TOKYO J. MATH.
VOL. 31, NO. 1, 2008

Hyperbolic Knots with a Large Number of Disjoint Minimal Genus Seifert Surfaces

Yukihiro TSUTSUMI*

Sophia University

Abstract. It is known that any genus one hyperbolic knot in the 3-dimensional sphere admits at most seven mutually disjoint and mutually non-parallel genus one Seifert surfaces. In this note, it is shown that for any integers $g > 1$ and $n > 0$, there is a hyperbolic knot of genus g in the 3-dimensional sphere which bounds n mutually disjoint and mutually non-parallel genus g Seifert surfaces.

1. Introduction

It is known that any genus one hyperbolic knot in the 3-dimensional sphere S^3 admits at most seven mutually disjoint and mutually non-parallel genus one Seifert surfaces [4]. In this paper, in contrast with the genus one case, we show the following:

THEOREM 1.1. *For any integers $g > 1$ and $n > 0$, there is a hyperbolic knot of genus g in the 3-dimensional sphere which bounds n mutually disjoint and mutually non-parallel genus g Seifert surfaces.*

2. Gluing lemmas

In this paper we say that a 3-manifold M (a knot K, a tangle T resp.) is hyperbolic if M (the exterior $E(K)$, $E(T)$ resp.) is irreducible, ∂-irreducible, atoroidal, anannular and not Seifert fibered. Through this paper, unless stated otherwise, all manifolds are assumed to be compact, and 3-manifolds are orientable. See [1] and [3] for basic terminology in 3-dimensional topology and knot theory which is not stated here.

We use the following lemmas in the proof of Theorem 1.1. The lemmas in this section are shown by a standard cut and paste argument.

LEMMA 2.1. *Let M be an irreducible and ∂-irreducible 3-manifold. Let F_1 and F_2 be disjoint homeomorphic surfaces in ∂M. If F_1 and F_2 are incompressible, then the manifold $M/(F_1 = F_2)$ obtained from M by identifying F_1 with F_2 is irreducible and ∂-irreducible.*

Received November 25, 2006
Mathematics Subject Classification: 57M25.
*Supported by JSPS Research Fellowships for Young Scientists.

PROOF. See [6, Lemma 2.1]. □

LEMMA 2.2. *Let M be an irreducible, ∂-irreducible, and atoroidal 3-manifold. Let F_1 and F_2 be disjoint homeomorphic surfaces in ∂M each component of which has negative Euler characteristic. Suppose that:*

- *$\partial M - (\partial F_1 \cup \partial F_2)$ is incompressible in M,*
- *there is no essential annulus A in M such that a component of ∂A is contained in F_1 and*
- *there is no essential annulus whose boundary is contained in $\partial M - (F_1 \cup F_2)$.*

Then the manifold M' obtained by gluing F_1 to F_2 is hyperbolic.

PROOF. See [6, Lemma 2.2]. □

3. Proof of Theorem 1.1

PROOF OF THEOREM 1.1. Let τ be a g-string tangle such that the exterior $E(\tau)$ is hyperbolic with a totally geodesic boundary and any "linking number" of the strings is null. Let $\tilde{\tau}$ be the $2g$-string tangle obtained from τ by multiplying each string of τ so that the "self linking number" of any band is zero. Let τ_i and $\tilde{\tau}_i$ be copies of τ and $\tilde{\tau}$ respectively. ($i = 1, 2, \ldots, n$.)

Let K be the knot illustrated in Figure 1. It is easy to see that K bounds a genus g Seifert surface S_1 as illustrated in Figure 1, and the Alexander polynomial $\Delta_K(t) = (-2t + 5 - 2t^{-1})^g$. Then we have that $g(K) = g$.

Let S_i be the genus g Seifert surface for K as illustrated in Figure 2. There are $g + 1$ annuli between S_i and S_{i+1} which cut off a 3-manifold homeomorphic to $E(\tau)$. Thus, the

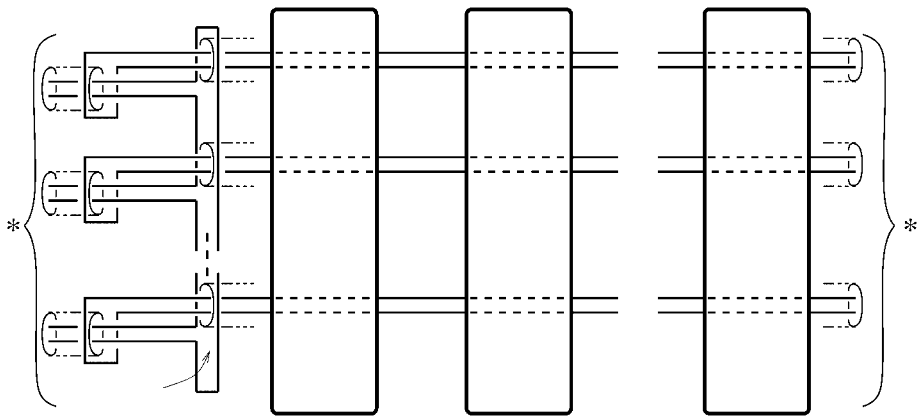

FIGURE 1. $K = \partial S_1$.

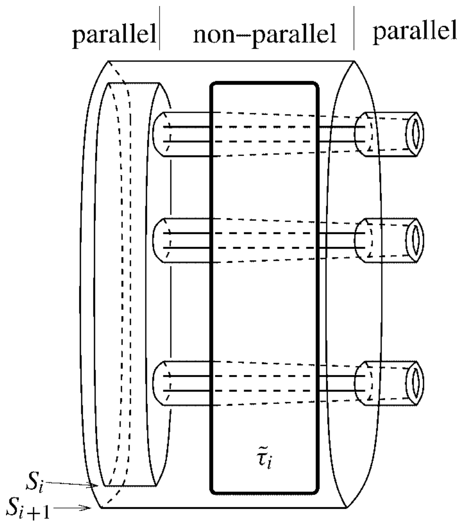

FIGURE 2. S_i and S_{i+1}.

region between S_i and S_{i+1} is not a product. Since S_n and S_{n+1} are not isotopic, we see that S_1 is notd isotopic to S_n. Hence $S_1, \ldots, S_n$ are mutually disjoint and mutually non-parallel.

In the remainder we show that K is a hyperbolic knot.

Note that K is contained in the handlebody H illustrated in Figure 3. Then, $E(K, S^3)$ is obtained from the exterior of the graph Γ in Figure 4 and the exterior $E(K, H)$ by identifying ∂H with $\partial E(\Gamma, S^3)$.

LEMMA 3.1. *$E(K, H)$ is irreducible, ∂-irreducible, atoroidal and there is no essential annulus whose boundary is contained in $\partial N(K, H)$.*

PROOF. In H there are g meridian disks $P_1, \ldots, P_g$ which cut the pair (K, H) into (T, B), where B is a 3-ball B and T is a string as in Figure 3. For the tangle Q in the 3-ball B' as in Figure 5, the double branched covering space Σ^2_Q is obtained from g Seifert fibered spaces each is homeomorphic to $S(D^2; 1/3, -1/3)$ by attaching $2g-2$ 1-handles in a certain way. Therefore $H_1(\Sigma^2_Q)$ is isomorphic to $(\mathbf{Z}/3\mathbf{Z}+\mathbf{Z})^g + \mathbf{Z}^{g-1}$. On the other hand, the double branched covering space Σ^2_T is obtained from Σ^2_Q by $g-1$ Dehn surgeries on $g-1$ disjoint knots in Σ^2_Q as the Montesinos tricks [2] about the $g-1$ bands illustrated in Figure 5. Hence $\mathrm{Tor}(H_1(\Sigma^2_T))$ cannot be eliminated by the $g-1$ Dehn surgery. Therefore T is non-trivial since the double branched covering space along the trivial tangle is torsion free. It is easy to see that T is almost trivial. Now we see from the fact that minimally knotted spatial graphs are totally knotted [5] that $E(T)$ is irreducible, ∂-irreducible and T is a prime tangle. Now it

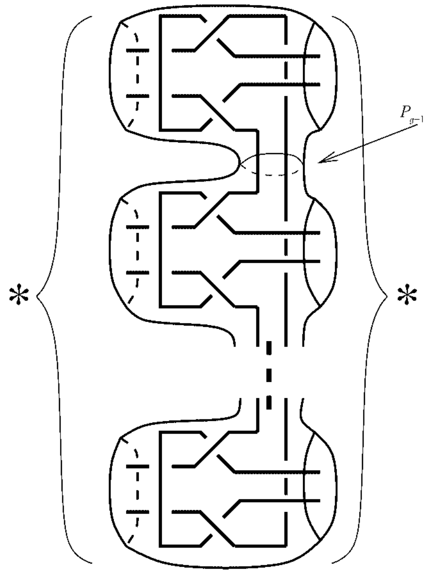

FIGURE 3. $H, B = \mathrm{cl}(H - N(P_1 \cup \cdots \cup P_g))$.

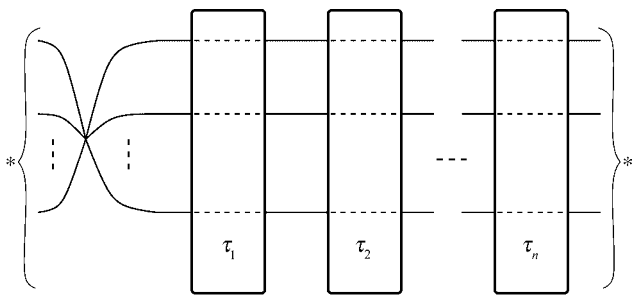

FIGURE 4. Γ.

is easy to see that $P_i \cap E(K, H)$ is incompressible and we have that $E(K, H)$ is irreducible and ∂-irreducible by Lemma 2.1.

There are g meridian disks $P_1, \ldots, P_g$ in H as noted before and $g - 1$ disks $P_{g+1}, \ldots, P_{2g-1}$ which decompose H into g 3-balls B_i together as in Figure 3. Notice that each tangle $(T_i, B_i) = (K \cap B_i, B_i)$ is trivial.

Suppose that there is an essential torus F in $E(K, H)$. We suppose that $F \cap \bigcup P_i$ is minimal among essential tori. Since $E(T_i, B_i)$ is a handlebody, F intersects some P_i essentially. That is, any component A of $F \cap B_i$ is an incompressible annulus. If A is a meridionally compressible annulus in B_i with respect to T_i, we see that T is not a prime tangle, a contradiction.

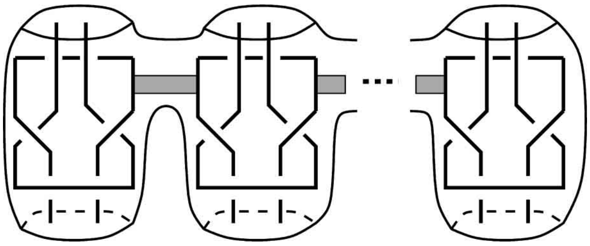

FIGURE 5. (Q, B').

Then we may assume that A is meridionally incompressible. There are $3g-1$ loops l_j in ∂B coming from ∂P_i as illustrated in Figure 3. It is easy to see that any two of l_i and l_j are not isotopic in $B-T$. Hence $F \cap \bigcup P_i$ is contained in one disk, say P_i, and ∂A is on the same side of P_i. We may assume that A is contained in B_i and any component of ∂A is not null homologous in $B_i - T_i$. Then ∂A cobounds an annulus A' in P_i, and $F' = A \cup A'$ is a torus. By the incompressibility of F and by the minimality of $F \cap \bigcup P_i$, we see that F' bounds a solid torus V and a component ℓ of $\partial A'$, which is isotopic to $l_j \subset \partial P_i$ in P_i, goes around V at least two times. This means that the homology class represented by ℓ is not primitive in $H_1(B_i - T_i)$. However the homology class of l_j is primitive in $H_1(B_i - T_i)$ since it is the sum of two elements represented by two longitudes of $\mathrm{cl}(B_i - N(T_i))$. This implies that $E(K, H)$ is atoroidal.

Suppose that there is an essential annulus A such that $\partial A \subset \partial N(K, H)$. If ∂A is meridional, then T is not a prime tangle, a contradiction. We may assume that each of $A \cap P_1, \ldots, A \cap P_g$ is essential. That is, $A \cap B$ is an essential rectangle R in $E(T, B)$. In this case, R becomes an essential disk in $E(T, B)$, a contradiction to the ∂-irreducibility of $E(T, B)$.

This completes the proof of Lemma 3.1. □

LEMMA 3.2. *$E(\Gamma)$ is hyperbolic.*

PROOF. Note that $E(\Gamma)$ is homeomorphic to the exterior of the tangle $\tau_1 + \tau_2 + \cdots + \tau_n$, where "+" denote the sum of tangles. There are $(g+1)$-punctured spheres $X_1, \ldots, X_{n-1}$ in $E(\Gamma)$ which cut $E(\Gamma)$ into $E(\tau_1), \ldots, E(\tau_n)$. Since each $E(\tau_i)$ is hyperbolic, there is no essential annulus A such that $\partial A \subset X_i \cap X_{i+1}$. Then we see from Lemma 2.2 that $E(\Gamma)$ is hyperbolic. This completes the proof of Lemma 3.2. □

Then by Lemmas 2.2, 3.1 and 3.2, we see that that $E(K)$ is hyperbolic. This completes the proof of Theorem 1.1.

References

[1] W. JACO, *Lectures on Three-Manifold Topology*, AMS. Conference board of Math. No. 43, 1977.

[2] J. M. MONTESINOS, Surgery on links and double branched coverings of S^3, Ann. Math. Studies, **84** (1975), 227–259.

[3] D. ROLFSEN, *Knots and links*, Publish or Perish, Inc, 1976
[4] Y. TSUTSUMI, Universal bounds for genus one Seifert surfaces for hyperbolic knots and surgeries with non-trivial JSJT-decompositions, Interdiscip. Inform. Sci. **9** (2003), 53–60.
[5] M. OZAWA and Y. TSUTSUMI, Minimally knotted spatial graphs are totally knotted, Tokyo J. Math. **26** (2003), 413–421.
[6] Y. TSUTSUMI, Excellent non-orientable spanning surfaces with distinct boundary slopes, Topology Appl. **139** (2004), 199–210.

Present Address:
DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE AND TECHNOLOGY,
KEIO UNIVERSITY,
HIYOSHI 3–14–1, KOHOKU-KU, YOKOHAMA, 223–8522 JAPAN.
e-mail: yukihiro@math.keio.ac.jp