_{L^2(0,a;x^\mu)} \leqq a \, |\partial_x u|_{L^2(0,a;x^{\mu+1})} \, .$$

The next lemma is crucial in our argument.

LEMMA 3.5 ([Ya2]).   *Let $p \geqq 1$ and $\mu > -1$. Then, for $u \in W^{1,p}(0, a; x^\mu)$ we have*

$$\int_0^a x^\mu |u(x)|^p dx \leqq C_1 \left(\int_0^a x^{\mu+1} |u(x)|^p dx + \int_0^a x^{\mu+p} |u'(x)|^p dx \right),$$

$$\sup_{x \in (0,a)} x^\mu |u(x)|^p \leqq C_2 \left(\int_0^a x^{\mu+1} |u(x)|^p dx + \int_0^a x^{\mu+p-1} |u'(x)|^p dx \right),$$

where the constants $C_i > 0$ depend on μ, a, p.

The following proposition is the Rellich type theorem in our weighted Sobolev space.

PROPOSITION 3.6 ([Ya-Na-Ma]).   *Let $\mu > -1$. Then any bounded set of $H_0^1(\Omega; x^\mu)$ is compact in $L^2(\Omega; x^\mu)$. Moreover, any bounded set of $H_0^1(0, a; x^\mu)$ is compact in $L^2(0, a; x^\mu)$.*

4.   Higher order energy estimates of (P)

In this section we derive the higher order a priori estimates of the Galerkin approximate solutions. From now on we will always assume $\mu = 0$ in our function spaces and the differential operator L_μ of the previous section. For simplicity, we abbreviate $L^p(0, a; x^0)$, $H^s(0, a; x^0)$ and $K^s(0, a; x^0)$ as L^p, H^s and K^s, respectively, and the inner product $(\cdot, \cdot)_{L^2(0,a;x^0)}$ and the norm $|\cdot|_{L^2(0,a;x^0)}$ as $(\cdot, \cdot)_{L^2}$ and $|\cdot|_{L^2}$. Also we write $|\cdot|_{L^p(0,a;x^0)}$ as $|\cdot|_{L^p}$. Furthermore we write

$$\|u\| = ((u, u))^{1/2} = \left(\int_0^a x \partial_x u(x)^2 dx \right)^{1/2},$$

which is the norm in H_0^1 and equivalent to $|\cdot|_{H^1}$ on account of Lemma 3.4.

Let us state key estimates by dividing into two subsections.

4.1.   Higher Order Energy Estimates of Solutions of (P)

PROPOSITION 4.1.   *Let $u = u(x, t) \in C^\infty(I; K^\infty(0, a; x^0))$ be a solution of (P). Then, for any $j = 0, 1, \ldots$, there exist constants A_j depending on the norms $|\partial_t^{j+1} u(\cdot, 0)|_{L^2}$, $\|\partial_t^j u(\cdot, 0)\|$ and T, and a constant β such that*

(4.1) $$|\partial_t^{j+1} u(\cdot, t)|_{L^2}^2 + \|\partial_t^j u(\cdot, t)\|^2 \leqq (|\partial_t^{j+1} u(\cdot, 0)|_{L^2}^2 + \|\partial_t^j u(\cdot, 0)\|^2 + A_j \tau) e^{\beta \tau}$$

holds on any interval $[0, \tau)$ of I.

PROOF.   We use Proposition 3.1 (i) to obtain

$$0 = (\partial_t^2 u + Lu + u^3, \partial_t u)_{L^2} = \frac{1}{2}\frac{d}{dt}(|\partial_t u|_{L^2}^2 + \|u\|^2) + \frac{1}{4}\frac{d}{dt}|u|_{L^4}^4,$$

which implies

$$|\partial_t u(\cdot, t)|_{L^2}^2 + \|u(\cdot, t)\|^2 + |u(\cdot, t)|_{L^4}^4 = |\partial_t u(\cdot, 0)|_{L^2}^2 + \|u(\cdot, 0)\|^2 + |u(\cdot, 0)|_{L^4}^4$$
$$\leqq |\partial_t u(\cdot, 0)|_{L^2}^2 + |u(\cdot, 0)|_{H^1}^2 + C|u(\cdot, 0)|_{H^1}^4$$

for all $t \in [0, \tau)$, where we used Proposition 3.2 in the last step.

Differentiating the equation in (P) j-times with respect to t, multiplying by $\partial_t^{j+1} u$, and integrating from 0 to a with respect to x, we obtain

$$(\partial_t^{j+2} u + L\partial_t^j u + \partial_t^j u^3, \partial_t^{j+1} u)_{L^2} = 0,$$

which implies

$$(4.2) \qquad \frac{d}{dt}(|\partial_t^{j+1} u|_{L^2}^2 + \|\partial_t^j u\|^2) = -2(\partial_t^j u^3, \partial_t^{j+1} u)_{L^2} \leqq |\partial_t^j u^3|_{L^2}^2 + |\partial_t^{j+1} u|_{L^2}^2.$$

Here we can write

$$\partial_t^j u^3 = 3u^2 \partial_t^j u + Q_j(u, \partial_t u, \ldots, \partial_t^{j-1} u),$$

where $Q_j(X_0, X_1, \ldots, X_{j-1})$ is a certain polynomial of degree 3 in variables $X_0, X_1, \ldots, X_{j-1}$. If we use the following inequality

$$(4.3) \qquad |fgh|_{L^2} \leqq C\|f\|\|g\|\|h\|, \quad f, g, h \in H_0^1,$$

then we get

$$(4.4) \qquad |u^2 \partial_t^j u|_{L^2} \leqq C\|u\|^2 \|\partial_t^j u\| \leqq C_0 \|\partial_t^j u\|,$$

$$(4.5) \qquad |Q_j(u, \partial_t u, \ldots, \partial_t^{j-1} u)|_{L^2} \leqq C Q_j(\|u\|, \|\partial_t u\|, \ldots, \|\partial_t^{j-1} u\|),$$

where we set

$$C_0 = C(|\partial_t u(\cdot, 0)|_{L^2}^2 + |u(\cdot, 0)|_{H^1}^2 + C|u(\cdot, 0)|_{H^1}^4).$$

The inequality (4.3) follows from Hölder's inequality, Proposition 3.2 and Lemma 3.4. Hence combining (4.2) with (4.4)–(4.5), we arrive at

(4.6)
$$\frac{d}{dt}(|\partial_t^{j+1} u|^2 + \|\partial_t^j u\|^2) \leqq |\partial_t^{j+1} u|_{L^2}^2 + C\{C_0 \|\partial_t^j u\| + Q_j(\|u\|, \|\partial_t u\|, \ldots, \|\partial_t^{j-1} u\|)\}.$$

Since $Q_1 = 0$, it follows from (4.6) with $j = 1$ that (4.1) holds for $j = 1$ on account of Gronwall's lemma. Moreover, we have

$$Q_2(\|u\|, \|\partial_t u\|) \leq C(\|u(0)\|, \|\partial_t u(0)\|).$$

Applying this bound to (4.6) and using again Gronwall's lemma, we conclude (4.1) holds for $j = 2$. Thus the estimate (4.1) for $j \geqq 3$ follows from the induction argument. The proof of Proposition 4.1 is now finished. □

4.2. Higher Order Estimates of Approximate Solutions. In what follows we construct the approximate solutions of (P) by using the Galerkin method (cf. [Wo-Ya]), and obtain the higher order energy estimates. For this purpose we need to impose the suitable smoothness on the initial data as in assumption (A).

Let us take the approximate solutions to (P) of the form

$$u_m(x, t) = \sum_{i=1}^{m} a_i^m(t)\, \phi_i(x), \quad m = 1, 2, \dots,$$

satisfying

$$(4.7) \qquad (\partial_t^2 u_m + L u_m + u_m^3, \phi_k)_{L^2} = 0, \quad k = 1, \dots, m,$$

with the initial condition

$$(4.8) \qquad u_m(x, 0) = \sum_{i=1}^{m} (\phi, \phi_i)_{L^2} \phi_i(x), \quad \partial_t u_m(x, 0) = \sum_{i=1}^{m} (\psi, \phi_i)_{L^2} \phi_i(x).$$

Hence (4.7)–(4.8) is IVP to a system of second-order ODEs with unknown functions $a_1^m(t), \dots, a_m^m(t)$. It follows from assumption (A) that

(4.9)

$$\left| \sum_{i=1}^{m} (\phi, \phi_i)\, \phi_i(\cdot) - \phi(\cdot) \right|_{H^4} \to 0, \quad \left| \sum_{i=1}^{m} (\psi, \phi_i)\, \phi_i(\cdot) - \psi(\cdot) \right|_{H^3} \to 0 \quad \text{as } m \to +\infty.$$

The next proposition plays a central role in our argument.

PROPOSITION 4.2. *Assume* (A). *Let u_m be the solutions of* (4.7)–(4.8). *Then we have*

$$(4.10) \qquad |\partial_t^{j+1} u_m(\cdot, t)|_{L^2} + \|\partial_t^j u_m(\cdot, t)\| \leqq C_{j+1}, \quad 0 \leqq j \leqq 3, \quad t \in I,$$

where the constant C_{j+1} depends on T, $|\phi|_{H^{j+1}}$ and $|\psi|_{H^j}$, but is independent of m.

PROOF. Differentiating equation (4.7) j-times with respect to t, multiplying by $\frac{d^{j+1}}{dt^{j+1}} a_k^m(t)$ and summing up from $k = 1$ to m, we have

$$(\partial_t^{j+2} u_m + L\partial_t^j u_m + \partial_t^j u_m^3, \partial_t^{j+1} u_m)_{L^2} = 0.$$

Hence we obtain the estimate (4.1) of Proposition 4.1 for u replaced by u_m, i.e.,

$$|\partial_t^{j+1} u_m(\cdot, t)|_{L^2}^2 + \|\partial_t^j u_m(\cdot, t)\|^2 \leqq (|\partial_t^{j+1} u_m(\cdot, 0)|_{L^2}^2 + \|\partial_t^j u_m(\cdot, 0)\|^2 + A_j \tau) e^{\beta \tau}$$

for $t \in [0, \tau)$. If we establish the following bounds independent of m,

$$(4.11) \qquad |\partial_t^{j+1} u_m(\cdot, 0)|_{L^2} + \|\partial_t^j u_m(\cdot, 0)\| \leqq C_{j+1}, \quad 0 \leqq j \leqq 3,$$

for $m = 1, 2, \ldots$, then we have the required estimate (4.10). These bounds will be given in Lemmas 4.4–4.5 below.   □

In what follows, we pay an attention to prove (4.11). The crucial point of the treatment of our cubic nonlinear term is given by the following lemma, which states that H^s forms an algebra for a suitable integer s.

LEMMA 4.3.   *Let $f, g, h \in K^s$ for some integer $1 \leqq s \leqq 4$. Then $fgh \in K^s$ and there exists a constant C such that*

(4.12)
$$|fgh|_{H^s} \leqq C|f|_{H^s}|g|_{H^s}|h|_{H^s} .$$

In particular, let $f, g \in K^s$ for $s = 1, 2$. Then $fg \in K^s$ and we have

(4.13)
$$|fg|_{H^s} \leqq C|f|_{H^s}|g|_{H^s} .$$

PROOF.   Since (4.13) is an immediate consequence of (4.12), we may prove only (4.12). In the following equation

$$|fgh|_{H^s}^2 = \sum_{k=0}^{s} \int_0^a x^k |(fgh)^{(k)}|^2 dx ,$$

it suffices to estimate only the term for $k = s$, since the other terms can be estimated in the similar way. We see from Leibniz rule that

$$(fgh)^{(s)} = \sum_{j=0}^{s} \sum_{l=0}^{j} C_{jls} f^{(j-l)} g^{(l)} h^{(s-j)} .$$

Hence we have

$$\int_0^a x^s |(fgh)^{(s)}|^2 dx \leqq \sum_{j=0}^{s} \sum_{l=0}^{j} C_{jls} \int_0^a x^s |f^{(j-l)}(x)|^2 |g^{(l)}(x)|^2 |h^{(s-j)}|^2 dx$$

$$= \int_0^a |f(x)|^2 |g(x)|^2 x^s |h^{(s)}(x)|^2 dx$$

(4.14)
$$+ \sum_{j=1}^{s} C_{j0s} \int_0^a x^j |f^{(j)}(x)|^2 |g(x)|^2 x^{s-j} |h^{(s-j)}(x)|^2 dx$$

$$+ \sum_{j=1}^{s} \sum_{l=1}^{j} C_{jls} \int_0^a x^{j-l} |f^{(j-l)}(x)|^2 \cdot x^l |g^{(l)}(x)| \cdot x^{s-j} |h^{(s-j)}(x)|^2 dx .$$

Here it follows from Lemma 3.5 that

$$\sup_{x \in (0,a)} |f(x)|^2 \leqq C \left(\int_0^a x|f(x)|^2 dx + \int_0^a x|f'(x)|^2 dx \right) \leqq C|f|_{H^1}^2 ,$$

and in a similar way, we have $\sup_{x\in(0,a)}|g(x)|^2 \leqq C|g|_{H^1}^2$. Therefore, the first term in the right-hand side of (4.14) can be estimated by

$$\int_0^a |f(x)|^2|g(x)|^2 x^s|h^{(s)}(x)|^2 dx \leqq \sup_{x\in(0,a)}|f(x)|^2 \sup_{x\in(0,a)}|g(x)|^2 \int_0^a x^s|h^{(s)}(x)|^2 dx$$

$$(4.15) \qquad\qquad\qquad\qquad \leqq C|f|_{H^1}^2|g|_{H^1}^2|h|_{H^s}^2 .$$

Next we have, for $j = 1, \ldots, s$,

$$\sup_{x\in(0,a)} x^{s-j}|h^{(s-j)}(x)|^2$$

$$(4.16) \qquad\qquad \leqq C\left(\int_0^a x^{s-j+1}|h^{(s-j)}(x)|^2 dx + \int_0^a x^{s-j+1}|h^{(s-j+1)}(x)|^2 dx\right)$$

$$\leqq C|h|_{H^s}^2 ,$$

and hence, the second term in the right-hand side of (4.14) can be estimated by

$$\sum_{j=1}^s C_{j0s} \int_0^a x^j|f^{(j)}(x)|^2|g(x)|^2 x^{s-j}|h^{(s-j)}(x)|^2 dx$$

$$(4.17) \qquad \leqq \sum_{j=1}^s C_{j0s} \sup_{x\in(0,a)}|g(x)|^2 \sup_{x\in(0,a)} x^{s-j}|h^{(s-j)}(x)|^2 \int_0^a x^j|f^{(j)}(x)|^2 dx$$

$$\leqq C|f|_{H^s}^2|g|_{H^1}^2|h|_{H^s}^2,$$

since $\sup_{x\in(0,a)}|g(x)|^2 \leqq C|g|_{H^1}^2$. In the similar way, since we have, for $1 \leqq l \leqq j \leqq s$,

$$\sup_{x\in(0,a)} x^{j-l}|f^{(j-l)}(x)|^2 \leqq C\left(\int_0^a x^{j-l+1}|f^{(j-l)}(x)|^2 dx + \int_0^a x^{j-l+1}|f^{(j-l+1)}(x)|^2 dx\right)$$

$$\leqq C(|f|_{H^{s-1}}^2 + |f|_{H^s}^2),$$

the last term in the right-hand side of (4.14) can be estimated by

(4.18)

$$\sum_{j=1}^s \sum_{l=1}^j C_{jls} \int_0^a x^{j-l}|f^{(j-l)}(x)|^2 \cdot x^l|g^{(l)}(x)| \cdot x^{s-j}|h^{(s-j)}(x)|^2 dx$$

$$\leqq \sum_{j=1}^s \sum_{l=1}^j C_{jls} \sup_{x\in(0,a)} x^{j-l}|f^{(j-l)}(x)|^2 \sup_{x\in(0,a)} x^{s-j}|h^{(s-j)}(x)|^2 \int_0^a x^l|g^{(l)}(x)| dx$$

$$\leqq C|f|_{H^s}^2|g|_{H^s}^2|h|_{H^s}^2 ,$$

where we used (4.16). Summarizing (4.14)–(4.18), we conclude the estimate (4.12).

It can be checked that $L^j(fgh) \in H_0^1$ for $j = 0, \dots, [(s-1)/2]$, provided $1 \leqq s \leqq 4$, and hence, we conclude $fgh \in K^s$ if $f, g, h \in K^s$. Finally, if $f, g \in K^s$ for $s = 1, 2$, then $fg \in K^s$, since $fg \in H_0^1$. The proof of Lemma 4.3 is complete. $\qquad\square$

Let us define a projector P_m of $L^2(0, a; x^0)$ into itself by the cut-off Bessel–Fourier series

$$P_m u = \sum_{i=1}^{m} u_i \, \phi_i \,, \quad u_i = (u, \phi_i) \,.$$

Note that P_m commutes the operators L and ∂_t, and satisfies $|P_m f|_{L^2} \leqq |f|_{L^2}$. Clearly (4.7) is equivalent to the following equation

$$(4.19) \qquad \partial_t^2 u_m + L u_m + P_m u_m^3 = 0 \,.$$

Differentiating (4.19) j-times with respect to t for $0 \leqq j \leqq 2$ and putting $t = 0$, we obtain

$$\partial_t^{j+2} u_m(x, 0) = -L \partial_t^j u_m(x, 0) - P_m \partial_t^j u_m(x, 0)^3 \,.$$

Then putting $j = 1, 2$ into this equation and using (4.19) again, we have the following useful equations:

$$(4.20) \qquad \partial_t^3 u_m(x, 0) = -L \partial_t u_m(x, 0) - P_m \partial_t u_m(x, 0)^3 \,,$$

$$(4.21) \qquad \partial_t^4 u_m(x, 0) = L^2 u_m(x, 0) + L P_m u_m(x, 0)^3 - P_m \partial_t^2 u_m(x, 0)^3 \,.$$

Now we are in a position to prove (4.11) by stating the following two lemmas.

LEMMA 4.4. *Let u_m be the solutions of (4.7)–(4.8). Then for $0 \leqq j \leqq 3$ we have*

$$(4.22) \qquad |\partial_t^{j+1} u_m(\cdot, 0)|_{L^2} \leqq C_{j+1} \,,$$

where C_{j+1} is a certain polynomial in $|\phi|_{H^{j+1}}$ and $|\psi|_{H^j}$, independent of m.

PROOF. (4.22) for $j = 0$ follows immediately from the energy conservation law. We divide the proof of (4.22) into three cases.

CASE: $j = 1$. In this case we can use (4.19) at $t = 0$, and it follows from Proposition 3.1 (iii) and (4.3) that

$$|\partial_t^2 u_m(\cdot, 0)|_{L^2} \leqq |L u_m(\cdot, 0)|_{L^2} + |u_m(\cdot, 0)^3|_{L^2} \leqq C(|u_m(\cdot, 0)|_{H^2} + \|u_m(\cdot, 0)\|^3)$$
$$\leqq C(|\phi|_{H^2} + |\phi|_{H^1}^3)$$

for large m, which proves (4.22) for $j = 1$.

CASE: $j = 2$. In this case, we can use (4.20), and using again Proposition 3.1 (iii) and (4.3), we have

$$|\partial_t^3 u_m(\cdot, 0)|_{L^2} \leqq |L \partial_t u_m(\cdot, 0)|_{L^2} + 3|u_m(\cdot, 0)^2 \partial_t u_m(\cdot, 0)|_{L^2}$$

$$\leqq C(|\partial_t u_m(\cdot,0)|_{H^2} + \|u_m(\cdot,0)\|^2 \|\partial_t u_m(\cdot,0)\|)$$
$$\leqq C(|\psi|_{H^2} + |\phi|^2_{H^1}|\psi|_{H^1})$$

for large m, which proves (4.22) for $j = 2$.

CASE: $j = 3$.   In this case, we can use (4.21), and it follows from Proposition 3.1 (iii) and (4.3) that

$$|\partial_t^4 u_m(\cdot,0)|_{L^2} \leqq |L^2 u_m(\cdot,0)|_{L^2} + |Lu_m(\cdot,0)^3|_{L^2} + 3|u_m(\cdot,0)^2\partial_t^2 u_m(\cdot,0)|_{L^2}$$
$$+ 6|u_m(\cdot,0)(\partial_t u_m(\cdot,0))^2|_{L^2}$$
$$\leqq C(|u_m(\cdot,0)|_{H^4} + |u_m(\cdot,0)^3|_{H^2}) + 3|u_m(\cdot,0)^2\partial_t^2 u_m(\cdot,0)|_{L^2}$$
$$+ 6|u_m(\cdot,0)(\partial_t u_m(\cdot,0))^2|_{L^2}\,.$$

Here we have, by Lemma 4.3,

$$|u_m(\cdot,0)^3|_{H^2} \leqq C|u_m(\cdot,0)|^3_{H^2} \leqq C|\phi|^3_{H^2}\,,$$

and by the estimate of case $j = 1$,

$$|u_m(\cdot,0)^2\partial_t^2 u_m(\cdot,0)|_{L^2} \leqq |u_m(\cdot,0)|^2_{L^\infty}|\partial_t^2 u_m(\cdot,0)|_{L^2} \leqq C|\phi|^2_{H^1}(|\phi|_{H^2} + |\phi|^3_{H^1})\,,$$
$$|u_m(\cdot,0)(\partial_t u_m(\cdot,0))^2|_{L^2} \leqq |u_m(\cdot,0)|_{L^2}|\partial_t u_m(\cdot,0)|^2_{L^\infty} \leqq C|\phi|_{L^2}|\psi|^2_{H^1}\,.$$

Combining the estimates obtained now, we arrive at the estimate (4.22) for $j = 3$. The proof of Lemma 4.4 is complete.   □

LEMMA 4.5.   *Let u_m be the solutions of (4.7)–(4.8). Then for $0 \leqq j \leqq 3$ we have*

(4.23) $$\|\partial_t^j u_m(\cdot,0)\| \leqq C_{j+1}\,,$$

where each C_{j+1} is a certain polynomial of $|\phi|_{H^{j+1}}$ and $|\psi|_{H^j}$, but is independent of m.

PROOF.   We note that there exists a constant $C > 1$ such that

(4.24) $$C^{-1}\|u\| \leqq |L^{1/2}u|_{L^2} \leqq C\|u\|$$

holds for any $u \in K^1$, which can be easily proved by using the spectral representation of L (see Appendix).

Since (4.23) for $j = 0, 1$ is trivial, it suffices to prove the cases $j = 2, 3$. In what follows we will use Lemma 4.3 without any comment.

THE CASE: $j = 2$.   Using (4.19) and (4.24), we have

$$\|\partial_t^2 u_m(\cdot,0)\| \leqq C(|L^{3/2}u_m(\cdot,0)|_{L^2} + |L^{1/2}u_m(\cdot,0)^3|_{L^2})$$
$$\leqq C(|u_m(\cdot,0)|_{H^3} + |u_m(\cdot,0)|^3_{H^1})$$
$$\leqq C(|\phi|_{H^3} + |\phi|^3_{H^1})\,,$$

which proves (4.23) for $j = 2$.

THE CASE: $j = 3$.   Using (4.20) and (4.24), we have

$$\|\partial_t^3 u_m(\cdot, 0)\| \leqq C(|L^{3/2}\partial_t u_m(\cdot, 0)|_{L^2} + |L^{1/2}\partial_t u_m(\cdot, 0)^3|_{L^2})$$

$$\leqq C(|\partial_t u_m(\cdot, 0)|_{H^3} + |\partial_t u_m(\cdot, 0)^3|_{H^1})$$

$$\leqq C(|\partial_t u_m(\cdot, 0)|_{H^3} + |u_m(\cdot, 0)^2 \partial_t u_m(\cdot, 0)|_{H^1})$$

$$\leqq C(|\partial_t u_m(\cdot, 0)|_{H^3} + |u_m(\cdot, 0)|_{H^1}^2 |\partial_t u_m(\cdot, 0)|_{H^1})$$

$$\leqq C(|\psi|_{H^3} + |\phi|_{H^1}^2 |\psi|_{H^1}),$$

which proves (4.23) for $j = 3$. The proof of Lemma 4.5 is now finished.   □

5. Proof of Main Theorem

Recalling the abbreviation $H^s = H^s(0, a; x^0)$ and $W^{s,p} = W^{s,p}(0, a; x^0)$, we shall prove here the following.

THEOREM 5.1.   *Suppose assumption* (A). *Then* (P) *has a unique solution u satisfying*

$$u \in C^3(I; L^2) \bigcap_{j=0}^{2} C^j(I; H^{3-j} \cap H_0^1).$$

The following remark is important.

REMARK 5.2.   In addition to the result of Theorem 5.1, it should be mentioned that our problem (P) is H^4 well-posed. More precisely, the solution u from Theorem 5.1 also satisfies

(5.1) $$u \in W^{4,\infty}(I; L^2) \bigcap_{j=0}^{3} W^{j,\infty}(I; H^{4-j} \cap H_0^1).$$

The proof of Theorem 5.1 will be done together with the proof of (5.1).

5.1. Compactness of Approximate Solutions

LEMMA 5.2.   *Let u_m be the approximate solutions of* (4.7)–(4.8). *Then there exists a limiting function u of $\{u_m\}$ along a subsequence such that, for $0 \leqq j \leqq 3$,*

$$\partial_t^j u \in H_0^1(\Omega; x^0) \cap L^\infty(I; H_0^1(0, a; x^0)), \quad \partial_t^4 u \in L^2(\Omega; x^0) \cap L^\infty(I; L^2(0, a; x^0)),$$

$$\partial_t^j u_m \to \partial_t^j u \quad \text{weakly in } H_0^1(\Omega; x^0),$$

$$\partial_t^j u_m \to \partial_t^j u \quad \text{weakly* in } L^\infty(I; H_0^1(0, a; x^0)),$$

$$\partial_t^j u_m \to \partial_t^j u \quad \text{strongly in } L^2(\Omega; x^0), \qquad\qquad \text{as } m \to +\infty,$$

$$\partial_t^j u_m \to \partial_t^j u \quad \text{strongly in } L^\infty(I; L^2(0, a; x^0)),$$

$$\partial_t^j u_m(x,t) \to \partial_t^j u(x,t) \quad \text{a.e. in } \Omega\,,$$

$$\partial_t^4 u_m \to \partial_t^4 u \quad \text{weakly in } L^2(\Omega; x^0)\,,$$

$$\partial_t^4 u_m \to \partial_t^4 u \quad \text{weakly* in } L^\infty(I; L^2(0,a; x^0))\,,$$

where each derivative $\partial_t^j u$ is taken in the distributional sense.

PROOF. Since $\{\partial_t^j u_m\}$ $(j = 0,\ldots,3)$ and $\{\partial_t^4 u_m\}$ are uniformly bounded in $H_0^1(\Omega; x^0)$ and $L^2(\Omega; x^0)$, respectively, on account of Proposition 4.2, it follows from Proposition 3.6 that there exists a limiting function w_j of $\{\partial_t^j u_m\}$ along a subsequence such that, for $0 \leqq j \leqq 3$,

$$\partial_t^j u_m \to w_j \quad \text{weakly in } H_0^1(\Omega; x^0)\,,$$

$$\partial_t^j u_m \to w_j \quad \text{weakly* in } L^\infty(I; H_0^1(0,a; x^0))\,,$$

$$\partial_t^j u_m \to w_j \quad \text{strongly in } L^2(\Omega; x^0)\,, \qquad \text{as } m \to +\infty\,,$$

$$\partial_t^j u_m \to w_j \quad \text{strongly in } L^\infty(I; L^2(0,a; x^0))\,,$$

$$\partial_t^4 u_m \to w_4 \quad \text{weakly in } L^2(\Omega; x^0)\,,$$

$$\partial_t^4 u_m \to w_4 \quad \text{weakly* in } L^\infty(I; L^2(0,a; x^0))\,,$$

and hence, each w_j $(j = 0,\ldots,3)$ belongs to $L^\infty(I; H_0^1(0,a; x^0))$ and further, w_4 belongs to $L^\infty(I; L^2(0,a; x^0))$. Put $u = w_0$. Then it can be readily checked that $\partial_t^j u = w_j$ for $j = 0,\ldots,4$ by the standard distributional argument. Finally, as is well-known, since the convergent sequence in $L^\infty(I; L^2(0,a; x^0))$ has a convergent subsequence a.e. in Ω, it follows that each $\partial_t^j u_m(x,t)$ converges to $\partial_t^j u(x,t)$ a.e. in Ω. Thus Lemma 5.2 is proved. $\square$

The following lemma gives the uniform convergence of $\{\partial_t^j u_m(\cdot,t)\}$ with respect to t.

LEMMA 5.3. *Let u_m be as in Lemma 5.2. Then the limiting function $\partial_t^j u$ of Lemma 5.2 belongs to $C(I; H_0^1(0,a; x^0))$ for $j = 0,1,2$ and $\partial_t^3 u$ belongs to $C(I; L^2(0,a; x^0))$, and there exists a subsequence, denoted by the same, of $\{u_m\}$ such that*

$$(5.2) \qquad \sup_{t \in I} |\partial_t^j u_m(\cdot,t) - \partial_t^j u(\cdot,t)|_{L^2} \to 0 \quad \text{for } 0 \leqq j \leqq 3 \text{ as } m \to +\infty\,.$$

PROOF. Since the sequence $\{\partial_t^j u_m(\cdot,t)\}$ is uniformly bounded on I in H_0^1 for $j = 0,1,2$, or even $j = 3$ by Proposition 4.2, we have, for $j = 0,1,2$.

(5.3)

$$|((\partial_t^j u_m(\cdot,t_1) - \partial_t^j u_m(\cdot,t_2), \rho))| \leqq \left| \int_{t_1}^{t_2} |((\partial_\tau^{j+1} u_m(\cdot,\tau), \rho))| \, d\tau \right| \leqq C_{j+2}|t_1 - t_2| \|\rho\|$$

for all $\rho \in H_0^1$ and all $t_1, t_2 \in I$. Letting $m \to +\infty$ in (5.3), we have

$$|((\partial_t^j u(\cdot, t_1) - \partial_t^j u(\cdot, t_2), \rho))| \leqq C_{j+2}|t_1 - t_2| \|\rho\|$$

for all $\rho \in H_0^1$ and all $t_1, t_2 \in I$, which is ensured by Lemma 5.2. Thus we get, by the Riesz theorem,

$$\|\partial_t^j u(\cdot, t_1) - \partial_t^j u(\cdot, t_2)\| \leqq C_{j+2}|t_1 - t_2|,$$

which proves $\partial_t^j u \in C(I; H_0^1)$ for $j = 0, 1, 2$.

Next, the sequence $\{\partial_t^j u_m(\cdot, t)\}$ is uniformly bounded on I in $L^2(0, a; x^0)$ for $j = 0, 1, 2, 3$, or even $j = 4$. Furthermore, it is also equi-continuous on I in $L^2(0, a; x^0)$ for $j = 0, 1, 2, 3$. In fact, we have, for $j = 0, 1, 2, 3$,

$$|\partial_t^j u_m(\cdot, t_1) - \partial_t^j u_m(\cdot, t_2)|_{L^2} \leqq \left| \int_{t_2}^{t_1} |\partial_\tau^{j+1} u_m(\cdot, \tau)|_{L^2} \, d\tau \right| \leqq C_{j+1}|t_1 - t_2|$$

for all $t_1, t_2 \in I$. Therefore, using the Ascoli–Arzelà theorem, we can choose a subsequence, denoted by the same, of $\{\partial_t^j u_m(\cdot, t)\}$ such that it converges uniformly to $\partial_t^j u \in C(I; L^2(0, a; x^0))$ on I strongly in $L^2(0, a; x^0)$. This proves (5.2). The proof of Lemma 5.3 is complete. □

We need the (uniform) convergence of $\partial_t^j u_m$ in $L^2(0, a; x^0)$ up to $j = 2$.

LEMMA 5.4. *Let u_m and u be as in Lemma 5.3. Then we have $\partial_t^j u^3 \in C(I; H_0^1(0, a; x^0))$ for $0 \leqq j \leqq 2$ and*

(5.4) $$\sup_{t \in I} |\partial_t^j u_m^3(\cdot, t) - \partial_t^j u^3(\cdot, t)|_{L^2} \to 0 \quad \text{for } 0 \leqq j \leqq 2 \quad \text{as } m \to +\infty.$$

PROOF. Noting

$$\partial_t u^3 = 3u^2 \partial_t u, \quad \partial_t^2 u^3 = 6u(\partial_t u)^2 + 3u^2 \partial_t^2 u,$$

we can apply Lemma 4.3 to each term in the right-hand sides, since $\partial_t^j u$ ($j = 0, 1, 2$) belongs to $C(I; H_0^1)$ by Lemma 5.3. Then we get $\partial_t^j u^3 \in C(I; H_0^1)$ for $j = 0, 1, 2$.

We note that there exists a constant C_{j+1} depending on the norms $|\phi|_{H^{j+1}}$ and $|\psi|_{H^j}$ such that

(5.5) $$\sup_{\substack{(x,t) \in \Omega \\ m=1,2,\dots}} |\partial_t^j u_m(x, t)| \leqq C_{j+1}, \quad j = 0, 1, 2,$$

(5.6) $$\sup_{t \in I} \|\partial_t^j u(\cdot, t)\| \leqq C_{j+1}, \quad j = 0, 1, 2.$$

The estimate (5.5) follows from Proposition 3.2 (ii), Lemma 3.4 and Proposition 4.2 as follows:

$$\sup_{x\in(0,a]}|\partial_t^j u_m(x,t)| \leqq C\|\partial_t^j u_m(\cdot,t)\| \leqq C_{j+1}\,,$$

and further, the estimates (5.6) follows from

$$\|\partial_t^j u(\cdot,t)\| \leqq \liminf_{m\to+\infty}\|\partial_t^j u_m(\cdot,t)\| \leqq C_{j+1}\,.$$

We prove (5.4) for each j by using Proposition 4.2, Lemmas 4.3 and 5.3, and the estimates (5.5)–(5.6) without any comment.

CASE: $j=0$. It follows that

$$|u_m(\cdot,t)^3 - u(\cdot,t)^3|_{L^2} \leqq |u_m(\cdot,t)-u(\cdot,t)|_{L^2}(\|u_m(\cdot,t)\|^2 + \|u_m(\cdot,t)\|\|u(\cdot,t)\| + \|u(\cdot,t)\|^2)$$

$$\leqq C|u_m(\cdot,t)-u(\cdot,t)|_{L^2} \to 0\,,$$

$$\text{uniformly in } t\in I,\quad m\to+\infty\,,$$

which proves (5.4) for $j=0$.

CASE: $j=1$. It follows that

$$|\partial_t u_m(\cdot,t)^3 - \partial_t u(\cdot,t)^3|_{L^2}$$

$$\leqq C(|(u_m(\cdot,t)^2 - (u(\cdot,t)^2)\partial_t u_m(\cdot,t)|_{L^2} + |u(\cdot,t)^2(\partial_t u_m(\cdot,t)-\partial_t u(\cdot,t))|_{L^2})$$

$$\leqq C(|u_m(\cdot,t)-u(\cdot,t)|_{L^2} + |\partial_t u_m(\cdot,t)-\partial_t u(\cdot,t)|_{L^2}) \to 0\,,$$

uniformly in $t\in I$ as $m\to+\infty$, which proves (5.4) for $j=1$.

CASE: $j=2$. In this case, noting

$$\partial_t^2 u_m^3 - \partial_t^2 u^3 = 6(u_m-u)(\partial_t u_m)^2 + 6u\{(\partial_t u_m)^2 - (\partial_t u)^2\}$$

$$+ 3u_m^2(\partial_t^2 u_m - \partial_t^2 u) + 3(\partial_t^2 u_m^2 - \partial_t^2 u^2)\,,$$

we have

$$|\partial_t^2 u_m(\cdot,t)^3 - \partial_t^2 u(\cdot,t)^3|_{L^2}$$

$$\leqq C(|u_m(\cdot,t)-u(\cdot,t)|_{L^2} + |\partial_t u_m(\cdot,t)-\partial_t u(\cdot,t)|_{L^2}$$

$$+ |\partial_t^2 u_m(\cdot,t)-\partial_t^2 u(\cdot,t)|_{L^2}) \to 0\,,$$

uniformly in $t\in I$ as $m\to+\infty$, which proves (5.4) for $j=2$. The proof of Lemma 5.4 is complete. $\square$

5.2. Proof of Theorem 5.1 and Remark 5.2. We differentiate (4.7) j-times with respect to $t\in I$ for $0\leqq j\leqq 2$, apply Proposition 3.1 (i) and let $m\to+\infty$. Then using Lemmas 5.2–5.4, we obtain

$$(\partial_t^{j+2} u(\cdot,t),\phi_k)_{L^2} + ((\partial_t^j u(\cdot,t),\phi_k)) + (\partial_t^j u(\cdot,t)^3,\phi_k)_{L^2} = 0\,,\quad k=1,2,\ldots\,.$$

Since the linear hull generated by $\{\phi_k\}$ is dense in $H_0^1(0, a; x^0)$, we obtain

$$(5.7) \qquad (\partial_t^{j+2} u(\cdot, t), v)_{L^2} + ((\partial_t^j u(\cdot, t), v)) + (\partial_t^j u(\cdot, t)^3, v)_{L^2} = 0$$

for all $v \in H_0^1(0, a; x^0)$. This means that $\partial_t^j u$ is a weak solution of (P).

To gain the regularity in space we need the following elliptic regularity theorem of Nirenberg type.

PROPOSITION 5.5.   *Let $s \in Z_+$ and $\mu > -1$. Assume that*

either $g \in H^s(0, a; x^\mu) \cap H_0^1(0, a; x^\mu)$ for $s \in N$, or $g \in L^2(0, a; x^\mu)$ for $s = 0$.

Then the elliptic BVP

$$(EP) \qquad\qquad L_\mu w = g, \quad w(a) = 0$$

has a unique solution w in $H^{s+2}(0, a; x^\mu) \cap H_0^1(0, a; x^\mu)$ satisfying

$$(5.8) \qquad\qquad |w|_{H^{s+2}(0,a;x^\mu)} \leqq C|g|_{H^s(0,a;x^\mu)},$$

where C is a constant depending on μ, s.

The proof of Proposition 5.5 is postponed and will be given in Appendix.

We have already known from Lemma 5.2 that $u \in W^{4,\infty}(I; L^2) \cap W^{3,\infty}(I; H_0^1)$. $u \in C^3(I; L^2)$ follows from Lemma 5.3. Hence it is sufficient to prove $\partial_t^j u \in C(I; H^{3-j} \cap H_0^1) \cap L^\infty(I; H^{4-j} \cap H_0^1)$ for $0 \leqq j \leqq 2$.

At first we prove $u \in C(I; H^3 \cap H_0^1)$. From (5.7) with $j = 0$, it follows that $w \equiv u(\cdot, t)$ is a weak solution of elliptic BVP

$$(5.9) \qquad\qquad Lw = g_1 = -(\partial_t^2 u(\cdot, t) + u(\cdot, t)^3), \quad w(a) = 0.$$

Applying Lemmas 5.3–5.4 to g_1 in (5.9), we have $g_1 \in C(I; H_0^1)$. Hence we conclude from Proposition 5.5 with $s = 0$ that $u(\cdot, t) \in H^3 \cap H_0^1$ and

$$|u(\cdot, t_1) - u(\cdot, t_2)|_{H^3} \leqq C\|g_1(\cdot, t_1) - g_1(\cdot, t_2)\| \quad \text{for all } t_1, t_2 \in I.$$

This proves $u \in C(I; H^3 \cap H_0^1)$.

Next we prove $u \in C^1(I; H^2 \cap H_0^1) \cap W^{1,\infty}(I; H^3 \cap H_0^1)$. It follows from (5.7) with $j = 1$ that $w \equiv \partial_t u(\cdot, t)$ is a weak solution of elliptic BVP

$$Lw = g_3 = -(\partial_t^3 u(\cdot, t) + \partial_t u(\cdot, t)^3), \quad w(a) = 0.$$

Since $\partial_t^3 u \in C(I; L^2) \cap L^\infty(I; H_0^1)$ by Lemmas 5.2–5.3, and since $\partial_t u^3 \in C(I; L^2) \cap L^\infty(I; H_0^1)$ by Lemmas 4.3, 5.2 and 5.4, we have $g_3 \in C(I; L^2) \cap L^\infty(I; H_0^1)$. Thus it follows from Proposition 5.5 with $s = 1$ that $\partial_t u \in H^3 \cap H_0^1$ and

$$|\partial_t u(\cdot, t_1) - \partial_t u(\cdot, t_2)|_{H^2} \leqq C_2|g_3(\cdot, t_1) - g_3(\cdot, t_2)|_{L^2} \quad \text{for all } t_1, t_2 \in I,$$

which proves $\partial_t u \in C(I; H^2 \cap H_0^1) \cap L^\infty(I; H^3 \cap H_0^1)$, hence $u \in C^1(I; H^2 \cap H_0^1) \cap W^{1,\infty}(I; H^3 \cap H_0^1)$.

To see $u \in C^2(I; H_0^1)$, we prepare, using the equation of (P),

$$\|\partial_t^2 u(\cdot, t_1) - \partial_t^2 u(\cdot, t_2)\| \leqq \|Lu(\cdot, t_1) - Lu(\cdot, t_2)\| + \|u(\cdot, t_1)^3 - u(\cdot, t_2)^3\|$$

$$\leqq C(|u(\cdot, t_1) - u(\cdot, t_2)|_{H^2} + \|u(\cdot, t_1) - u(\cdot, t_2)\|)$$

for all $t_1, t_2 \in I$. Then it follows from the continuity of u on I in H^2 that u belongs to $C^2(I; H_0^1)$.

Now we prove $u \in W^{2,\infty}(I; H^2 \cap H_0^1)$. It follows from (5.7) with $j = 2$ that $w \equiv \partial_t^2 u(\cdot, t)$ is a weak solution of elliptic BVP

$$Lw = g_2 = -(\partial_t^4 u(\cdot, t) + \partial_t^2 u(\cdot, t)^3), \quad w(a) = 0.$$

Since $g_2 \in L^\infty(I; L^2)$ by Lemmas 5.2 and 5.4, we conclude from Proposition 5.5 with $s = 0$ that $\partial_t^2 u(\cdot, t) \in L^\infty(I; H^2 \cap H_0^1)$, hence, $u \in W^{2,\infty}(I; H^2 \cap H_0^1)$.

It remains to prove $u \in L^\infty(I; H^4 \cap H_0^1)$. Let us go back to (5.9). Since $\partial_t^2 u \in L^\infty(I; H^2 \cap H_0^1)$ by the above argument, it follows from Lemma 4.3 that $u^3 \in L^\infty(I; H^2 \cap H_0^1)$, and hence, $g_1 \in L^\infty(I; H^2 \cap H_0^1)$. Then applying again Proposition 5.5 with $s = 2$ to (5.9), we have $u \in L^\infty(I; H^4 \cap H_0^1)$.

Summarizing the above argument, we conclude that the weak solution u of (P) is, in fact, the strong solution and satisfies

$$u \in C^3(I; L^2) \bigcap_{j=0}^{2} C^j(I; H^{3-j} \cap H_0^1), \quad u \in W^{4,\infty}(I; L^2) \bigcap_{j=0}^{3} W^{j,\infty}(I; H^{4-j} \cap H_0^1).$$

The proof of uniqueness is standard. For the completeness we give it. Let u, v be two solutions of (P) with the same initial-boundary condition. Put $U = u - v$. Then U satisfies the following problem:

(5.10) $\qquad \partial_t^2 U + LU + u^3 - v^3 = 0, \quad U(x, 0) = \partial_t U(x, 0) = 0, \quad U(a, t) = 0.$

Multiplying (5.10) by $\partial_t U$ and integrating, we get

$$\frac{1}{2}\frac{d}{dt}(|U(t)|_{L^2}^2 + \|U(t)\|^2) = -(u(t)^3 - v(t)^3, \partial_t U(t))_{L^2}.$$

Since

$$(u(t)^3 - v(t)^3, \partial_t U(t))_{L^2} \leqq \int_0^a |u(t)^3 - v(t)^3||\partial_t U(t)|dx$$

$$\leqq C(|u(t)|_{L^\infty}^2 + |v(t)|_{L^\infty}^2)\int_0^a |U(t)||\partial_t U(t)|dx$$

$$\leqq C\|U(t)\||\partial_t U(t)|_{L^2},$$

we have

$$\frac{1}{2}\frac{d}{dt}(|U(t)|_{L^2}^2 + \|U(t)\|^2) \leqq C\|U(t)\|\,|\partial_t U(t)|_{L^2}\,.$$

By Gronwall's lemma, we conclude that $|U(t)|_{L^2}^2 + \|U(t)\|^2 = 0$ for all $t \geq 0$, i.e., $u(t) = v(t)$. Thus Theorem 5.1 and Remark 5.2 are proved. □

PROOF OF MAIN THEOREM COMPLETED. For the proof of our main theorem it suffices to prove the following.

LEMMA 5.6. $X_3 \equiv C^3(I; L^2)\bigcap_{j=1}^3 C^{3-j}(I; H^j \cap H_0^1)$ is embedded in $C^2((0, a] \times I)$.

PROOF. Let $u \in X_3$, i.e., $u \in C^j(I; H^{3-j} \cap H_0^1)$ for $j = 1, 2, 3$, and $u \in C^3(I; L^2)$. Then we prove that each $\partial_x^\alpha \partial_t^j u(x, t)$, $0 \leqq \alpha + j \leqq 2$, coincides with the derivatives in the classical sense.

Let $\delta > 0$ be any fixed constant. Then $H^s(\delta, a; x^0)$, which is defined by the same way as H^s, is isomorphic to the usual Sobolev space $H^s(\delta, a)$. By the Sobolev imbedding theorems, fixing $t \in I$, we can claim that $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous on the interval $[\delta, a]$ of x for $0 \leqq j \leqq k$, $0 \leqq \alpha \leqq 2 - k$. Hence $\partial_x^\alpha \partial_t^j u(x, t)$ is also continuous on $(0, a]$ for any fixed $t \in I$ on account of the arbitrariness of δ. Furthermore, by the usual Sobolev inequality we have

$$(5.11) \qquad |\partial_x^\alpha \partial_t^j u(x, t)| \leqq C_\delta\, |\partial_t^j u(\cdot, t)|_{H^{3-k}(\delta, a; x^0)} \leqq C_\delta\, |\partial_t^j u(\cdot, t)|_{H^{3-k}}\,,$$

for all $(x, t) \in [\delta, a] \times I$ and $0 \leqq j \leqq k$, $0 \leqq \alpha \leqq 2 - k$, where $C_\delta > 0$ depends on δ. Moreover $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous in $x \in [\delta, a]$ for any fixed $t \in I$. Using (5.11), we have

$$\left|\frac{1}{h}(\partial_x^\alpha \partial_t^{j-1} u(x, t+h) - \partial_x^\alpha \partial_t^{j-1} u(x, t)) - \partial_x^\alpha \partial_t^j u(x, t)\right|$$

$$\leqq C_\delta \left|\frac{1}{h}(\partial_t^{j-1} u(\cdot, t+h) - \partial_t^{j-1} u(\cdot, t)) - \partial_t^j u(\cdot, t)\right|_{H^{3-k}} \to 0 \quad (h \to 0)\,.$$

Therefore $\partial_x^\alpha \partial_t^j u(x, t)$ is a classical derivative with respect to $(x, t) \in [\delta, a] \times I$ for $0 \leqq j \leqq k$, $0 \leqq \alpha \leqq 2 - k$. Since δ is arbitrary, $\partial_x^\alpha \partial_t^j u(x, t)$ is a classical derivative in $(x, t) \in (0, a] \times I$.

Finally we prove that each $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous on $(0, a] \times I$. Let $(x_0, t_0) \in (0, a] \times I$ be arbitrarily fixed. Take $\delta \in (0, x_0)$. Notice that, for $(x, t) \in [\delta, a] \times I$,

$$|\partial_x^\alpha \partial_t^j u(x, t) - \partial_x^\alpha \partial_t^j u(x_0, t_0)|$$

$$\leqq |\partial_x^\alpha \partial_t^j u(x, t) - \partial_x^\alpha \partial_t^j u(x, t_0)| + |\partial_x^\alpha \partial_t^j u(x, t_0) - \partial_x^\alpha \partial_t^j u(x_0, t_0)|\,.$$

Then the first term of the right-hand side tends to 0 as $t \to t_0$, since $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous on I for any fixed $x \in (0, a]$ by (5.11). Since we have already known that $\partial_x^\alpha \partial_t^j u(x, t)$ is

continuous on $(0, a]$ for any fixed $t \in I$, the second term of the right-hand side tends to 0 as $x \to x_0$. Thus $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous at (x_0, t_0). In conclusion, $\partial_x^\alpha \partial_t^j u(x, t)$ is continuous in $(0, a] \times I$ on account of the arbitrariness of (x_0, t_0). This proves $u \in C^2((0, a] \times I)$. The proof of Lemma 5.6 is complete. $\qquad\qquad\square$

6. Appendix

In this appendix let us prove Proposition 5.5. We consider the eigenvalue problem for $L_\mu, \mu > -1$,

$$(6.1) \qquad \begin{cases} L_\mu \phi(x) = \lambda \phi(x), & x \in (0, a), \\ \phi(a) = 0. \end{cases}$$

Then it was shown in [Ya1] that (6.1) has the eigenvalues λ_j and the corresponding eigenfunctions $\phi_j(x)$ which can be written by

$$\lambda_j = \frac{\mu_j^2}{4(\mu + 1)a}, \quad \phi_j(x) = \frac{1}{a^{1/2} J_{\mu+1}(\mu_j)} \frac{J_\mu(\mu_j \sqrt{x/a})}{x^{\mu/2}},$$

where $\{\mu_j : j \in N\}$ is the set of all positive zero points of the μ-order Bessel function $J_\mu(x)$ with $\mu_1 < \mu_2 < \dots$. For several properties of the Bessel functions and their zero points, see [T] and [W]. It should be noted that $\{\phi_j\}$ *is CONS in* $L^2(0, a; x^\mu)$ *and a complete and orthogonal system in* $H_0^1(0, a; x^\mu)$.

Now let us expand $f \in L^2(0, a; x^\mu)$ into the Bessel–Fourier series

$$f = \sum_j f_j \phi_j, \quad f_j = (f, \phi_j)_{L^2(0,a;x^\mu)}.$$

Let us introduce a subspace $W^\alpha(0, a)$ ($\alpha \geq 0$) of $L^2(0, a; x^\mu)$ whose element f satisfies

$$\sum_j \lambda_j^\alpha f_j^2 < +\infty.$$

Then $W^\alpha(0, a)$ is a Hilbert space with inner product

$$(f, g)_{W^\alpha(0,a)} = \sum_j \lambda_j^\alpha f_j g_j,$$

where $f = \sum f_j \phi_j$ and $g = \sum g_j \phi_j$. The crucial point of $W^s(0, a)$ is given by the following proposition.

PROPOSITION 6.1 ([Ya1]). *Let $s \in Z_+$ and $\mu > -1$. Then $W^s(0, a)$ is isomorphic to $K^s(0, a; x^\mu)$, i.e.,*

$$C_0 |f|_{H^s(0,a;x^\mu)} \leq |f|_{W^s(0,a)} \leq C_1 |f|_{H^s(0,a;x^\mu)},$$

holds for $f \in K^s(0, a; x^\mu)$, where $C_0, C_1 > 0$ depend on s, μ.

We can also define the fractional power of L_μ by

$$L_\mu^\alpha f = \sum_j \lambda_j^\alpha f_j \phi_j$$

for all $\alpha \geq 0$. Then it was shown in [Ya1] that there exists a constant $C > 1$ such that

$$C^{-1}|f|_{W^{2\alpha}} \leq |L^\alpha f|_{L^2} \leq C|f|_{W^{2\alpha}}$$

for all $f \in W^\alpha(0, a)$.

Now we consider BVP

(EP) $$L_\mu w = g, \quad w(a) = 0.$$

At first we prove the unique solvability to (EP) in $K^s(0, a; x^\mu)$.

LEMMA 6.2. *If $g \in K^s(0, a; x^\mu)$ for $s \in Z_+$, then (EP) has a unique solution w in $K^{s+2}(0, a; x^\mu)$ satisfying (5.8).*

PROOF. Substituting $g = \sum_j g_j \psi_j$ and $w = \sum_j w_j \psi_j$ into (EP) and comparing each of the Fourier coefficients, we have $w_j = g_j/\lambda_j$. Hence we obtain

$$\sum_j \lambda_j^{s+2}|w_j|^2 \leq \sum_j \lambda_j^s|g_j|^2.$$

By using Proposition 6.1, w belongs to $K^{s+2}(0, a; x^\mu)$ and (5.8) holds. The uniqueness is clear from (5.8). □

PROOF OF PROPOSITION 5.5. We prove Proposition 5.5 by the induction argument. The case $s = 1$ is clear from Lemma 6.2. Assuming that the statement of Proposition 5.5 holds for $s - 1$ ($s \geq 2$), we will prove the statement of Proposition 5.5 for s by establishing the regularity of solutions inside of $(0, a)$ and near $x = a$. Let us introduce a function space B of all functions $\alpha(x)$ of C^∞ in $[0, a]$ such that

$$\alpha(x) = \begin{cases} 1, & x \in [0, \xi], \\ 0, & x \in [\eta, a], \end{cases}$$

for some ξ, η with $0 < \xi < \eta < a$.

As the first step we check the regularity of solutions in the interior of $(0, a)$. More precisely, we will see $\alpha w \in K^{s+2}(0, a; x^\mu)$ for any $\alpha \in B$ by using *so called* bootstrap method. It follows from the equation in (EP) that

(6.2) $$L_\mu(\alpha w) = -\alpha g - \frac{2\partial_x \alpha}{\mu + 1}(x \partial_x w) - (L_\mu \alpha)w.$$

Since w belongs to $H^{s+1}(0, a; x^\mu)$ by the assumption on induction, both $x \partial_x w$ and w belong to $H^s(0, a; x^\mu)$. In fact, it is enough to show that

(6.3) $$\int_0^a x^{\mu+j}\{\partial_x^j(x \partial_x w)\}^2 dx < +\infty, \quad j = 0, 1, \ldots, s.$$

It follows from the Leibniz formula that

$$\partial_x^j(x\,\partial_x w) = x\partial_x^{j+1}w + j\partial_x^j w\,,$$

and hence, we have

$$\int_0^a x^{\mu+j}(x\,\partial_x^{j+1}w)^2 dx = \int_0^a x^{\mu+j+2}(\partial_x^{j+1}w)^2 dx < +\infty\,,$$

$$\int_0^a x^{\mu+j}(j\partial_x^j w)^2 dx < +\infty\,,$$

which proves (6.3). Clearly if $\alpha \in B$ and $f \in H^s(0,a;x^\mu)$, then αf belongs to $K^s(0,a;x^\mu)$. Hence $\beta x \partial_x w \in K^s(0,a;x^\mu)$ for any $\beta \in B$, which implies that the right-hand side of (6.2) belongs to $K^s(0,a;x^\mu)$. Therefore, using Lemma 6.2, we get $\alpha w \in K^{s+2}(0,a;x^\mu)$ for any $\alpha \in B$.

As the second step, we check that (5.8) holds for w replaced by αw. Applying (5.8) in Lemma 6.2 to (6.2), we have

$$
\begin{aligned}
(6.4)\quad |\alpha w|_{H^{s+2}(0,a;x^\mu)} &\leqq C\left| -\alpha g - \frac{2\partial_x\alpha}{\mu+1}(x\partial_x w) - \left(\frac{x}{\mu+1}\partial_x^2\alpha + \partial_x\alpha\right)w\right|_{H^s(0,a;x^\mu)} \\
&\leqq C(|g|_{H^s(0,a;x^\mu)} + |x\partial_x w|_{H^s(0,a;x^\mu)} + |w|_{H^s(0,a;x^\mu)}) \\
&\leqq C(|g|_{H^s(0,a;x^\mu)} + |w|_{H^{s+1}(0,a;x^\mu)}) \\
&\leqq C(|g|_{H^s(0,a;x^\mu)} + |g|_{H^{s-1}(0,a;x^\mu)}) \\
&\leqq C|g|_{H^s(0,a;x^\mu)}\,,
\end{aligned}
$$

with a certain constant C depending on α and m, where we have used the assumption on induction.

As the third step, we check the regularity of w at the boundary $x = a$ and prove the estimate (5.8). Given a constant $\delta \in (0,a)$, let us define a set E_δ of all $\gamma \in C^\infty((0,a])$ such that

$$\gamma(x) = \begin{cases} 1\,, & x \in [a-\delta/2, a]\,, \\ 0\,, & x \in [0, a-\delta]\,. \end{cases}$$

Then (6.2) holds even for α replaced by γ in the interval $(a-\delta, a)$. In this case L_μ is not degenerate and is a strongly elliptic operator in $(a-\delta, a)$. Hence the regularity theorem on strongly elliptic operator (Nirenberg theorem) tells us that γw belongs to $H^{s+2}(0,a;x^\mu)$ for any $\gamma \in E_\delta$. In order to obtain the estimate of $|\gamma w|_{H^{s+2}(0,a;x^\mu)}$, we use the representation of the solutions of equation in (EP):

$$w(x) = -\int_a^x \frac{\mu+1}{s^{\mu+1}}\left(\int_a^s y^\mu g(y)dy\right)ds\,.$$

Using this representation formula, we can write the derivatives of $w(x)$ by

$$(6.5) \qquad w^{(j)}(x) = \sum_{k=0}^{j-2} Q_k(x) g^{(k)}(x) + Q_j(x) \left(\int_a^x y^\mu g(y) dy \right)$$

for $j = 1, \ldots, s+2$, where $Q_k(x)$ are the polynomials of $1/x$. Up to now, it is sufficient to estimate the top term $|x^{\frac{s+2}{2}} (\gamma w)^{(s+2)}|_{L^2(0,a;x^\mu)}$, since the lower derivatives can be treated more easily. Since

$$|x^{\frac{s+2}{2}} (\gamma w)^{(s+2)}|_{L^2(0,a;x^\mu)}^2 = \int_{a-\delta}^a x^{\mu+s+2} |(\gamma w)^{(s+2)}|^2 dx \leqq C \int_{a-\delta}^a x^{\mu+s+2} \sum_{j=0}^{s+2} |w^{(j)}|^2 dx ,$$

it follows from (6.5) that

$$|x^{\frac{s+2}{2}} (\gamma w)^{(s+2)}|_{L^2(0,a;x^\mu)}^2 \leqq C \int_{a-\delta}^a x^{\mu+s+2} \sum_{j=0}^s |g^{(j)}|^2 dx \leqq C |g|_{H^s(0,a;x^\mu)}^2 .$$

Thus we get the required estimate

$$(6.6) \qquad |\gamma w|_{H^{s+2}(0,a;x^\mu)} \leqq C |g|_{H^s(0,a;x^\mu)} .$$

Writing $w = \alpha w + (1-\alpha)w$, $\alpha \in B$, $1 - \alpha \in E_\delta$ for some $\delta \in (0, a)$, we conclude that w belongs to $H^{s+2}(0, a; x^\mu)$. Moreover, we obtain (5.8). In fact, we see from (6.4) and (6.6) that

$$|w|_{H^{s+2}(0,a;x^\mu)} \leqq |\alpha w|_{H^{s+2}(0,a;x^\mu)} + |(1-\alpha)w|_{H^{s+2}(0,a;x^\mu)} \leqq C |g|_{H^s(0,a;x^\mu)} .$$

This completes the proof of Proposition 5.5. $\qquad\qquad\qquad\qquad\qquad \square$

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