TOKYO J. MATH.
VOL. 31, NO. 2, 2008

# An Appendix to the Weierstrass Representation of a Maximal Spacelike Surface in $\mathbb{L}^3$

Maria Luiza LEITE

*Universidade Federal de Pernambuco*

(Communicated by K. Shinoda)

**Abstract.** In this note we uncover some analytical properties of the Weierstrass pair $\{g, \eta\}$ representing an oriented and connected maximal surface in Minkowski space $\mathbb{L}^3$, proving that $g$ is holomorphic with values in the unit disk $\mathbb{D}$ and $\eta$ is a holomorphic 1-form that never vanishes.

In his paper *Maximal surfaces in the* 3*-dimensional Minkowski space* $\mathbb{L}^3$, O. Kobayashi obtains the Weierstrass representation of a *maximal surface* $M$ *of* $1^{st}$ *kind* in terms of a pair $\{g, \eta\}$, where $g$ is a meromorphic function satisfying $|g| \neq 1$ and $\eta$ is a holomorphic 1-form such that $g^2\eta$ is a holomorphic 1-form on $M$ which does not vanish on the zeroes of $\eta$ (see [1], Theorem 1.1). Assuming $M$ is connected and the unit normal is lying on the hyperbolic plane $\mathbb{H}^2 \subset \mathbb{L}^3$, modeled by the upper leaf of the quadric surface $r^2 + s^2 - t^2 = -1$ in $\mathbb{R}^3$, we prove in Theorem 1 that $g$ is actually holomorphic, $|g| < 1$ and the holomorphic 1-form $\eta$ is never null.

It is natural that the normal map of a maximal immersion into $\mathbb{L}^3$ is represented by a holomorphic function $g : M \to \mathbb{D}$, in analogy to the normal map of a minimal immersion into Euclidean space, which is represented by a holomorphic function $g : M \to \mathbb{C} \cup \{\infty\}$.

THEOREM 1. *Let* $\mathbf{F} : M \to \mathbb{L}^3$ *be a maximal immersion of an oriented and connected surface, with unit normal in* $\mathbb{H}^2 = \{(r, s, t) : r^2 + s^2 - t^2 = -1, t \geq 1\}$. *Considering the Riemann surface structure in* $M$ *compatible with the normal choice, there corresponds a Weierstrass pair* $\{g, \eta\}$, *such that* $g : M \to \mathbb{D}$ *is holomorphic*, $\eta$ *is a holomorphic 1-form without zeroes, each component of* $(1 + g^2, i(1 - g^2), 2g)\ \eta$ *has no real periods and the immersion satisfies*

$$\mathbf{F} = (1/2)\Re\left(\int (1 + g^2, i(1 - g^2), 2g)\ \eta\right).$$

Received December 14, 2006; revised January 25, 2007
1991 *Mathematics Subject Classification*: 53C50 (Primary)
*Key words and phrases*: maximal surfaces, Weierstrass representation

*In complex coordinates* $z \in \Omega \subset \mathbb{C}$, *one has that* $2\mathbf{F}_z = (\Phi_1, \Phi_2, \Phi_3)$ *is holomorphic*,

$$|\Phi_1 - i\Phi_2| > 0\,, \quad g = \frac{\Phi_3}{\Phi_1 - i\Phi_2} \quad \textit{and} \quad \eta = (\Phi_1 - i\Phi_2)dz\,.$$

*Moreover*, $g = \Pi \circ \mathbf{N}$, *where* $\Pi(r, s, t) = \frac{r+is}{1+t}$ *maps* $\mathbb{H}^2$ *isometrically onto the unit disk* $\mathbb{D}$ *with the Poincaré metric* $ds^2 = (2|dz|/1 - |z|^2)^2$.

PROOF. Using coordinates and the notation $\mathbf{F}_z = \frac{1}{2}(\mathbf{F}_x - i\mathbf{F}_y)$, $\mathbf{F}_{\bar{z}} = \frac{1}{2}(\mathbf{F}_x + i\mathbf{F}_y)$, let us recall that $\Delta\mathbf{F} = 4\mathbf{F}_{z\bar{z}} = (2HE)\mathbf{N}$, where $H$ denotes the mean curvature; in particular, the immersion is maximal iff $\mathbf{F}_z$ is holomorphic in every system of complex coordinates. Therefore $(\Phi_1, \Phi_2, \Phi_3)$ is a triple of local holomorphic functions (see [1]).

The hypothesis that the complex structure is compatible with the unit normal means that the Minkowski vector product $\mathbf{R}(\mathbf{F}_x \times \mathbf{F}_y)$ is a positive multiple of $\mathbf{N}$; here $\times$ denotes the usual vector product in $\mathbb{R}^3$, and $\mathbf{R}$ stands for the reflection $\mathbf{R}(r, s, t) = (r, s, -t)$. It follows from $\mathbf{F}_x - i\mathbf{F}_y = (\Phi_1, \Phi_2, \Phi_3)$ that $\mathbf{R}(\mathbf{F}_x \times \mathbf{F}_y) = \Im(\Phi_2\overline{\Phi_3}, \Phi_3\overline{\Phi_1}, -\Phi_1\overline{\Phi_2})$, hence

$$\Im(\Phi_1\overline{\Phi_2}) < 0 \tag{1}$$

Since $|\Phi_1 \pm i\Phi_2|^2 = |\Phi_1|^2 + |\Phi_2|^2 \pm 2\Im(\Phi_1\overline{\Phi_2})$, one uses (1) to obtain that

$$|\Phi_1 - i\Phi_2|^2 - |\Phi_1 + i\Phi_2|^2 = -4\Im(\Phi_1\overline{\Phi_2}) > 0 \;\Rightarrow$$

$$|\Phi_1 - i\Phi_2| > |\Phi_1 + i\Phi_2|\,. \tag{2}$$

Recall from [K] that

$$\Phi_1^2 + \Phi_2^2 - \Phi_3^2 = (\Phi_1 - i\Phi_2)(\Phi_1 + i\Phi_2) - \Phi_3^2 = 0\,, \tag{3}$$

$$|\Phi_1|^2 + |\Phi_2|^2 - |\Phi_3|^2 = \mathbf{F}_x.\mathbf{F}_x + \mathbf{F}_y.\mathbf{F}_y > 0\,. \tag{4}$$

Letting $\mathcal{Z} \subset \Omega$ denote the zero set of $\Phi_3$, we claim that

(i) $|\Phi_1 - i\Phi_2| > 0 = |\Phi_3| = |\Phi_1 + i\Phi_2|$ in $\mathcal{Z}$.

(ii) $|\Phi_1 - i\Phi_2| > |\Phi_3| > |\Phi_1 + i\Phi_2| > 0$ in $\Omega - \mathcal{Z}$.

Indeed, it is immediate from (3) that

$$|\Phi_1 - i\Phi_2||\Phi_1 + i\Phi_2| = |\Phi_3|^2\,. \tag{5}$$

Combining (2) and (5), one sees that the the zeroes of $|\Phi_1 + i\Phi_2|$ and $|\Phi_3|$ coincide. Clearly, (i) holds. Outside $\mathcal{Z}$, one has that $|\Phi_1 - i\Phi_2| > |\Phi_1 + i\Phi_2| > 0$, so (5) gives us (ii).

Having proved the claim, one concludes that $f = \Phi_1 - i\Phi_2 \neq 0$ everywhere in $\Omega$, $g = \frac{\phi_3}{\phi_1 - i\phi_2}$ is holomorphic and $|g| < 1$ are true.

The other assertions in the theorem are well known (see [1]): $g$ and $\eta$ are globally defined,

$$\mathbf{F}_x - i\mathbf{F}_y = (1/2)(1 + g^2, i(1 - g^2), 2g)f\,, \quad \mathbf{F}_x.\mathbf{F}_x = \mathbf{F}_y.\mathbf{F}_y = (1 - |g|^2)^2|f|^2/4 > 0\,,$$

$$\mathbf{R}(\mathbf{F}_x \times \mathbf{F}_y) = (1-|g|^2)|f|^2/4\ (2\Re(g), 2\Im(g), 1+|g|^2) \Rightarrow$$

$$\mathbf{N} = \frac{(2\Re(g), 2\Im(g), 1+|g|^2)}{1-|g|^2} = \Pi^{-1} \circ g\,, \quad \mathbf{N}.\mathbf{N} = -1\,,$$

since $(2\Re(g), 2\Im(g), 1+|g|^2).(2\Re(g), 2\Im(g), 1+|g|^2) = -(1-|g|^2)^2$ holds in $\mathbb{L}^3$.

Standard computations show that the stereographic projection $\Pi : \mathbb{H}^2 \to \mathbb{D}$ is conformal and induces the Poincaré metric on the unit disk. One may prove directly that the metric on $\mathbb{H}^2$ induced by $\mathbb{L}^3$ is hyperbolic. □

REMARK 1. Minor adjustments are necessary in order to describe the examples in Kobayashi paper within the convention that the normal map be represented by a holomorphic map $g : M \to \mathbb{D}$. The first one is to replace the Weierstrass pair $\{g, \eta\}$ given in [1] by $\{1/g, g^2\eta\}$ on any connected component where $|g| > 1$ holds. The second one concerns a *maximal surface of* $2^{nd}$ *kind*, alternatively represented by a Weierstrass pair $\{g, \eta\}$, with $g$ satisfying $\Re(g) \neq 0$ (see Corollary 1.3 of [1]); to obtain the same immersion, it suffices to change the pair $\{g, \eta\}$ into $\left\{\frac{1-g}{1+g}, -(1+g^2)\eta/2\right\}$, knowing that $\left|\frac{1-g}{1+g}\right| \neq 1 \Leftrightarrow \Re(g) \neq 0$.

REMARK 2. Two connected components of a same maximal surface described in [1] may be congruent, or totally non-congruent. For instance, the so-called *Enneper maximal surface of* $1^{st}$ *kind* is the disjoint union of two connected maximal surfaces which are not congruent, even at a small level; the component represented by $g(z) = z,\ |z| < 1,\ \eta = dz$, has each family of curvature lines mapped by the normal map into a family of equidistant curves in $\mathbb{H}^2$, in contrast to the other one, represented by $g(z) = 1/z,\ |z| > 1,\ \eta = z^2 dz$, for which a subfamily of curvature lines is mapped into a family of geodesic circles converging to one horocycle in $\mathbb{H}^2$. On the other hand, the so-called *Enneper maximal surface of* $2^{nd}$ *kind* consists of two congruent components, corresponding to $g(z) = \frac{1-z}{1+z},\ \eta = (-1/g')dz$, with $|g| < 1$ (resp. $|g| > 1$) on the domain $\Re(z) > 0$ (resp. $\Re(z) < 0$); the change $z \to -z$ yields the congruence.

## References

[1] O. KOBAYASHI, Maximal surfaces in the 3-dimensional Minkowski space $\mathbb{L}^3$, Tokyo J. Math. **6** (1983), 297–309.

*Present Address*:
DEPARTAMENTO DE MATEMÁTICA,
UNIVERSIDADE FEDERAL DE PERNAMBUCO,
RECIFE, 50.740–540, PE, BRASIL.
*e-mail*: mll@dmat.ufpe.br