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Duality of Weights, Mirror Symmetry and Arnold's Strange Duality

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Abstract. A duality of weight systems which corresponds to Batyrev's toric mirror symmetry is given. It is shown that Arnold's strange duality for exceptional unimodal singularities reduces to this duality.

0.1. Introduction. The hypersurfaces in weighted projective spaces often appear as important examples in the context of mirror symmetry. In this paper, we describe the relation between polar duality and duality of weight systems.

The duality of weights partly suggests why [10] produced a mirror symmetric phenomenon using only a resolution of weighted hypersurfaces in weighted $\mathbf{P}^4$. In fact, it is shown that those examples in weighted 4-spaces correspond to some reflexive polytopes [9].

As an application, we will show that Arnold's strange duality for fourteen exceptional unimodal singularities with $\mathbf{C}^*$-action reduces to polar duality (Theorem 4.3.9).

K. Saito defined weight systems and used them effectively for isolated hypersurface singularities with a $\mathbf{C}^*$-action [35]. The relation between weight systems and those singularities is as follows: take a germ of an analytic function $f : \mathbf{C}^n \to \mathbf{C}$ with $f(0) = 0$, which determines a germ of a hypersurface singularity $(\{f = 0\}, 0) \subset (\mathbf{C}^n, 0)$. The function f can be expressed as a weighted homogeneous polynomial of n variables $(x_1, \dots, x_n)$ after some suitable analytic change of coordinates near the origin if and only if there is a holomorphic tangent vector field D such that $Df = f$ [34]. In this case, we can assign to f a weight system $W = (wt(x_1), \dots, wt(x_n); wt(f))$.

We associate some hypersurface singularity with a $\mathbf{C}^*$-action on a compact complex surface with trivial canonical sheaf in the following manner.

Let $X_0 \subset \mathbf{C}^n$ be a hypersurface singularity with $\mathbf{C}^*$-action whose weights are all positive. Such an X_0 is known to be an algebraic variety defined by a weighted homogeneous polynomial [29].

Let X_t be its Milnor fiber (if it exists; e.g. when X_0 is an isolated singularity). We will not restrict ourselves to isolated singularities. Let $\bar{X}$ be the natural compactification of X_t in $\mathbf{P}(1, a_1, \dots, a_n)$ by sending the points $(x_1, \dots, x_n)$ of $\mathbf{C}^n$ to the points $(1 : x_1 : \dots : x_n)$.

We treat only the case where the total degree h is greater than the sum of the weights $\sum_{i=1}^n a_i$. We denote the difference $h - \sum_{i=1}^n a_i$ by a_0. Our main interest is the case $a_0 = 1$.

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Let $\hat{X}$ be the image of $\bar{X}$ by the $\mathbf{Z}/a_0\mathbf{Z}$-quotient: $\mathbf{P}(1, a_1, \ldots, a_n) \to \mathbf{P}(a_0, a_1, \ldots, a_n)$. By the adjunction formula, the canonical sheaf of $\hat{X}$ is trivial. We denote a crepant resolution of $\hat{X}$ by X if it exists, in which case $\hat{X}$ has only canonical Gorenstein singularities.

In the case where X_0 is an isolated singularity, it is known that the Milnor lattice or exceptional lattice of resolution and the exceptional lattice at infinity enjoy some dual property [24]. We will show how polar duality of polytopes related to X descends to the duality of weight systems.

When $n = 3$, X is a $K3$ surface. First we assume X is a $K3$ surface corresponding to one of Arnold's singularities [30] [15].

The transcendental lattice T_X decomposes as $L_G \oplus U$, where the dual graph of L_G is a tree of (-2)-elements with three branches and U is the even unimodular hyperbolic lattice of rank 2. We will write $L_D := S_X$ in this case. Then the whole cohomology ring decomposes as $H^*(X, \mathbf{C}) = H^{0,0} \oplus (L_G \oplus U \oplus L_D)_{\mathbf{C}} \oplus H^{2,2}$, where the subscript $\mathbf{C}$ means the tensor product by $\mathbf{C}$. The mirror map should interchange $(L_G)_{\mathbf{C}}$ and $(L_D)_{\mathbf{C}}$, $H^{0,0} \oplus H^{2,2}$ and $U_{\mathbf{C}}$, respectively, since the vector space $(L_G)_{\mathbf{C}}$ corresponds to the tangent space of the deformation space of complex structures with a fixed Picard number and $(L_D)_{\mathbf{C}}$ corresponds to a deformation of complexified algebraic Kähler structures. This is an explanation due to [3] why mirror symmetry for algebraic $K3$ surface is said to correspond to Arnold's duality.

Roan [32] applied orbifold constructions for four of the dual pairs and constructed nonlinear coordinate changes for them. Borcea [8] used Batyrev's polar dual construction for the reflexive pyramids and calculated some of the dual weights.

We shall give a proof that polar dual of polytopes corresponding to Arnold's singularities gives corresponding the dual singularities.

More generally, take a $K3$ surface X which is the minimal resolution of a toric hypersurface with only rational double points. Let L_D be the restriction of the toric divisors of the ambient space and L_0 be the orthogonal complement $L_D^{\perp}$ in S_X. Then $(S_X)_{\mathbf{Q}}$ decomposes as a direct sum $(L_D)_{\mathbf{Q}} \oplus (L_0)_{\mathbf{Q}}$.

The reason why the full algebraic lattice and L_G does not interchange is that the induced complexified Kähler class is trivial on the L_0-part while the embedded deformation has the full dimension. This L_0-part is a generalization of $H_n^{1,1}$ in [32], where Roan showed a similar mirror property for a quotient construction. Batyrev pointed out in a non-published version of [5] that for general Δ, $\operatorname{rk} L_D(\Delta) + \operatorname{rk} L_D(\Delta^*)$ can be smaller than 20. For the 14 exceptional unimodal singularities, we found polytopes for which the sum above equals 20.

Since the mirror of a generic marked $K3$ surface seems to be again a generic one, mirror phenomena for special marked $K3$ surfaces are of particular interest, where 'special' means e.g. the surface is algebraic and its (complexified) Kähler class is also ample.

For general $K3$ surfaces L_0 is defined as the orthogonal complement of the fixed lattice by the automorphism group which fixes $H^{2,0}$ and the (complexified) Kähler class. We show this group is finite using Nikulin's results on the automorphism of a $K3$ surface and Torelli's theorem. H. Tsuji kindly informed the author that the finiteness also follows from Yau's Theorem.

We refer to [3] [21] and their references for an introduction to some background and [26] [40] for general information about mirror symmetry. Dolgachev [14] gave relevant general results and examples using lattice theory. Since the preprint version of our paper appeared, several extensions (e.g. [17] [18]) have been published. The current status of mirror symmetry can be seen in e.g. [20] and [40].

In Section 1, we shall formulate a dual correspondence of reflexive pairs in **Q**-Fano toric projective varieties after reviewing some results of toric geometry. In Section 2, we prepare the language of weight systems according to K. Saito and introduce our notion of duality of weight systems. Our duality partly coincides with Saito's duality. In Section 3, we state the relation between duality of weight systems and Batyrev's toric mirror symmetry using polar polytopes. We find that our construction is partly a generalization of those of Batyrev [5], Borcea [8] and Berglund-Katz [7] for reflexive simplices and pyramids. In Section 4, we review some general results on $K3$ surfaces, and we discuss some lattice theory and automorphisms. We also construct a monomial divisor mirror map of Aspinwall-Greene-Morrison in the case of $K3$ surfaces as toric hypersurfaces and establish Arnold's strange duality. We add some examples which are not Arnold's singularities.

0.2. Notation. We use the following notation throughout the paper.
$\mathbf{N}$: the monoid of nonnegative integers.
$\mathbf{Z}_+$: the semigroup of positive integers.
$\mathbf{Z}$: the ring of integers.
$\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$: the fields of rational, real, complex numbers.
$\mathbf{T}$: n-dimensional complex torus $(\mathbf{C}^*)^n$.

1. Polar duality.

1.1. Reflexive polytopes. We fix here some notation concerning the geometry of convex bodies and review some results. We refer to [28] for general treatment of toric geometry and geometry of convex bodies.

Let M be a free $\mathbf{Z}$-module of rank n. We naturally embed M in the $\mathbf{Q}$-vector space $M_{\mathbf{Q}} := M \otimes_{\mathbf{Z}} \mathbf{Q}$. Similarly for the dual module $N := M^\vee = \mathrm{Hom}(M, \mathbf{Z})$. We denote the natural pairing by $\langle\, , \,\rangle : M \times N \to \mathbf{Z}$, which we also use for its $\mathbf{Q}$-extensions.

DEFINITION 1.1.1. A subset Δ in $M_{\mathbf{Q}}$ is called a *convex polytope* if it is a convex hull of a finite subset of $M_{\mathbf{Q}}$. Δ is *integral* if the finite subset can be taken from M. A *face* of Δ is a nonempty intersection with a hyperplane whose closed half space contains the whole of Δ. We denote the *k-skeleton* of Δ by $\Delta^{[k]}$, which is the union of all faces of Δ whose dimension is no more than k. We call a codimension-one face a *facet*. We denote $|M \cap \Delta|$ and $|M \cap \mathrm{Int}\,\Delta|$ by $l(\Delta)$ and $l^*(\Delta)$, respectively.

REMARK 1.1.2. We use similar terminology for subsets of $N_{\mathbf{Q}}$. We shall sometimes omit the word "convex" in this paper since we always assume this property for polytopes. A face of a polytope Δ is also a polytope.

DEFINITION 1.1.3. Let K, K_1 and K_2 be subsets of $M_{\mathbf{Q}}$ and $c \in \mathbf{Q}$. $cK := \{cx \in M_{\mathbf{Q}} \mid x \in K\}$, $K_1 + K_2 := \{x_1 + x_2 \in M_{\mathbf{Q}} \mid x_i \in K_i \text{ for } i = 1, 2\}$.

DEFINITION 1.1.4. K be a subset in $M_{\mathbf{Q}}$. The *polar dual* of K is the following subset of $N_{\mathbf{Q}}$: $K^*=\{y \in N_{\mathbf{Q}} \mid \langle x, y\rangle \geq -1 \text{ for all } x \in K\}$.

LEMMA 1.1.5. (1) *K^* is a convex set containing* 0. *If $K_1 \subset K_2$ then $K_1^* \supset K_2^*$.*

(2) *If Δ is an n-dimensional convex polytope with* $0 \in \operatorname{Int}\Delta$, *then Δ^* is also an n-dimensional convex polytope with* $0 \in \operatorname{Int}\Delta^*$. *Moreover*, $\Delta^{**} = \Delta$ *by the natural identification $M^{\vee\vee} = M$.*

DEFINITION 1.1.6. For an n-dimensional polytope Δ in $M_{\mathbf{Q}}$, define an n-dimensional projective variety

$$\mathbf{P}_{\Delta, M} = \mathbf{P}_{\Delta} := \operatorname{Proj} \bigoplus_{k=0}^{\infty} \mathbf{C}\langle k\Delta \cap M\rangle T^k ,$$

where $\mathbf{C}\langle k\Delta \cap M\rangle$ denotes the $\mathbf{C}$-vector space generated the elements of M in $k\Delta$, and the multiplication comes from the addition of M. $(\mathbf{P}_{\Delta}, \mathcal{O}(1))$ is the polarized variety associated to the polytope Δ.

DEFINITION 1.1.7. Let Δ be an n-dimensional integral polytope with $0 \in \operatorname{Int}\Delta$. Δ is said to be *reflexive* if one of the following equivalent conditions is satisfied:

(1) Δ^* is also an integral polytope;
(2) there exists a finite subset $\{y_1, \dots, y_k\}$ in N such that $\Delta = \{x \in M_{\mathbf{Q}} \mid \langle x, y_i\rangle \geq -1 \text{ for } 1 \leq i \leq k\}$;
(3) for each facet δ of Δ, there exists an integral element $y \in N$ such that $\delta \subset \{x \in M_{\mathbf{Q}} \mid \langle x, y\rangle = -1\}$;
(4) for each facet δ, there are no points of M between the origin and the hyperplane containing δ.

If Δ is reflexive, Δ^* is reflexive. We cite here some results in [5].

THEOREM 1.1.8. *Let Δ be an n-dimensional integral polyhedron in $M_{\mathbf{Q}}$, $\mathbf{P}_{\Delta}$ the corresponding n-dimensional projective toric variety*, *and $\mathcal{F}(\Delta)$ the family of projective Δ-regular hypersurfaces $\overline{Z}_f$ in $\mathbf{P}_{\Delta}$. Then the following conditions are equivalent*:

(1) *the family $\mathcal{F}(\Delta)$ of Δ-regular hypersurfaces in $\mathbf{P}_{\Delta}$ consists of Calabi-Yau varieties with canonical singularities*;
(2) *the ample invertible sheaf $\mathcal{O}_{\Delta}(1)$ on the toric variety $\mathbf{P}_{\Delta}$ is anticanonical(i.e. $\mathbf{P}_{\Delta}$ is a toric Fano variety with Gorenstein singularities)*;
(3) *Δ contains only one integral point m_0 in its interior, and $(\Delta - m_0, M)$ is a reflexive pair.*

THEOREM 1.1.9. *For any reflexive polyhedron* Δ *of dimension* $n \geq 4$, *the Hodge number* $h^{n-2,1}(\widehat{Z}_f)$ *of a MPCP-desingularization of a* Δ*-regular Calabi-Yau hypersurface* $\overline{Z}_f \subset \mathbf{P}_\Delta$ *equals the Picard number* $h^{1,1}(\widehat{Z}_g)$ *of a MPCP-desingularization of a* Δ^**-regular projective Calabi-Yau hypersurface* $\overline{Z}_g \subset \mathbf{P}_{\Delta^*}$ *corresponding to the dual reflexive polyhedron* Δ^*.

In the above two theorems, "polyhedron" is what we call "polytope" in this paper. We reserve the word "polyhedron" for a three dimensional polytope. An MPCP-desingularization is a maximal projective crepant partial toric embedded resolution.

1.2. Dual correspondence for toric projective varieties. In this section, we shall treat a family of anticanonical members in a $\mathbf{Q}$-Fano toric variety $\mathbf{P}_\Delta$ where Δ may not be integral. We will consider not only the quotient family corresponding to the same Δ but also specializations corresponding to sub-Newton polytopes.

DEFINITION 1.2.1. Let $\mathcal{P}_n$ be the set of n-dimensional pairs (Δ, M) where M is a free $\mathbf{Z}$-module of rank n and Δ is an n-dimensional polytope in $M_\mathbf{Q}$ which contains 0 in its interior, such that $\mathbf{P}_{\Delta,M}$ is $\mathbf{Q}$-Fano and Δ represents the anticanonical divisor.

We usually assume that $n \geq 2$. Let $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ be two elements of $\mathcal{P}_n$. We denote the dual groups $(\overline{M})^\vee$ by $\underline{N}$ and $(\overline{N})^\vee$ by $\underline{M}$. We also denote the polar duals $\overline{\Delta}^*$ and $\overline{\nabla}^*$ by $\underline{\nabla}$ and $\underline{\Delta}$, respectively.

DEFINITION 1.2.2. A $\mathbf{Q}$-linear isomorphism $\sigma : \underline{M}_\mathbf{Q} \to \overline{M}_\mathbf{Q}$ such that $\sigma(\underline{\Delta}) \subset \overline{\Delta}$, is called a *correspondence* between $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ if $\sigma(\underline{\Delta}) \cap \overline{M}$ generates $\overline{M}$ and $\underline{\Delta}$ has a nonempty intersection with each facet of $\overline{\Delta}$.

If $\sigma(\underline{M}) = \overline{M}$ and ${}^t\sigma$ is a correspondence, we call σ a *dual correspondence*.

We shall identify $\underline{M}_\mathbf{Q}$ and $\overline{M}_\mathbf{Q}$, $\overline{N}_\mathbf{Q}$ and $\underline{N}_\mathbf{Q}$ by σ and ${}^t\sigma$, respectively. We shall sometimes omit σ.

When $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ have a dual correspondence σ, we will use $M = \overline{M} = \sigma(\underline{M})$ and $N = \overline{N} = {}^t\sigma(\underline{N})$.

A basic observation is the following

PROPOSITION 1.2.3. *Assume that there exists a dual correspondence between two elements* $(\overline{\Delta}, M)$ *and* $(\overline{\nabla}, N)$ *of* $\mathcal{P}_n$. *Let* (Δ, M) *be a reflexive pair such that* $\underline{\Delta} \subset \Delta \subset \overline{\Delta}$.

Then there is a natural birational map $\Phi : \mathbf{P}_{\overline{\Delta}} \dashrightarrow \mathbf{P}_\Delta$ *coming from the inclusion between the polytopes. A general anticanonical member of* $\mathbf{P}_\Delta$ *is birational to the corresponding member of* $\mathbf{P}_{\overline{\Delta}}$.

Similarly, from $\underline{\nabla} \subset \Delta^* \subset \overline{\nabla}$, *a general anticanonical member of* $\mathbf{P}_{\Delta^*}$ *is birational to that of* $\mathbf{P}_{\overline{\nabla}}$.

PROOF. We denote the sublinear system of the anticanonical linear system of $\mathbf{P}_{\overline{\Delta}}$ corresponding to the polytope Δ by $\Lambda(\Delta)$.

By definition, any nonempty nonzero subsystem Λ is free from base points on the n-dimensional orbit of $\mathbf{T}$.

Since the set $\Delta \cap M$ generates the whole M, $\Lambda(\Delta)$ defines a birational map $\overline{\varphi} : \mathbf{P}_{\overline{\Delta}} \dashrightarrow \mathbf{P}^{l(\Delta)-1}$. The image V is the projective variety $\operatorname{Proj} \bigoplus_{k=0}^{\infty} \left(\sum_{i=1}^{k} (\Delta \cap M) \right)$, which is associated to the graded algebra generated by $\Delta \cap M$.

Each facet of $\overline{\Delta}$ corresponds to an $(n-1)$-dimensional orbit of $\mathbf{T}$ on $\mathbf{P}_{\overline{\Delta}}$, which has nonempty intersection with Δ, since Δ contains $\underline{\Delta}$. Thus $\Lambda(\Delta)$ is free from base components and the birational transform of a hyperplane section of V in $\mathbf{P}_{\overline{\Delta}}$ is a member of $\Lambda(\Delta)$. In particular, a general member of $\Lambda(\Delta)$ is an irreducible divisor if $n > 1$.

On the other hand, the anticanonical complete linear system on $\mathbf{P}_{\Delta}$ is free from base points, since Δ is an integral polytope, and defines a morphism φ to the same V, which is birational by the same reason as above. Take $\Phi := \varphi^{-1} \circ \overline{\varphi}$. □

REMARK 1.2.4. Thus, in the situation above, if a Calabi-Yau variety $\bar{Z}_{\Delta}$ is a specialization of $\bar{Z}_{\Delta'}$, then $\bar{Z}_{\Delta^*}$ is a generalization of $\bar{Z}_{\Delta'^*}$. As a result, we have more Kähler moduli and less complex structure moduli for the desingularization $\tilde{Z}_{\Delta}$, and the opposite for $\tilde{Z}_{\Delta^*}$. We believe that this relation holds also outside toric varieties.

REMARK 1.2.5. $\mathbf{P}_{\Delta}$ is isomorphic to V if $n \leq 3$. Note that we have unique minimal models in this case.

PROPOSITION 1.2.6. *Let Δ be a reflexive polytope of dimension at most three, and $(\mathbf{P}_{\Delta}, \mathcal{O}(1))$ be the polarized variety associated to it. Then $\mathcal{O}(1)$ is simply generated and defines a projective normal embedding in $\mathbf{P}^{l(\Delta)-1}$.*

PROOF. The proof follows from the following

PROPOSITION 1.2.7. *Let Δ be a reflexive polytope in $M_{\mathbf{Q}}$ whose dimension is at most three, and k be a positive integer. Then for any integral point $P \in k\Delta$, there exist integral points $P_i \in \Delta$ $(i = 1, \ldots, k)$ with $P = P_1 + \cdots + P_k$.*

PROOF. We assume $n = 3$, since the other cases are similar and easier. For each 2-dimensional face δ of Δ, we have a subdivision by integral simplices (triangles) such that

(1) each simplex is elementary, that is, it does not contain points of M other than the vertices,
(2) each simplex has common points with other simplices only on its edges, and
(3) the union of all simplices covers δ.

We note that an elementary simplex of dimension two is regular. That means, if we denote the vertices of the simplex by Q_0, Q_1 and Q_2 and the plane containing δ by α, $Q_1 - Q_0$ and $Q_2 - Q_0$ spans $\alpha \cap M$. Let C be the closed cone over the simplex with apex at 0. Then any integral point in C can be written as an integral combination of the Q_i's, since Δ is reflexive.

REMARK 1.2.8. In higher dimensions, there exist elementary non-regular simplices.

In general for an ample divisor A on an algebraic $K3$ surface, it is known that $3A$ is very ample.

COROLLARY 1.2.9. *For a general toric hypersurface $\bar{Z}$ of dimension* 1 *or* 2, *the induced morphism $\bar{Z} \to \mathbf{P}^{l(\Delta)-2}$ is a projectively normal embedding.*

PROOF. $H^1(\mathbf{P}_\Delta, \mathcal{O}(k))$ vanishes for $k \geq 0$. □

In Section 3, we shall apply these notions to the case of weighted projective spaces with one particular choice of a homogeneous coordinate.

2. Duality for weight systems

2.1. Weighted projective space. We fix here some notations. For general properties of weighted projective spaces, see for example, [25] [13] [19].

DEFINITION 2.1.1. Let $a_0, \ldots, a_n$ be fixed positive integers. We denote by $\mathbf{P}(a_0, \ldots, a_n)$ the projective variety Proj $\mathbf{C}[x_0, \ldots, x_n]$ where the degree of x_i is a_i. We call it a *weighted projective space* of weight $(a_0, \ldots, a_n)$.

For any positive integer k, there is a natural isomorphism $\mathbf{P}(a_0, \ldots, a_n) \cong \mathbf{P}(ka_0, \ldots, ka_n)$.

DEFINITION 2.1.2. A weight $(a_0, \ldots, a_n)$ is *reduced* if $\gcd(a_0, \ldots, a_n) = 1$.

We assume the weight is reduced. Let k be $\gcd(a_1, \ldots, a_n)$. Then $\mathbf{P}(a_0, a_1, \ldots, a_n) \cong \mathbf{P}(a_0, a_1/k, \ldots, a_n/k)$, which leads to the following

DEFINITION 2.1.3. A reduced weight $(a_0, \ldots, a_n)$ is *well-formed* if for all i, $\gcd(a_j)_{j \neq i} = 1$.

We shall usually treat only well-formed weights.

2.2. Weight systems. We refer to [35] for general results on weight systems, especially for $n = 3$.

DEFINITION 2.2.1. An $(n+1)$-tuple of positive integers $W = (a_1, \ldots, a_n; h)$ is called a *system of weights* or simply a *weight system* or a *weight*. We always assume that $h \in \sum_{i=1}^n \mathbf{N}a_i$. We call the integers a_i *weights* of W and the last weight h the *degree* of W.

DEFINITION 2.2.2. W is said to be *reduced* if $\gcd(a_1, \ldots, a_n, h) = 1$.

REMARK 2.2.3. If W is reduced then $\gcd(a_1, \ldots, a_n) = 1$, since $h \in \sum_{i=1}^n \mathbf{N}a_i$.

DEFINITION 2.2.4. $W = (a_1, \ldots, a_n; h)$ and $W' = (a'_1, \ldots, a'_n; h')$ are *equivalent* if for some rational number k and a permutation $\sigma \in \mathfrak{S}_n$, we have $ka_{\sigma(i)} = a'_i$ $(1 \leq i \leq n)$ and $kh = h'$.

For each equivalence class of weight systems there is exactly one reduced weight system satisfying $a_1 \leq a_2 \leq \cdots \leq a_n$. We usually choose this one.

2.3. Duality for weight systems. Let n be a positive integer. Take a weight system $W_a = (a_1, \dots, a_n; h)$. Assume that $h \in \sum_{i=1}^n \mathbf{N}a_i$ as always, and that W_a is reduced, for the sake of brevity. We denote the integral vector ${}^t(a_1, \dots, a_n)$ by a.

DEFINITION 2.3.1. $a_0 := h - \sum_{i=1}^n a_i$.

Assume also that $a_0 \neq 0$.

For a rational monomial $X_0^{m_0} \cdots X_n^{m_n}$ of degree h, we assign an n-tuple of integers $(\alpha_1, \dots, \alpha_n) = (m_1 - 1, \dots, m_n - 1)$. Note that $X_0 \cdots X_n$ corresponds to the origin. Such n-tuples constitute the following set:

DEFINITION 2.3.2.

$$M(W_a) := \left\{ (\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^{\oplus n} \;\middle|\; \sum_{i=1}^n a_i(\alpha_i + 1) \equiv h \mod a_0 \right\}.$$

LEMMA 2.3.3. *$M(W_a)$ is a subgroup of $\mathbf{Z}^{\oplus n}$ of index $|a_0|$.*

PROOF. This follows from $M(W_a) = \{(\alpha_1, \dots, \alpha_n) \in \mathbf{Z}^{\oplus n} \mid \sum_{i=1}^n a_i \alpha_i \equiv 0 \mod a_0\}$ and $\gcd(a_1, \dots, a_n) = 1$. $\square$

REMARK 2.3.4. If W_a and W_b are equivalent, $M(W_a)$ and $M(W_b)$ are the same up to the permutation of the coordinates. We sometimes abbreviate this set as M in this section.

Let $C = (c_{ij})$ be an $n \times n$-matrix whose elements are nonnegative integers. Let B be the $n \times n$-matrix $(c_{ij} - 1)$.

LEMMA 2.3.5. *Assume $Ca = {}^t(h, \dots, h)$. Then*

(1) $(\det C)/h = (\det B)/a_0$, *and this is an integer.*

(2) *The following three conditions are equivalent:* (a) $\{(c_{i1} - 1, \dots, c_{in} - 1) \mid 1 \leq i \leq n\}$ *is a basis of M*, (b) $|\det B| = |a_0|$, *and* (c) $|\det C| = h$.

PROOF. Note that for $1 \leq i \leq n$, $(c_{i1} - 1, \dots, c_{in} - 1) \in \pi(a_0) \cap M$ and $(c_{i1}, \dots, c_{in}) \in \pi(h)$, where $\pi(t)$ is the hyperplane defined by $\sum_{i=1}^n a_i \alpha_i = t$. The rest is clear. $\square$

REMARK 2.3.6. We will say a few words for the case where W_a is not reduced. Let d_0 be $\gcd(a_1, \dots, a_n)$. Then M is a subgroup of index $|a_0|/d_0$, and the row vectors of B is a basis $\iff |\det B| = |a_0|/d_0 \iff |\det C| = h/d_0$.

Let $W_a = (a_1, \dots, a_n; h)$ and $W_b = (b_1, \dots, b_n; k)$ be two weight systems.

DEFINITION 2.3.7. An integer matrix $C \in M_n(\mathbf{N})$ is said to be a *weighted magic square* of weight $(W_a; W_b)$ if $C^t(a_1, \dots, a_n) = {}^t(h, \dots, h)$ and $(b_1, \dots, b_n)C = (k, \dots, k)$.

REMARK 2.3.8. In the case $a_1 = \cdots = a_n = b_1 = \cdots = b_n = 1$ and $h = k$, C is called an *integer stochastic matrix* or *magic square*, as in classical combinatorics theory [38].

DEFINITION 2.3.9. A weighted magic square C is *primitive* if $|\det C| = h/\gcd(a_1, \dots, a_n) = k/\gcd(b_1, \dots, b_n)$.

REMARK 2.3.10. C is primitive if and only if the row vectors of B span $M(W_a)$ and the column vectors span $M(W_b)$.

DEFINITION 2.3.11. When there exists a primitive weighted magic square C of weight $(W_a; W_b)$, we say W_a and W_b are *dual* with respect to C.

Dual weight systems are *strongly dual* if all rows and columns of C contain 0.

REMARK 2.3.12. If W_a and W_b are reduced dual weights, it follows that $h = k$, $\sum_{i=1}^n a_i = \sum_{i=1}^n b_i$ and $a_0 = b_0$.

A permutation on weights $(a_1, \dots, a_n)$ (resp. $(b_1, \dots, b_n)$) interchanges the corresponding columns (resp. rows) of C. From this definition, one can calculate the dual weights.

The following Proposition is convenient for the calculation.

PROPOSITION 2.3.13. *Let $W_a = (a_1, \dots, a_n; h)$ and $W_b = (b_1, \dots, b_n; h)$ be dual weight systems with a_0 be $h - \sum_{i=1}^n a_i > 0$ and let $C = (c_{ij})$ be the corresponding weighted magic square. Let Θ be the $(n-1)$-simplex in $M(W_a)_{\mathbf{Q}}$ with vertices $\{(c_{i1} - 1, \dots, c_{in} - 1) | 1 \le i \le n\}$. Then*

(1) $\sum_{j=1}^n c_{ij} a_j = h$ *for* $1 \le i \le n$,

(2) $\dfrac{a_0}{h - a_0}(1, \dots, 1) \in \operatorname{Int} \Theta$ *and*

(3) *Θ is elementary, i.e., $\Theta \cap M(W_a)$ are the set of all vertices of Θ.*

Conversely, for a given $(a_1, \dots, a_n) \in \mathbf{Z}^{\oplus n}$, if there exists an $(n-1)$-simplex Θ with vertices $\{(c_{i1} - 1, \dots, c_{in} - 1) | 1 \le i \le n\}$ which satisfies 1, 2 and 3 above, then there exists a weight system $W_b = (b_1, \dots, b_n; h)$ such that C is a weighted magic square of weight $(W_a; W_b)$. If W_b is reduced, then it is a dual weight system.

PROOF. The assertions (1) and (3) are trivial. The remaining assertion (2) is equivalent to saying that the origin sits inside the n-simplex Δ which is the cone over Θ with apex $(-1, \dots, -1)$. This is equivalent to saying that all the coefficients of the linear relation between the vertices of Δ have the same sign. Since we can take nothing but b_i's as coefficients, $b_0{}^t(-1, \dots, -1) + \sum_{i=1}^n b_i{}^t(c_{i1} - 1, \dots, c_{in} - 1) = 0$, and (2) follows.

We will prove the converse. From the argument above, the b_i's are determined as positive integers up to ratio. We will take the reduced b_i's first. We define $k := \sum_{i=0}^n b_i$ and $W_b := (b_1, \dots, b_n; k)$. Then C is a weighted magic square of weight $(W_a; W_b)$. We note that $|\det C|$ is a multiple of k, and by 3, $|\det C| = h$. Thus multiplying the b_i's by some integer, we can achieve $h = k$. In particular, if this new W_b is reduced then C is primitive since $k = |\det C|$. □

DEFINITION 2.3.14. A weight system W is *self-dual* if W and W are dual.

REMARK 2.3.15. If one can take a symmetric matrix as C, then W is self-dual. In general, there may be several (a finite number of) dual weight systems as shown in the Proposition below. Similarly, self-duality does not mean that the given weight system is the only dual weight system.

2.4. Examples. First we show examples of the case $a_0 = -1$, which should be regarded as mirror symmetry for log Calabi-Yau manifolds.

PROPOSITION 2.4.1. *The weight systems* $A_{l-1} = \{(1, k, l-k; l),\ 1 \leq k \leq l/2)\}$ $(l \geq 2)$ *are closed under duality for each* l. $D_{l+1} = (2, l-1, l; 2l)$ $(l \geq 3)$, $E_6 = (3, 4, 6; 12)$, $E_7 = (4, 6, 9; 18)$ *and* $E_8 = (6, 10, 15; 30)$ *are self-dual.*

PROOF. The case A_{l-1}.

For an arbitrary positive integer j such that $jk < l$, we can take C as

$$\begin{pmatrix} l-jk & j & 0 \\ k & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then the dual weight system is $(1, j, l-j; l)$. These are the only choice for C up to permutation.

For D_{l+1}(l: odd), D_{l+1}(l: even), E_6, E_7 and E_8, take the following matrices as C:

$$\begin{pmatrix} (l+1)/2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \begin{pmatrix} l/2 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 2 \\ 0 & 3 & 0 \\ 2 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

They are all symmetric and unique up to the permutation of the rows. They are strongly dual. □

Next, we will treat the case of the exceptional unimodal singularities [1]. They are all weighted homogeneous with $a_0 = 1$.

THEOREM 2.4.2. *Let* $W = (a_1, a_2, a_3; h)$ *be a weight system which corresponds to one of the* 14 *exceptional unimodal singularities. Then* W *has a unique dual weight system* W^* *up to equivalence. This* W^* *is the weight system which corresponds to Arnold's strange duality.*

PROOF. We list below the choices of C. These are, in fact, the unique choices for primitive C up to permutation. They are automatically strongly dual.

Let us take coordinates x, y and z. In the Table below, each monomial $x^\alpha y^\beta z^\gamma$ in the column C appears as a row vector (α, β, γ) of C. For example, in the case E_{14}, $C = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$, and which is nothing but the transposed matrix ${}^tC$ for Q_{10}.

We list also C_0, which shows the vertices of the maximum Newton polytope of polynomials of degree h of x, y and z, as well as the Gabrielov numbers and the Dolgachev numbers [1]. □

TABLE 1. (0) Arnold's singularities.

class	a_1	a_2	a_3	h	C_0			C			Gab	Dol
E_{12}	6	14	21	42	x^7	y^3	z^2	x^7	y^3	z^2	2 3 7	2 3 7
E_{13}	4	10	15	30	x^5y	y^3	z^2	x^5y	y^3	z^2	2 3 8	2 4 5
Z_{11}	6	8	15	30	x^5	xy^3	z^2	x^5	xy^3	z^2	2 4 5	2 3 8
E_{14}	3	8	12	24	x^8	y^3	z^2	x^4z	y^3	z^2	2 3 9	3 3 4
Q_{10}	6	8	9	24	x^4	y^3	xz^2	x^4	y^3	xz^2	3 3 4	2 3 9
Z_{12}	4	6	11	22	x^4y	xy^3	z^2	x^4y	xy^3	z^2	2 4 6	2 4 6
W_{12}	4	5	10	20	x^5	y^4	z^2	x^5	z^2	y^2z	2 5 5	2 5 5
Z_{13}	3	5	9	18	x^6	xy^3	z^2	x^3z	xy^3	z^2	2 4 7	3 3 5
Q_{11}	4	6	7	18	x^3y	y^3	xz^2	x^3y	y^3	xz^2	3 3 5	2 4 7
W_{13}	3	4	8	16	x^4y	y^4	z^2	x^4y	z^2	y^2z	2 5 6	3 4 4
S_{11}	4	5	6	16	x^4	xz^2	y^2z	x^4	xz^2	y^2z	3 4 4	2 5 6
Q_{12}	3	5	6	15	x^5	y^3	xz^2	x^3z	y^3	xz^2	3 3 6	3 3 6
S_{12}	3	4	5	13	x^3y	xz^2	y^2z	x^3y	xz^2	y^2z	3 4 5	3 4 5
U_{12}	3	4	4	12	x^4	y^3	z^3	x^4	y^2z	yz^2	4 4 4	4 4 4

REMARK 2.4.3. K. Saito defines a duality of weights using a duality of poset diagrams coming from the eigenvalues of the monodromy of minimally elliptic singularities [37]. His duality also reproduces Arnold's duality and he also computed the dual weights for the 49 weight systems corresponding to the minimally elliptic singularities which are not simple elliptic. His results include the uniqueness of duality for isolated singularities.

As shown below, our duality for $n = 3$ coincides with Saito's in the cases: (0) the fourteen unimodal singularities and (3) $a_0 \geq 2$. For cases (2) $a_0 = 1$ and modality m is more than one, our duality gives new dual weights, where nonisolated singularities appear as dual partners of isolated singularities. These of course also correspond to the polar duality, as shown in Section 5.

It is natural to treat the simple elliptic cases (1) for $n = 2$, since these correspond to weighted elliptic curves. Their list is the same as that of the weighted 2-spaces which correspond to reflexive polytopes. The weights are self-dual for $n = 2$.

Notice that the singularities of dual weights are not dual in the sense of Arnold's duality but their (explicit) specializations enjoy this duality as shown in later sections.

PROPOSITION 2.4.4. (1) $(1, 1; 3)$, $(1, 2; 4)$, $(2, 3; 6)$ *are self-dual.*

(2) $(2, 2, 3; 8)$, $(2, 2, 5; 10)$, $(2, 3, 4; 10)$, $(2, 4, 7, 14)$ *and* $(2, 6, 9, 18)$ *are self-strongly dual*; $(2, 3, 6; 12)$ *and* $(2, 4, 5; 12)$ *are strongly dual.* $(2, 3, 3; 9)$ *has no strongly dual weight.*

(3) *Duality of weights for minimally elliptic weight systems with* $a_0 > 1$ *coincides with Saito's duality* [37].

PROOF. Here we give the full list of primitive C up to permutation in Tables 1–3. The notation of classes is after [1] and [22].

The * in (2) signifies that the dual weight is not strongly dual, hence the singularity corresponding to the smaller triangle is reducible. Other weight systems are strongly dual. □

TABLE 2. (1) Simple elliptic singularities.

class	C		a_1	a_2	h	dual	
P_8	x^2y	xy^2	1	1	3	1	1
X_9	y^2	x^2y	1	2	4	1	2
J_{10}	x^3	y^2	2	3	6	2	3

TABLE 3. (2) Minimal elliptic singularities with $a_0 = 1$ and $m > 1$

class	C			a_1	a_2	a_3	h	dual		
$V_{1,0} = V_{15}$	xz^2	yz^2	x^2y^2	2	2	3	8	2	2	3
	x^2y^2	xy^3	xz^2				*	1	2	4
N_{16}	x^3y^2	x^2y^3	z^2	2	2	5	10	2	2	5
$U_{1,0} = U_{14}$		—		2	3	3	9		—	
	y^2z	x^3y	yz^2				*	1	3	4
$S_{1,0} = S_{14}$	y^2z	x^2y^2	xz^2	2	3	4	10	2	3	4
	xz^2	x^3z	y^2z				*	1	3	5
$W_{1,0} = W_{15}$	y^2z	x^3y^2	z^2	2	3	6	12	2	4	5
	z^2	x^3z	y^2z				*	1	4	6
$Q_{2,0} = Q_{14}$	y^3	x^2y^2	xz^2	2	4	5	12	2	3	6
$Z_{1,0} = Z_{15}$	xy^3	x^3y^2	z^2	2	4	7	14	2	4	7
$J_{3,0} = J_{16}$	y^3	x^3y^2	z^2	2	6	9	18	2	6	9

REMARK 2.4.5. The four weight systems in (3): $(6, 8, 13; 32)$, $(6, 16, 21; 48)$, $(6, 16, 27; 54)$ and $(8, 10, 15; 40)$ are not in the list of Reid [31] [19], hence do not correspond to $K3$ surface with cyclic quotient singularities.

For the weights with some C but without a dual weight (e.g. W_{17}), C does not satisfy primitivity for the dual weight. On the contrary, they enjoy a dual correspondence defined in Section 1, except for $(5, 6, 15; 30)$.

All the dual weights are strongly dual.

REMARK 2.4.6. There are 41 weight systems with $a_0 = 1$ in Reid's list and 19 of them are not minimally elliptic. A similar calculation shows that $(1, 1, 1; 4)$, $(1, 1, 2; 5)$, $(1, 2, 2; 6)$, $(1, 2, 3; 7)$ and $(1, 2, 4; 8)$ are self-dual; $(1, 1, 3; 6)$ and $(1, 2, 2; 6)$ are dual;

$(1, 2, 4; 8)$, $(1, 3, 4; 9)$, $(1, 3, 5; 10)$ and $(1, 4, 6; 12)$ have dual weight systems which are minimally elliptic as in Proposition 2.4.4 (2); and all ten weight systems with $a_1 > 1$ do not have dual weight systems.

3. Relation of duality of weight systems and polar duality

3.1. We shall show in this section that the duality of weights for $a_0 = 1$ is a special case of the dual correspondence defined in Section 1. We will always choose one coordinate for the compactification parameter.

TABLE 4. (3) Minimal elliptic singularities with $a_0 > 1$.

class	C			a_1	a_2	a_3	h	dual		
V'_{18}		—		3	3	4	12	—	—	—
U_{16}	x^5	y^2z	yz^2	3	5	5	15	3	5	5
S_{16}	x^4y	xz^2	y^2z	3	5	7	17	3	5	7
W_{17}	x^5y	z^2	y^2z	3	5	10	20	—	—	—
Q_{16}	x^4z	y^3	xz^2	3	7	9	21	3	7	9
Z_{17}	x^4z	xy^3	z^2	3	7	12	24	—	—	—
E_{18}	x^5z	y^3	z^2	3	10	15	30	—	—	—
${}_2V^*_{18}$	xy^3	x^3z	yz^2	4	5	7	19	4	5	7
${}_1V^*_{18}$	x^3z	y^4	xz^2	4	5	8	20	4	5	8
N_{19}	x^3z	xy^4	z^2	4	5	12	24	—	—	—
S_{17}	x^6	xz^2	y^2z	4	7	10	24	—	—	—
W_{18}	x^7	y^2z	z^2	4	7	14	28	4	7	14
Q_{17}	x^5y	y^3	xz^2	4	10	13	30	—	—	—
Z_{18}	x^6y	xy^3	z^2	4	10	17	34	4	10	17
E_{19}	x^7y	y^3	z^2	4	14	21	42	—	—	—
${}_3V^*_{19}$	y^4	x^3z	yz^2	5	6	9	24	—	—	—
${}_2N_{20}$	y^5	x^3z	z^2	5	6	15	30	—	—	—
V'_{20}	x^3z	z^3	xy^3	6	7	9	27	6	7	9
${}_2V^*_{19}$	x^5	xy^3	yz^2	6	8	11	30	—	—	—
${}_1V^*_{19}$	x^4y	y^4	xz^2	6	8	13	32	4	7	16
${}_1N_{20}$	x^5y	xy^4	z^2	6	8	19	38	6	8	19
Q_{18}	x^8	y^3	xz^2	6	16	21	48	3	16	24
Z_{19}	x^9	xy^3	z^2	6	16	27	54	4	18	27
E_{20}	x^{11}	y^3	z^2	6	22	33	66	6	22	33

V'_{21}	z^3	y^4	x^3z	8	9	12	36	8	9	12
V^*_{20}	y^4	x^5	yz^2	8	10	15	40	5	8	20
N_{21}	y^5	x^5y	z^2	8	10	25	50	8	10	25

Let $W_a = (a_1, \dots, a_n; h)$ be a weight system with $h \in \sum_{i=1}^n \mathbf{N}a_i$ and $a_0 := h - \sum_{i=1}^n a_i$ as usual. We assume $a_0 > 0$ in this section.

Let $\mathbf{P}(a)$ be the weighted projective space of weight $(a_0, a_1, \dots, a_n)$ and $X_0, \dots, X_n$ be the natural homogeneous coordinates of $\mathbf{P}(a)$. Then the n-dimensional complex torus $\mathbf{T}$ is embedded equivariantly in $\mathbf{P}(a)$ as the locus $\{F_0 := \prod_{i=0}^n X_i \neq 0\}$.

Let $M(W_a)$ be the group defined in Section 3. $M(W_a)$ is nothing but the abelian group of exponents of rational monomials, which appears in the context of toric constructions. For a point $P = (\alpha_1, \dots, \alpha_n)$ on $M(W_a)$, we write the corresponding monic monomial by $F_P = \prod_{i=0}^n X_i^{\alpha_i+1}$, where α_0 is determined by $\sum_{i=0}^n a_i\alpha_i = 0$.

DEFINITION 3.1.1. Let $\overline{\Delta}(a)$ be the n-simplex in $M(W_a)_{\mathbf{Q}}$ which is the convex hull of the following vertices: $(-1 + h/a_1, -1, \dots, -1)$, $\dots$, $(-1, \dots, -1, -1 + h/a_n)$ and $(-1, \dots, -1)$.

$(\overline{\Delta}, \overline{M}) = (\overline{\Delta}(a), M(W_a))$ is an element of $\mathcal{P}_n$ with $\mathbf{P}_{\overline{\Delta}, \overline{M}} \cong \mathbf{P}(a)$.

LEMMA 3.1.2. *Let $(\overline{\Delta}, M)$ be as above. Then its dual $(\overline{\Delta}^*, M^*)$ is integral and the vertices generate M^*.*

PROOF. For the sake of simplicity, we assume that W_a is reduced. Recall that $M = \{(\alpha_1, \dots, \alpha_n) | \sum_{i=1}^n a_i\alpha_i \equiv 0 \mod a_0\}$. Let $M' := \mathbf{Z}^{\oplus n}$, which contains M as a $\mathbf{Z}$-submodule of index a_0. If one computes the dual of $(\overline{\Delta}, M')$, it is an n-simplex with vertices $v_1 := (1, 0, \dots, 0)$, $\dots$, $v_n := (0, \dots, 0, 1)$ and $v_0 := (-a_1/a_0, \dots, -a_n/a_0)$ in $(M')^*_{\mathbf{Q}}$. Thus $v_1, \dots, v_n$ generates $(M')^*$. Since $(M')^*$ is a submodule of M^*, Those n vertices are also contained in M^*. On the other hand, v_0 also belongs to M^* since $\sum_{i=1}^n (-a_i/a_0)\alpha_i \in \mathbf{Z}$ for any $(\alpha_1, \dots, \alpha_n) \in M$.

The order of v_0 in $M^*/(M')^*$ is exactly a_0 since $(a_1, \dots, a_n)$ is reduced. Thus the last statement follows. □

REMARK 3.1.3. By the previous proof, the submodule $(M')^*$ generated by $v_1, \dots, v_n$ has index a_0 in M.

Let $(\overline{\nabla}, \overline{N}) = (\overline{\Delta}(b), M(W_b))$ be another pair. We assume W_b is also reduced, for the sake of simplicity.

We restrict here the correspondence σ in Section 1 as follows: (*)

(1) σ sends the vertex of $(\overline{\Delta})^*$ which corresponds to the divisor $\{X_0 = 0\}$ to the vertex of $\overline{\nabla}$ which corresponds to the point $(1 : 0 : \cdots : 0)$ (i.e., $X_1 = \cdots = X_n = 0$).

(2) similarly for ${}^t\sigma$.

PROPOSITION 3.1.4. *Assume that $(\overline{\Delta}, \overline{M})$ and $(\overline{\nabla}, \overline{N})$ has a dual correspondence σ which satisfies the condition $(*)$ above.*

Let $(c_{i1} - 1, \dots, c_{in} - 1)$ $(1 \leq i \leq n)$ be the vertices other than $(-1, \dots, -1)$ of $\sigma(\overline{\nabla}^*)$ in $\overline{M}$. Let B be the $n \times n$-matrix $(c_{ij} - 1)$.

Then

(1) $a_0|h$, $b_0|h$ *and* $|\det B| = a_0 b_0$,

(2) *If* $a_0 = b_0 = 1$, W_a *and* W_b *are dual weight systems.*

PROOF. (1) As $\sigma(\overline{N}^*) = \overline{M}$ and $\sigma(-b_1/b_0, \ldots, -b_n/b_0) = (-1, \ldots, -1)$, $(-1, \ldots, -1) \in \overline{M}$ follows. Thus $a_0|h$. Similarly for $b_0|h$. Since $\sigma(1, 0, \ldots, 0)$, $\ldots$, $\sigma(0, \ldots, 0, 1)$ generates a submodule of index b_0 in $\overline{M}$, $|\det B| = a_0 b_0$.

(2) Let C be (c_{ij}). By condition (*), C is a weighted magic square of weight $(W_a; W_b)$ and primitive by $|\det B| = 1$ using Lemma 2.3.5. □

Thus we have

THEOREM 3.1.5. *Assume two weight systems* W_a *and* W_b *with* $a_0 = b_0 = 1$ *are dual. Then*

(1) *there exists an identification* $M = (\overline{N})^* = \overline{M}$ *such that* $\overline{\nabla}^*$ *is an integral simplex and a subcone of* $\overline{\Delta}$ *with a common apex* $(-1, \ldots, -1)$ *and a common plane* $\pi = \left\{\sum_{i=1}^{n} a_i \alpha_i = a_0\right\}$ *of facets. The vertices of* $\overline{\nabla}^*$ *on* π *generate* M. *Similarly for the induced identification* $N = (\overline{M})^* = \overline{N}$.

(2) *If the linear subsystem of the anticanonical divisor of* $\mathbf{P}(a)$ *corresponds to some reflexive polytope* Δ *such that* $\overline{\nabla}^* \subset \Delta \subset \overline{\Delta}$, *then the family corresponding to* Δ^* *is the linear subsystem of the anticanonical divisor of* $\mathbf{P}(b)$. *This relation is inclusion-reversing with respect to polytopes.*

REMARK 3.1.6. The facet above is the Newton polytope for an $(n-1)$-dimensional hypersurface singularity X_0 with $\mathbf{C}^*$-action. $\bar{Z}$ is the toric compactification in $\mathbf{P}(a)$ of a deformation Z of X_0.

REMARK 3.1.7. Let Δ be the *full Newton polytope* of $\mathbf{P}(a)$, that is the convex hull of all vertices corresponding to the monomial anticanonical divisors. We have the associated projective variety $\mathbf{P}_\Delta$ as usual. In general, Δ does not contain all the generators of the ring of regular functions of $\mathbf{P}(a)$. Thus $\mathbf{P}(a)$ and $\mathbf{P}_\Delta$ are not isomorphic in general.

REMARK 3.1.8. By [5] (Theorem 5.4.5, Corollary 5.4.6), for a reflexive simplex $\Delta = \overline{\Delta}$, that is, for the case $a_i|h$ for all i, $\mathbf{P}_\Delta$ is a weighted projective variety whose weight is the coefficients $(a_0, \ldots, a_n)$ of the unique linear relation $\sum_{i=0}^{n} a_i P_i = 0$ $(a_i \in \mathbf{Z}_+$, $\gcd(a_0, \ldots, a_n) = 1)$ among the vertices $\{P_i\}$. Moreover, the polar dual family of deformation of Fermat-type hypersurfaces is the quotient of a subfamily by $\pi_1(\Delta, M)$ [5](Corollary 5.5.6).

REMARK 3.1.9. Borcea's construction [8] does not assume the property (*) which arises from singularity theory. As he has pointed out, for a given reflexive polytope, its realization as a family of weighted hypersurface involves some choice. Nevertheless, for Arnold's singularities, our duality of weights is unique and compatible with that of singularity theory.

4. Application for $n = 3$

4.1. $K3$ **surfaces.** We refer the reader to [6] for more complete information about $K3$ surfaces and [3] for string theory on $K3$ surfaces.

Let X be a $K3$ surface, that is, a compact complex surface with trivial canonical bundle and irregularity $h^1(\mathcal{O}_X) = 0$. It is well known that any $K3$ surface is simply connected, Kähler and symplectic. The only nonzero Hodge numbers of X are $h^{0,0} = h^{2,0} = h^{0,2} = h^{2,2} = 1$ and $h^{1,1} = 20$.

First we recall here some properties of lattices.

DEFINITION 4.1.1. Let L be a free $\mathbf{Z}$-module of finite rank with a symmetric bilinear $\mathbf{Z}$-valued pairing $\langle\, ,\, \rangle$. Such an L is called a *lattice*. L is *even* if for each $x \in L$, $\langle x, x\rangle$ is an even integer, and *unimodular* if the discriminant is ± 1. The *index* of L is the triplet $(\lambda_+, \lambda_0, \lambda_-)$ of the numbers of positive, null and negative eigenvalues of the corresponding matrix. L is said to be *positive definite*, *negative definite*, *definite*, *indefinite*, respectively, if the corresponding matrix is so.

Since $\pi_1(X) = \{1\}$, we can identify $H^2(X, \mathbf{Z})$ and its natural image in $H^2(X, \mathbf{C})$. $H^2(X, \mathbf{Z})$ is naturally regarded as a lattice with pairing given by the cup product, and it is an even unimodular lattice of index $(3, 0, -19)$, so is isomorphic to $L := (-E_8)^{\oplus 2} \oplus U^{\oplus 3}$. Here, $-E_8$ is the even unimodular negative definite lattice of rank 8, whose matrix is (-1) times that of the lattice corresponding to the Dynkin diagram E_8. U is the even unimodular indefinite lattice of rank 2, whose matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

In general, the algebraic lattice S_X and the transcendental lattice T_X of a compact complex surface X are defined as sublattices in (the image of) $H^2(X, \mathbf{Z})$ as $S_X := H^{1,1}(X) \cap H^2(X, \mathbf{Z})$ and $T_X := S_X^{\perp}$ in $H^2(X, \mathbf{Z})$. Here $\perp$ denotes the orthogonal complement. The rank of S_X is called the *Picard number* of X and denoted by $\rho(X)$.

Ruan and Tian have constructed the *quantum cohomology ring* with $\mathbf{Z}$-coefficients of symplectic manifolds in [33]. As a graded $\mathbf{Z}$-module, the quantum cohomology ring is the same as the classical cohomology ring, but in general, the multiplicative structures are different. The mirror map μ is an involution of moduli of Calabi-Yau manifolds with Kähler structure which induces an isomorphism between cohomology groups, interchanging so-called the A-model intersection numbers described by the quantum cohomology ring and the B-model intersection numbers coming from the Gauss-Manin connection.

The following is an easy modification of [33] (Example 8.4).

PROPOSITION 4.1.2. *Let X be a compact Kähler surface with trivial canonical bundle. Then the quantum cohomology ring of X is isomorphic to the classical cohomology ring of X.*

Thus for a $K3$ surface or a 2-dimensional complex torus, the A-model intersection number is the classical intersection product. On the other hand, the B-model intersection numbers for those surfaces are also the intersection product by Hodge theory. Thus the mirror map preserves the usual cup product.

4.2. Fixed set of the mirror map. First we shall prove a finiteness result for a special automorphism group of a marked $K3$ surface.

THEOREM 4.2.1. *Let X be a $K3$ surface with a Kähler class ω, L be the second integral cohomology lattice $H^2(X, \mathbf{Z})$ and G be a subgroup of* $\mathrm{Aut}\, X$*. Assume that the induced action of G on L fixes the cohomology class of a nonzero holomorphic 2-form and ω. Then*

(1) *The fixed lattice L^G contains T_X.*

(2) *The orthogonal complement of $(L^G)^\perp$ in L is a negative definite sublattice of S_X.*

(3) *G is a finite group.*

REMARK 4.2.2. If the algebraic dimension $a(X)$ of X is not one, the theorem was essentially proved in [27].

REMARK 4.2.3. ω can be assumed to be a complexified Kähler class, since the action of automorphisms preserves the real structure of L.

First we note the following elementary fact.

LEMMA 4.2.4. *Let L be a lattice and K be a subset of L. Let ϕ be an isometry of L.*

(1) *If $\phi(K) = K$ then $\phi(K^\perp) = K^\perp$.*

(2) *If ϕ acts as the identity on K, then for any $x \in L$, $\phi(x) - x \in K^\perp$.*

COROLLARY 4.2.5. *Let X be a compact Kähler surface and $g \in$ $\mathrm{Aut}\, X$*. *Then $g^*(S_X) = S_X$ and $g^*(T_X) = T_X$.*

PROOF OF THEOREM. First note that the kernel of the multiplication of a nonzero holomorphic 2-form Ω on $H^2(X, \mathbf{R})$ is $H^{1,1} \cap H^2(X, \mathbf{R})$.

Take an arbitrary element $x \in T_X$ and $g \in G$. Then $y := g^*x - x \in T_X$. By the previous lemma, $y \in T_X \cap \Omega^\perp \cap \omega^\perp = T_X \cap S_X \cap \omega^\perp$. Since $y \in T_X \cap S_X$, we have $\langle y, y\rangle = 0$. On the other hand, since $S_X \cap \omega^\perp$ is negative definite by the signature theorem, one has $y = 0$. Thus $g^*x = x$. This proves (1).

(2) follows from $(L^G)^\perp \subset (T_X)^\perp \cap \Omega^\perp \cap \omega^\perp = S_X \cap \omega^\perp$.

Next we shall prove (3).

Since $(L^G)^\perp$ is negative definite and finitely generated, the isometry group $O((L^G)^\perp)$ is a finite group.

Let $j : G \to O((L^G)^\perp)$ be the natural group homomorphism. We claim that j is injective.

Suppose the induced action of an element $g \in G$ on $(L^G)^\perp$ is identity. For an arbitrary $x \in L$, $y := g^*x - x$ belongs to $(L^G)^\perp$. Thus $g^*y = y$. Then for an integer n, we have $(g^n)^*x = x + ny$ and since g^* is isometry, $\langle (g^n)^*x, (g^n)^*x\rangle = \langle x, x\rangle$. This leads to $y = 0$ by definiteness of $(L^G)^\perp$. Thus g^* is the identity on L.

Since g^* is, by definition, an effective Hodge isometry on L, g is the identity element in $\mathrm{Aut}\, X$ by Torelli's theorem. □

REMARK 4.2.6. The structure of G can be found in the list of [27] once one knows that G is a finite algebraic automorphism group. Also, $(L^G)^\perp$ does not contain square (-2)-elements.

COROLLARY 4.2.7. *$L^G + (L^G)^\perp$ is a sublattice in L of finite index and $L_{\mathbf{Q}} = (L^G)_{\mathbf{Q}} \oplus (L^G)^\perp_{\mathbf{Q}}$.*

PROOF. For any $x \in L$, there is an orthogonal decomposition whose coefficients are in $\frac{1}{|G|}\mathbf{Z}$:

$$x = \frac{1}{|G|}(x_1 \oplus x_2)\,, \quad x_1 = \sum_{g\in G} g^* x \in (L^G)\,, \quad x_2 = \sum_{g\in G}(x - g^* x) \in (L^G)^\perp\,.$$

□

Take an arbitrary $h \in O((L^G)^\perp)$. For arbitrary element x in L, we have a decomposition $x = x_1 + x_2$ in $L_{\mathbf{Q}}$, where $x_1 \in L^G$ and $x_2 \in (L^G)^\perp$. We define an action of $O((L^G)^\perp)$ on $L_{\mathbf{Q}}$ by $h(x) := x_1 + h(x_2)$.

PROPOSITION 4.2.8. *If $h(L) = L$, there exists a unique $g \in \mathrm{Aut}\, X$ which satisfies: $g^*\omega = \omega$, $g^*\Omega = \Omega$ and $h = g^*$.*

PROOF. Since ω and Ω belong to $(L^G)_{\mathbf{C}}$, h is an effective Hodge isometry on L. Thus the result follows from Torelli's theorem. □

REMARK 4.2.9. G is canonically isomorphic to the subgroup of $O(L)$ which fixes ω and Ω.

Next we define a sublattice $\langle\omega\rangle$ such that ω is 'general' in $(\langle\omega\rangle)_{\mathbf{C}}$.

DEFINITION 4.2.10. For a fixed $\alpha \in L_{\mathbf{C}}$, we denote by $\langle\alpha\rangle$ the minimum primitive sublattice of L such that $(\langle\alpha\rangle)_{\mathbf{C}}$ contains α.

REMARK 4.2.11. In fact, $\langle\alpha\rangle$ is determined by the set of all rational equalities which the coefficients of α satisfy, for an appropriate basis of L. When $\alpha \in L$, $\langle\alpha\rangle$ is nothing but $\mathbf{Z}\alpha$. This definition works for an arbitrary torsion-free $\mathbf{Z}$-module L and its extension of coefficients by any field of characteristic 0.

$\langle\omega\rangle$ is independent of the choice of the complex structure X.

LEMMA 4.2.12. *Let $\{x_i\}$ be a basis of $\langle\alpha\rangle$.*

(1) *If $\alpha = \sum_i a_i x_i$ ($a_i \in \mathbf{C}$) then the a_i's are linearly independent over $\mathbf{Q}$.*

(2) *$\alpha^\perp$ coincides with $\langle\alpha\rangle^\perp$ in L.*

PROOF. The first statement follows directly from the definition. The second one follows from the first one. □

PROPOSITION 4.2.13. *If $\alpha \in L^G$, then $\langle\alpha\rangle$ is contained in L^G.*

PROOF. Take a **Z**-basis $\{x_i\}$ of $\langle\alpha\rangle$ and write α as an **R**-linear combination $\sum_i a_i x_i$. From the minimality of $\langle\alpha\rangle$, the a_i's are linearly independent over **Q**. Let g be an element of G. Then we have $\alpha = \sum_i a_i g^* x_i$. The lattice generated by $\{g^* x_i\}$ is also primitive, thus it coincides with $\langle\alpha\rangle$ from the minimality. We can write $g^* x_i$ as a **Z**-combination $\sum_j b_{ij} x_j$. From the equality $\sum_i a_i g^* x_i = \alpha = \sum_j a_j x_j$ and torsion-freeness of L, we get $b_{ij} = \delta_{ij}$. Thus $g^* x_i = x_i$ and we are done.

REMARK 4.2.14. Thus we have $L^G \supset \langle\omega\rangle + \langle\Omega\rangle$. By adding to ω a small general element in $V := (L^G)_{\mathbf{C}} \cap ((\langle\omega\rangle + \langle\Omega\rangle)^{\perp})_{\mathbf{C}}$, one can arrange $L^G = \langle\omega\rangle + \langle\Omega\rangle$, since V is rationally defined and $V \cap L$ is negative definite.

REMARK 4.2.15. The lattice L_D in the next section is nothing but $\langle\omega\rangle$ when ω is the restriction of the cohomology class of a 'general' ample **R**-divisor of an embedded variety, and the lattice L_0 is $\langle\omega\rangle^{\perp} \cap \langle\Omega\rangle^{\perp}$ in L, and in fact, $(L^G)^{\perp}$ in L.

When ω is general in S_X, $\langle\omega\rangle = S_X$ and $L_0 = 0$.

4.3. Mirror symmetry for $K3$ toric hypersurfaces. In this section, we will treat the algebraic $K3$ surfaces which are the minimal resolutions of toric hypersurfaces.

DEFINITION 4.3.1. A *polyhedron* is a polytope of dimension three.

For a given reflexive polyhedron Δ, we use Θ and Γ for 2- and 1-dimensional faces of Δ. The dual face Γ^* of Γ is a 1-dimensional face of Δ^*, so a summation over Γ is same as that over Γ^*.

We denote by Z the intersection of a Δ-regular anticanonical section of $\mathbf{P}_\Delta$ and the n-dimensional orbit of the torus, and by $\bar{Z}$ its closure, namely the section itself. $\bar{Z}$ has only canonical Gorenstein singularities by Theorem 1.1.8 due to Batyrev. Let $\tilde{Z}$ be the minimal resolution of $\bar{Z}$. $X = \tilde{Z}$ is an algebraic $K3$ surface.

We define the following sublattices of $H^2(X, \mathbf{Z})$ for a toric hypersurface $K3$ surface X.

DEFINITION 4.3.2. $L_D := \mathrm{Im}(H^{1,1}(\tilde{\mathbf{P}}_\Delta) \to H^{1,1}(X)) \cap H^2(X, \mathbf{Z})$, $L_0 := L_D^{\perp}$ in S_X, where $\tilde{\mathbf{P}}_\Delta$ is a toric resolution of $\mathbf{P}_\Delta$.

From a result of [12] [4] [2], we get

PROPOSITION 4.3.3. *Let Δ be a three-dimensional reflexive polytope and $\tilde{Z}$ be the $K3$ surface which is the minimal resolution of a Δ-regular anticanonical section of $\mathbf{P}_\Delta$. Then*

(1) $L_D(\tilde{Z}) = \mathbf{Z}[\Delta^{*[1]} \cap N]/N$,

(2) $L_0(\tilde{Z}) = \bigoplus_{\dim \Gamma = 1} \mathbf{Z}(\mathrm{Int}\, \Gamma \cap M) \times (\mathrm{Int}\, \Gamma^* \cap N)$,

(3) $\mathrm{rk}\, T_{\tilde{Z}} - 2 = l(\Delta^{[1]}) - 3$.

In particular,

(4) $\mathrm{rk}\, L_D(\tilde{Z}) = l(\Delta^{*[1]}) - 3$,

(5) $\operatorname{rk} L_0(\tilde{Z}) = \sum_{\dim \Gamma=1} l^*(\Gamma^*) l^*(\Gamma)$.

PROOF. Following the same argument for the calculation of $H^{1,1}_{toric}$ in [2], one can show that $L_D(\tilde{Z}) = \mathbf{Z}[\Delta^{*[1]} \cap N]/N$ and in particular, $\operatorname{rk} L_D(\tilde{Z}) = l(\Delta^{*[1]}) - 3$.

By [12] (5.11), we have $\dim H^2(Z) = l(\Delta^{[1]}) - 1$, which is generated by vanishing cycles whose image is $\operatorname{rk} T_{\tilde{Z}}$.

The toric compactification $\mathbf{T} \to \mathbf{P}_\Delta$ adds a divisor for each vertex of Δ^*.

A toric resolution of the ambient space produces the divisors corresponding to the integral points P of $\Delta^* - ((\Delta^*)^{(0)} \cup \{0\})$. We denote by C_P the support of the center of blowing up in $\mathbf{P}_\Delta$ corresponding to P. If P is in the interior of some 2-dimensional face of Δ^*, C_P is a point which is disjoint from $\bar{Z}$, since $\bar{Z}$ is Δ-regular.

If P is in some $\operatorname{Int} \Gamma^*$, C_P is a curve in $\mathbf{P}_\Delta$ which intersects normally with $\bar{Z}$ at $(l^*(\Gamma)+1)$ points. Since each toric blowing-up corresponding to P creates one ambient divisor, its orthogonal complement increases L_0 by $l^*(\Gamma)$ divisors. Thus we have the description of L_0 and $\operatorname{rk} L_0(\tilde{Z}) = \sum_\Gamma l^*(\Gamma) l^*(\Gamma^*)$. □

REMARK 4.3.4. In fact, we can easily determine the lattices L_D and L_0 from the data (Δ, M). For instance, the dual graph of L_D is obtained from $\Delta^{*[1]}$ by removing three integral points which consist a basis of N. Each remaining vertex is a (-2)-curve and each remaining integral point on a edge Γ is a sum of $(l^*(\Gamma^*)+1)$ disjoint (-2)-curves. Each edge in Γ joining integral points shows the intersection number of the two end-points is $(l^*(\Gamma^*)+1)$.

We conjecture that the same rule of intersection product holds for L_G and $\Delta^{[1]}$.

COROLLARY 4.3.5. *For a fixed reflexive polyhedron* Δ, $\operatorname{rk} T_{\tilde{Z}}$, $L_D(\tilde{Z})$ *and* $L_0(\tilde{Z})$ *are independent of the choice of* Δ*-regular hypersurface* Z.

We will thus write them as $L_D(\Delta)$ and $L_0(\Delta)$, respectively.

COROLLARY 4.3.6. $\operatorname{rk} T(\Delta) - 2 = \operatorname{rk} L_D(\Delta^*)$, $L_0(\Delta) = L_0(\Delta^*)$.

REMARK 4.3.7. Since the rank of H^2 is 22, $\operatorname{rk} T(\Delta) + \operatorname{rk} L_D(\Delta) + \operatorname{rk} L_0(\Delta) = 22$. That formula gives a relation $\sum_\Gamma (l^*(\Gamma)+1)(l^*(\Gamma^*)+1) = 24$.

We have $\operatorname{rk} T(\Delta) - 2 = l(\Delta^{[1]}) - 3 = \operatorname{rk}(\Delta^{[1]} \cap M)/M$, which is the expected dimension for $H^{1,1}_{poly}$ in [2].

REMARK 4.3.8. In the case of Arnold's strange duality, the lattice structure of $L_G(\Delta)$ is isomorphic to that of $L_D(\Delta^*)$.

In general, $L_G \oplus U$ is not isomorphic as a lattice over $\mathbf{Z}$ to the Milnor lattice but it seems so over $\frac{1}{|G|}\mathbf{Z}$. This may be worth further investigation.

THEOREM 4.3.9. *Let* X_0 *be one of Arnold's singularities and* $W_a = (a_1, a_2, a_3; h)$ *be the corresponding weight system.*

(1) *There exists a unique dual weight system* W_b.

(2) W_b *coincides with the weight system of the dual singularity in the sense of Arnold.*

(3) *The full Newton polyhedron, namely the convex hull of* $\overline{\Delta}(a) \cap M(W_a)$ *in* $M(W_a)_{\mathbf{Q}}$ *is a reflexive polyhedron containing* $(\overline{\nabla})^*$ *with the dual correspondence above.*

(4) *Let* Δ *be any reflexive polyhedron in* $\overline{\Delta}(a)$ *which contains* $(\overline{\nabla})^*$. *The intersection with* Δ *and the plane* $\sum_{i=1}^{3} a_i\alpha_i = 1$ *is a facet which determines the equivalent singularity to* X_0.

(5) Δ^* *is a reflexive polyhedron in* $\overline{\nabla}$ *containing* $(\overline{\Delta})^*$.

(6) *There exists a* Δ *such that* $L_0(\Delta) = 0$.

PROOF. Parts (1), (2) and (5) have been done in previous sections. (3) is well-known and (4) is an easy case-by-case check. For 6, we give an example of a choice of Δ with $L_0 = 0$. For dual singularities, these reflexive polytopes are dual to each other. For self-dual singularities, these examples satisfy $\Delta^* \cong \Delta$. These choices of Δ are not unique in general. □

REMARK 4.3.10. One should choose the polyhedron rather carefully, since the restricted Kähler class ω of a general Kähler class of $\tilde{\mathbf{P}}_\Delta$ is sometimes not general in $(S_X)_{\mathbf{R}}$. Namely, there might be some non-zero (non-effective) integral algebraic 2-cocycle α such that $\omega \cup \alpha = 0$. This is where L_0 appears. Thus it is necessary to resort to a subpolytope of the full Newton polytope.

class	a_1	a_2	a_3	h	vertices of Δ
E_{12}	6	14	21	42	W^{42}, X^7, Y^3, Z^2
E_{13}	4	10	15	30	$W^{30}, W^6X^6, X^5Y, Y^3, Z^2$
Z_{11}	6	8	15	30	$W^{30}, W^6Y^3, X^5, XY^3, Z^2$
E_{14}	3	8	12	24	$W^{24}, W^6X^6, X^4Z, Y^3, Z^2$
Q_{10}	6	8	9	24	$W^{24}, W^6Z^2, X^4, Y^3, XZ^2$
Z_{12}	4	6	11	22	$W^{22}, W^6X^4, W^4Y^3, X^4Y, XY^3, Z^2$
W_{12}	4	5	10	20	$W^{20}, W^{10}Y^2, W^2X^2Y^2, X^5, Y^2Z, Z^2$
Z_{13}	3	5	9	18	$W^{18}, W^6X^4, W^3Y^3, X^3Z, XY^3, Z^2$
Q_{11}	4	6	7	18	$W^{18}, W^6X^3, W^4Z^2, X^3Y, Y^3, XZ^2$
W_{13}	3	4	8	16	$W^{16}, W^4X^4, W^4Y^3, X^4Y, Y^4, Z^2$
S_{11}	4	5	6	16	$W^{16}, W^6Y^2, W^4Z^2, X^4, XZ^2, Y^2Z$
Q_{12}	3	5	6	15	$W^{15}, W^6X^3, W^3Z^2, X^3Z, XZ^2, Y^3$
S_{12}	3	4	5	13	$W^{13}, W^4X^3, W^3Z^2, WY^3, X^3Y, XZ^2, Y^2Z$
U_{12}	3	4	4	12	$W^{12}, W^4Y^2, W^4Z^2, X^4, Y^2Z, YZ^2$

4.4. Examples. We give some illuminating examples other than Arnold's singularities.

EXAMPLE 4.4.1. Let us consider a pair of dual weight systems $W_a = (2, 3, 6; 12)$ and $W_b = (2, 4, 5; 12)$. In these cases $a_0 = b_0 = 1$. We will denote the homogeneous coordinates by $W = X_0$, $X = X_1$, $Y = X_2$ and $Z = X_3$ and the inhomogeneous ones by small letters. A polytope is represented by its vertex set $\langle$vertices$\rangle$.

The weighted magic square C is $\begin{pmatrix} 0 & 2 & 1 \\ 3 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ whose determinant is (-12). Thus $B = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$, $-B^{-1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

We use $(\alpha_1, \dots, \alpha_n)$ for the coordinates coming from $M(W_a) = \mathbf{Z}^{\oplus n}$ and $[\beta_1, \dots, \beta_n]$ for those from $M(W_b)$. Then $[\beta_1, \dots, \beta_n]B = (\alpha_1, \dots, \alpha_n)$ holds for the vertices in $M_{\mathbf{Q}}$ and the hyperplanes in $N_{\mathbf{Q}}$.

$\overline{\Delta} = \langle W^{12}, X^6, Y^4, Z^2 \rangle$. The coordinates of the vertices are : $(-1, -1, -1)$, $(5, -1, -1)$, $(-1, 3, -1)$, $(-1, -1, 1)$; $[-2, -3, -5]$, $[-2, 2, 1]$, $[2, 0, -1]$, $[0, 0, 1]$, respectively. This is a reflexive simplex. In fact, the dual is given by $(\overline{\Delta})^* = \langle W^{12}, Y^3, X^2Y^2, XZ^2 \rangle$.

Let X_0 be the singularity $\{x^6 + y^4 + z^2 = 0\}$ in $\mathbf{C}^3$. This is an isolated hypersurface singularity with Milnor number $\mu = 15$. In the table of Arnold [1], X_0 belongs to class $W_{1,0}$.

$Z = \{x^6 + y^4 + z^2 + 1 = 0\}$ is a smooth surface in $\mathbf{C}^3$ thus a Milnor fiber of X_0.

Ebeling [16] has calculated Milnor lattices of isolated bimodal singularities. The Milnor lattice $H_2(Z)$ is a direct sum of U and the lattice $L_G(\overline{\Delta})$ represented by Figure 1,

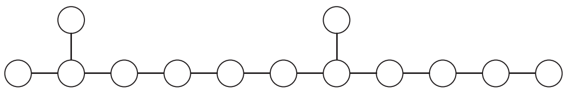

FIGURE 1.

where the circles are vanishing cycles.

The resolution graph of $X \to \bar{Z}$ is as in Figure 2.

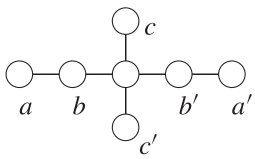

FIGURE 2.

where a circle represents a smooth rational curve with self-intersection (-2) and the central curve is $\{W = 0\}$. L_D is generated by the central curve and three (-4)-elements $a + a'$, $b + b'$ and $c + c'$. L_0 is generated by three (-4)-elements $e_1 := a - a'$, $e_2 := b - b'$ and $e_3 := c - c'$. The set of all the (-4)-elements of L_0 are $\{\pm e_1, \pm e_2, \pm e_3, \pm(e_1 + e_2)\}$ and the orthogonal group $O(L_0)$ preserves this set. The elements of $O(L_0)$ which extend to elements of $O(L)$

as in 4.2 are : id, $\sigma(e_1 \mapsto e_1, e_2 \mapsto e_2, e_3 \mapsto -e_3)$, $\tau(e_1 \mapsto -e_1, e_2 \mapsto -e_2, e_3 \mapsto e_3)$ and $\sigma\tau$. Thus $G \cong \mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}/2\mathbf{Z}$ and in fact each isomorphism is realized by a multiplication of coordinates: $\sigma(W : X : Y : Z) = (iW : X : iY : -Z)$, $\tau(W : X : Y : Z) = (\zeta^2 W : \zeta X : Y : -Z)$ where $i := \exp(\pi i/2)$ and $\zeta := \exp(\pi i/3)$.

Next we consider the dual polyhedron $\overline{\Delta}^*$. The singularity $\{y^3 + x^2y^2 + xz^2 = 0\}$ has a singular curve $\{x = 0\}$. We have a Milnor fiber $\{y^3 + x^2y^2 + xz^2 + 1 = 0\}$ and its weighted compactification $\bar{Z} = \{Y^3 + X^2Y^2 + XZ^2 + W^{12} = 0\}$ in $\mathbf{P}(1, 2, 4, 5)$. In this case, the restriction of the divisor $\{X = 0\}$ decomposes as a sum of three divisors $E_i : \{X = Y + \zeta^i W^4 = 0\}$ $(i = 0, 1, 2)$ where ζ is a nontrivial cubic root of unity. The other intersections with singular 2-dimensional strata of $\mathbf{P}$ are irreducible.

The singular set of $\bar{Z}$ consists of one D_8 at $\{W = Y = Z = 0\}$, one A_4 at $\{W = X = Y = 0\}$ and one A_1 at $\{W = X^2 + Y = Z = 0\}$. The resolution graph at infinity (including the divisor $W = 0$) becomes as in Figure 3.

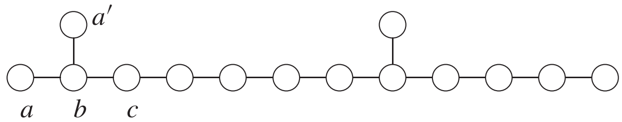

FIGURE 3.

This graph is one vertex longer than that of $L_G(\overline{\Delta})$. L_0 is generated by $a - a'$, $E_0 - E_1$ and $E_1 - E_2$. L_D is generated by $a + a'$ and other circles. Changing the base by $a + a' \to a' + a + b$ (other circles are the same), we have the same graph as that of $L_G(\overline{\Delta})$.

Let us take other polyhedra. In our duality, every reflexive polyhedron in $M_{\mathbf{Q}}$ should contain $\overline{\nabla}^* = \langle W^{12}, X^3Y^2, Y^2Z, Z^2 \rangle$, or rather $\nabla^* = \langle W^{12}, Y^2W^6, X^3Y^2, Y^2Z, Z^2 \rangle$, where ∇ is the maximum Newton polyhedron in N, that is the convex hull of $\overline{\nabla} \cap N$.

Take $\Delta = \langle W^{12}, W^3Y^3, W^2X^5, WXY^3, X^3Y^2, X^3Z, Y^2Z, Z^2 \rangle$, which has the same L_G as $\overline{\Delta}$ and $L_0(\Delta) = 0$. Thus Δ represents a Milnor fiber of the singularity of class $W_{1,0}$ and the dual graph of L_D coincides with the resolution graph as above. Its dual Δ^* is $\langle W^{12}, W^7Z, W^6XY, W^5XZ, W^3YZ, X^2Y^2, Y^3, XZ^2 \rangle$.

EXAMPLE 4.4.2. The weight system $W = (2, 3, 4; 10)$ is self-dual. Take the following six reflexive polyhedra (cf. Figure 4):

$$\begin{aligned}
\Delta_1 &= \langle W^{10}, W^2X^4, X^3Z, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle\,,\\
\Delta_2 &= \langle W^{10}, W^4X^3, WX^3Y, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle\,,\\
\Delta_3 &= \langle W^{10}, W^6X^2, W^2X^2Z, X^2Y^2, XZ^2, Y^2Z, W^2Z^2, WY^3 \rangle\,,\\
\Delta_4 &= \langle W^{10}, W^6X^2, W^2X^2Z, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle\,,\\
\Delta_5 &= \langle W^{10}, W^8X, W^3X^2Y, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle\,, \quad \text{and}\\
\Delta_6 &= \langle W^{10}, W^4XZ, X^2Y^2, XZ^2, Y^2Z, W^6Z, W^4Y^2 \rangle\,.
\end{aligned}$$

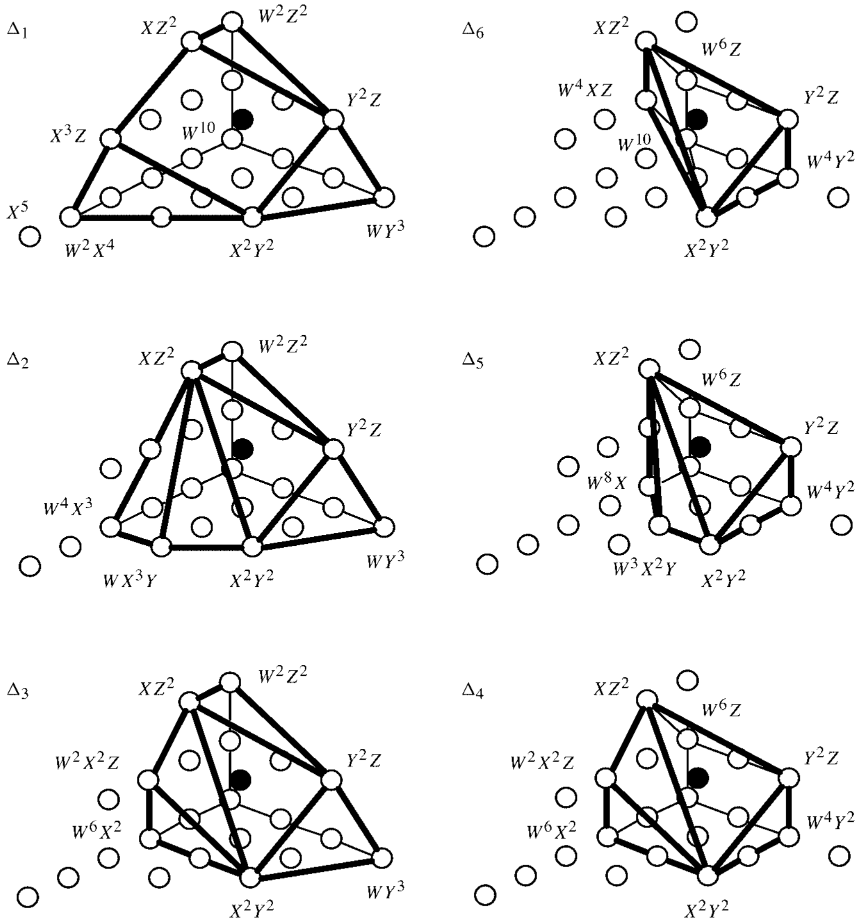

FIGURE 4. $W = (2, 3, 4; 10)$. The black circle represents the origin $XYZW$.

These polyhedra all satisfy $L_0(\Delta_i) = 0$ and $\Delta_i^* \cong \Delta_{7-i}$. Δ_1 corresponds to the Milnor fiber of the singularity of class $S_{1,0}$.

The dual graphs of $L_D(\Delta_i)$ is as in Figure 5.

There are inclusions of reflexive polyhedra: $\Delta_i \supset \Delta_j$ for $i < j$ and thus specializations of the families of weighted hypersurfaces. As the family specializes, $2A_1$ singularities at infinity get 'closer' to A_3, D_4, D_5 and finally to D_6 singularities.

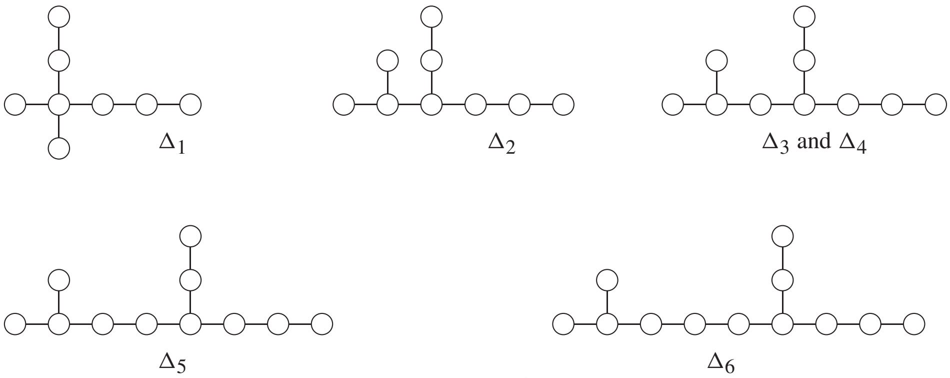

FIGURE 5.

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