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A Sharp Form of the Uniqueness of the Solution to Nonlinear Totally Characteristic Partial Differential Equations

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Abstract. The paper deals with the following nonlinear partial differential equation $(t\partial/\partial t)^m u = F(t, x, \{(t\partial/\partial t)^j(\partial/\partial x)^\alpha u\}_{j+\alpha\le m, j<m})$ with $(t, x) \in \mathbf{C}^2$ in the complex domain. Under the assumption that the equation is of totally characteristic type, the uniqueness of the solution was first proved in [4]. The present paper gives a sharp form of this uniqueness theorem.

1. Introduction and main result

Notations: $(t, x) \in \mathbf{C}_t \times \mathbf{C}_x$, $\mathbf{N} = \{0, 1, 2, \ldots\}$, and $\mathbf{N}^* = \{1, 2, \ldots\}$. Let $m \in \mathbf{N}^*$, set $N = \#\{(j, \alpha) \in \mathbf{N} \times \mathbf{N}\,;\ j + \alpha \le m,\ j < m\}$ (that is, $N = m(m + 3)/2$), and denote the complex variables $z \in \mathbf{C}^N$ by $z = \left\{z_{j,\alpha}\right\}_{j+\alpha\le m, j<m}$.

In this paper we will consider the following nonlinear singular partial differential equation:

$$\left(t\frac{\partial}{\partial t}\right)^m u = F\left(t, x, \left\{\left(t\frac{\partial}{\partial t}\right)^j\left(\frac{\partial}{\partial x}\right)^\alpha u\right\}_{\substack{j+\alpha\le m\\ j<m}}\right), \tag{1.1}$$

where $F(t, x, z)$ is a function of the variables (t, x, z) defined in a neighborhood Δ of the origin of $\mathbf{C}_t \times \mathbf{C}_x \times \mathbf{C}_z^N$, and $u = u(t, x)$ is the unknown function. Set $\Delta_0 = \Delta \cap \{t = 0, z = 0\}$, and set also $I_m = \{(j, \alpha) \in \mathbf{N} \times \mathbf{N}\,;\ j + \alpha \le m,\ \ j < m\}$ and $I_m(+) = \{(j, \alpha) \in I_m\,;\ \alpha > 0\}$.

We impose the following conditions on $F(t, x, z)$:

A$_1$) $F(t, x, z)$ is a holomorphic function on Δ;

A$_2$) $F(0, x, 0) \equiv 0$ on Δ_0;

A$_3$) $\dfrac{\partial F}{\partial z_{j,\alpha}}(0, x, 0) = O(x^\alpha)$ (as $x \longrightarrow 0$) for all $(j, \alpha) \in I_m(+)$.

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Then, the equation (1.1) is called a *nonlinear totally characteristic type* partial differential equation. By the condition A_3) we have $(\partial F/\partial z_{j,\alpha})(0,x,0) = x^{\alpha} c_{j,\alpha}(x)$ for some holomorphic functions $c_{j,\alpha}(x)$ ($(j,\alpha) \in I_m$).

We set

$$L(\lambda, \rho) = \lambda^m - \sum_{\substack{j+\alpha \le m \\ j<m}} c_{j,\alpha}(0)\, \lambda^j \rho(\rho-1)\cdots(\rho-\alpha+1)\,, \tag{1.2}$$

$$L_m(X) = X^m - \sum_{\substack{j+\alpha = m \\ j<m}} c_{j,\alpha}(0)\, X^j\,, \tag{1.3}$$

and denote by $c_1, \dots, c_m$ the roots of the equation $L_m(X) = 0$ in X. If we factorize $L(\lambda, l)$ into the form

$$L(\lambda, l) = (\lambda - \lambda_1(l)) \cdots (\lambda - \lambda_m(l))\,, \quad l \in \mathbf{N}\,, \tag{1.4}$$

by renumbering the subscript i of $\lambda_i(l)$ suitably we have

$$\lim_{l \to \infty} \frac{\lambda_i(l)}{l} = c_i \quad \text{for } i = 1, \dots, m\,.$$

If $\mathrm{Re}\, c_i < 0$ holds for all $i = 1, \dots, m$, we have $\mathrm{Re}\lambda_i(l) \longrightarrow -\infty$ (as $l \longrightarrow \infty$); in this case we can define

$$\beta = \max \left[0, \max_{\substack{1 \le i \le m \\ l \ge 0}} \mathrm{Re}\, \lambda_i(l) \right]. \tag{1.5}$$

Let us recall the result in [4]. We denote:

- $\mathcal{R}(\mathbf{C} \setminus \{0\})$ the universal covering space of $\mathbf{C} \setminus \{0\}$,
- $S_\theta = \{t \in \mathcal{R}(\mathbf{C} \setminus \{0\})\,;\ |\arg t| < \theta\}$ a sector in $\mathcal{R}(\mathbf{C} \setminus \{0\})$,
- $S_\theta(r) = \{t \in S_\theta\,;\ 0 < |t| < r\}$,
- $D_R = \{x \in \mathbf{C}\,;\ |x| \le R\}$.

We also denote by $\widetilde{\mathcal{S}}_+$ the set of all $u(t,x)$ satisfying the following i) and ii): i) $u(t,x)$ is a holomorphic function on $S_\theta(r) \times D_R$ for some $\theta > 0$, $r > 0$ and $R > 0$; and ii) $|u(t,x)| = O(|t|^{\sigma})$ uniformly on D_R (as $t \longrightarrow 0$ in $S_\theta(r)$) for some $\sigma > 0$.

THEOREM 1 ([4], Theorem 2). *Assume the conditions* A_1), A_2), A_3) *and*

$$\mathrm{Re}\, c_i < 0 \quad \textit{for } i = 1, \dots, m\,. \tag{1.6}$$

If $u_1(t,x)$ *and* $u_2(t,x)$ *are solutions of* (1.1) *belonging in the class* $\widetilde{\mathcal{S}}_+$ *and if they satisfy*

$$\max_{x \in D_R} |(u_1 - u_2)(t,x)| = O(|t|^a) \quad (\textit{as } t \longrightarrow 0 \textit{ in } S_\theta(r)) \tag{1.7}$$

for some $a > \beta$, *we have* $u_1 = u_2$ *in* $\widetilde{\mathcal{S}}_+$.

The following theorem is the main result of this paper, in which the assumption (1.7) is weakened into the form (1.8).

THEOREM 2. *Assume the conditions* A$_1$), A$_2$), A$_3$) *and* (1.6). *If* $u_1(t,x)$ *and* $u_2(t,x)$ *are solutions of* (1.1) *belonging in the class* $\widetilde{\mathcal{S}}_+$ *and if they satisfy*

$$\Big(\Big(\frac{\partial}{\partial x}\Big)^l (u_1 - u_2)\Big)(t,0) = O(|t|^a) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r)) \quad \text{for any } l \in \mathbf{N} \tag{1.8}$$

for some $a > \beta$, *we have* $u_1 = u_2$ *in* $\widetilde{\mathcal{S}}_+$.

REMARK 1. If $u_1 \in \widetilde{\mathcal{S}}_+$ and $u_2 \in \widetilde{\mathcal{S}}_+$ hold, by the definition we have

$$\max_{x \in D_R} |(u_1 - u_2)(t,x)| = O(|t|^s) \quad (\text{as } t \longrightarrow 0 \text{ in } S_\theta(r))$$

for some $s > 0$. If $s > \beta$ we can use Theorem 1; but, if $s \le \beta$ we need some additional condition like (1.7) or (1.8).

In the study of solutions of nonlinear totally characteristic type partial differential equations, the following situation often occurs: we can check the condition (1.8), but it is very difficult to check the condition (1.7). This is the reason why the author needs to publish this paper. The application of Theorem 2 will be given in the forthcoming paper.

See also [1], [2] and [3], in which the uniqueness of the solution is obtained for other types of nonlinear partial differential equations.

2. Pseudo-differential operators of Euler type

In the proof of Theorem 2, we will use the same notations as in [4]; in particular, we recall here the notations X_R, $C^0([0,T], X_R)$, S_k and $S_k([0,T], X_R)$.

For a formal power series $f(x) = \sum_{l \ge 0} f_l x^l \in \mathbf{C}[[x]]$, we set

$$|f|(x) = \sum_{l \ge 0} |f_l| x^l \quad \text{and} \quad |f|_\rho = |f|(\rho) = \sum_{l \ge 0} |f_l| \rho^l . \tag{2.1}$$

Let $R > 0$. Using this norm, we define X_R by

$$X_R = \{ f(x) \in \mathbf{C}[[x]] \, ; \, |f|_R < \infty \} .$$

It is easy to see that X_R is a Banach space with the norm $|\cdot|_R$. We denote by $C^0([0,T], X_R)$ the space of all continuous functions $f(t,x)$ on $[0,T]$ with values in X_R, which is also a Banach space with the norm $\|f\| = \max_{t \in [0,T]} |f(t)|_R$. For $m \in \mathbf{N}^*$ we denote by $C^m([0,T], X_R)$ the space of all C^m functions $f(t,x)$ on $[0,T]$ with values in X_R.

For a sequence $\lambda(l)$ ($l = 0, 1, 2, \ldots$) of complex numbers, we define the operator $\lambda(\theta)$: $\mathbf{C}[[x]] \longrightarrow \mathbf{C}[[x]]$ by the following:

$$\mathbf{C}[[x]] \ni f = \sum_{l \geq 0} f_l x^l \longmapsto \lambda(\theta) f = \sum_{l \geq 0} f_l \lambda(l) x^l \in \mathbf{C}[[x]] . \tag{2.2}$$

If $\lambda(\rho)$ is a mapping from $\mathbf{N}$ into $\mathbf{C}$, we can define an operator $\lambda(\theta) : \mathbf{C}[[x]] \longrightarrow \mathbf{C}[[x]]$. In particular, if $\lambda(\rho)$ is a function defined on $\mathbf{R}_+ = \{\rho \in \mathbf{R} ; \rho \geq 0\}$, we have an operator $\lambda(\theta) : \mathbf{C}[[x]] \longrightarrow \mathbf{C}[[x]]$. If $\lambda(\rho)$ is a polynomial in ρ, we easily see that $\lambda(\theta) = \lambda(x(d/dx))$ holds as an operator from $\mathbf{C}[[x]]$ into $\mathbf{C}[[x]]$. Thus, our operator $\lambda(\theta)$ can be regarded as a generalization of a differential operator of Euler type. From now, we will call this operator $\lambda(\theta)$ as *a pseudo-differential operator of Euler type*.

If a pseudo-differential operator $\lambda(\theta) : \mathbf{C}[[x]] \longrightarrow \mathbf{C}[[x]]$ satisfies

$$|\lambda(l)| \leq C(1 + l)^k \quad (l = 0, 1, 2, \ldots) \tag{2.3}$$

for some $C \geq 0$ and $k \geq 0$, we say that $\lambda(\theta)$ is *a pseudo-differential operator of order k*. We denote by S_k the set of all such pseudo-differential operators of order k as above.

Similarly, for a sequence $a(t, x; l) \in C^0([0, T], X_R)$ ($l = 0, 1, 2, \ldots$) we define the operator $a(t, x; \theta)$ by the following:

$$f(t, x) = \sum_{l \geq 0} f_l(t) x^l \longmapsto a(t, x; \theta) f(t, x) = \sum_{l \geq 0} a(t, x; l) f_l(t) x^l . \tag{2.4}$$

We often write $a(t; \theta) f(t)$ instead of $a(t, x; \theta) f(t, x)$. By the definition we have:

LEMMA 1. *For any $f(t, x) = \sum_{l \geq 0} f_l(t) x^l \in C^0([0, T], X_R)$ we have*

$$|a(t; \theta) f(t)|_R \leq \sum_{l \geq 0} |a(t; l)|_R \, |f_l(t)| \, R^l$$

where $|a(t; l)|_R$ is the norm of $a(t, x; l) \in \mathbf{C}[[x]]$ for fixed (t, l).

In view of Lemma 1, we say that $a(t, x; \theta)$ *is a pseudo-differential operator of order k* (≥ 0) *with symbol in $C^0([0, T], X_R)$*, if it satisfies

$$|a(t; l)|_R \leq C(1 + l)^k, \quad 0 \leq t \leq T \text{ and } l = 0, 1, 2, \ldots \tag{2.5}$$

for some $C > 0$. We denote by $S_k([0, T], X_R)$ the set of all the pseudo-differential operators of order k with symbol in $C^0([0, T], X_R)$.

3. Proof of Theorem 2

As is seen in [4], Theorem 1 is reduced to a uniqueness result in some linear pseudo-differential equations. Let us recall its reduced linear case.

Let $T > 0$, $R > 0$, and let

1) $\lambda_i(\theta) \in S_1$ ($i = 1, \ldots, m$),

2) $a_j(t,x;\theta) \in S_{m-j}([0,T],X_R)$ $(j < m)$,

3) $b_{q,j}(t,x;\theta) \in S_{m-q-j}([0,T],X_R)$ $(q+j \leq m, q > 0)$,

and set

$$\Theta_0 = 1\,,$$

$$\Theta_1 = \Big(t\frac{\partial}{\partial t} - \lambda_1(\theta)\Big)\,,$$

$$\Theta_2 = \Big(t\frac{\partial}{\partial t} - \lambda_2(\theta)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_1(\theta)\Big)\,,$$

$$\cdots\cdots\cdots$$

$$\Theta_m = \Big(t\frac{\partial}{\partial t} - \lambda_m(\theta)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_{m-1}(\theta)\Big)\cdots\Big(t\frac{\partial}{\partial t} - \lambda_1(\theta)\Big)\,.$$

Let $\mu \in \mathbf{R}$, and let us consider the following linear pseudo-differential equation:

$$\Theta_m u = \sum_{j<m} a_j(t,x;\theta)\Theta_j u + \sum_{\substack{q+j\leq m\\ q>0}} b_{q,j}(t,x;\theta)\Big(t^{\mu}\frac{\partial}{\partial x}\Big)^q \Theta_j u\,. \tag{3.1}$$

Suppose:

c$_1$) there are $b \geq 0$ and $c > 0$ such that $b - \mathrm{Re}\lambda_i(l) \geq c\,l$ holds for all $l \in \mathbf{N}$ and $i = 1,\ldots,m$,

c$_2$) for any $i = 0,1,\ldots,m-1$ we have

$$\sup_{\substack{0\leq t\leq T_0\\ l\geq 0}} \frac{|a_j(t;l)|_{R_0}}{(1+l)^{m-j}} = o(1) \quad (\text{as } T_0 \longrightarrow 0 \text{ and } R_0 \longrightarrow 0)\,,$$

c$_3$) $\mu > 0$.

Then, the proof of Theorem 1 was reduced to proving

PROPOSITION 1 ([4], Theorem 3*). *Assume the conditions* c$_1$), c$_2$) *and* c$_3$). *If* $u(t,x)$ *is a solution of* (3.1) *belonging in the class* $C^m((0,T],X_R)$ *and if it satisfies*

$$\Big|\Big(t\frac{\partial}{\partial t}\Big)^j u(t)\Big|_R = O(t^a) \quad (\textit{as } t \longrightarrow 0) \qquad \textit{for } \; j = 0,1,\ldots,m-1 \tag{3.2}$$

for some $a > b$, *we have* $u(t,x) \equiv 0$ *on* $(0,\varepsilon] \times D_r$ *for some* $\varepsilon > 0$ *and* $r > 0$.

Thus, by the same reduction as in [4] we see that to prove Theorem 2 it is sufficient to show the following result.

PROPOSITION 2. *Assume the conditions* c$_1$), c$_2$) *and* c$_3$). *If* $u(t,x)$ *is a solution of* (3.1) *belonging in the class* $C^m((0,T],X_R)$ *and if there are* $s > 0$ *and* $a > b$ *such that*

$$\Big|\Big(t\frac{\partial}{\partial t}\Big)^j u(t)\Big|_R = O(t^s) \;\; (\textit{as } t \longrightarrow 0) \;\; \textit{for } \; j = 0,1,\ldots,m-1, \textit{ and} \tag{3.3}$$

$$\Big(\Big(t\frac{\partial}{\partial t}\Big)^j\Big(\frac{\partial}{\partial x}\Big)^l u\Big)(t,0) = O(t^a) \;\; (\textit{as } t \longrightarrow 0) \tag{3.4}$$

for any $l \in \mathbf{N}$ *and* $j = 0, 1, \dots, m-1$,

we have $u(t,x) \equiv 0$ *on* $(0,\varepsilon] \times D_r$ *for some* $\varepsilon > 0$ *and* $r > 0$.

First we note the following two lemmas.

LEMMA 2. *Let* $\delta > 0$. *For a function* $w(t,x) \in C^1((0,T), X_R)$ *we define* Jw *by* $(Jw)(t,x) = w(t, t^\delta x)$. *We have*:

1) $Jw \in C^1((0,T), X_{R_1})$ *holds for any* $0 < R_1 < R/T^\delta$;

2) $J\circ\Big(t\dfrac{\partial}{\partial t}\Big)w = \Big(t\dfrac{\partial}{\partial t} - \delta x\dfrac{\partial}{\partial x}\Big)\circ Jw$;

3) $J\circ\Big(x\dfrac{\partial}{\partial x}\Big)w = \Big(x\dfrac{\partial}{\partial x}\Big)\circ Jw$, *more generally, for any pseudo-differential operator* $\lambda(\theta)$ *we have* $J\circ\lambda(\theta)w = \lambda(\theta)\circ Jw$;

4) $J\circ\Big(\dfrac{\partial}{\partial x}\Big)w = \Big(t^{-\delta}\dfrac{\partial}{\partial x}\Big)\circ Jw$.

LEMMA 3. *Let* $a(t,x;\theta) \in S_k([0,T], X_R)$ *and* $\delta > 0$. *Then for any* $0 < R_1 < R/T^\delta$ *we have* $a(t, t^\delta x; \theta) \in S_k([0,T], X_{R_1})$ *and* $|a(t, t^\delta x; l)|_{R_1} \le |a(t;l)|_R$ $(l = 0, 1, \dots)$.

The proof of these lemmas is easy, and so we may omit the details. Now let us give a proof of Proposition 2.

PROOF OF PROPOSITION 2. Let $u(t,x) \in C^m((0,T], X_R)$ be a solution of (3.1) satisfying the conditions (3.3) and (3.4) for some $s > 0$ and $a > b$. Take a sufficiently small $\delta > 0$ and set $u^*(t,x) = u(t, t^\delta x)$. Take any $0 < R_1 < R/T^\delta$ and fix it. Then, by Lemma 2 we see that $u^*(t,x) \in C^m((0,T], X_{R_1})$ and that u^* satisfies the following equation:

$$\Theta_m^* u^* = \sum_{j<m} a_j(t, t^\delta x;\theta)\Theta_j^* u^* + \sum_{\substack{q+j\le m\\ q>0}} b_{q,j}(t, t^\delta x;\theta)\Big(t^{\mu-\delta}\frac{\partial}{\partial x}\Big)^q \Theta_j^* u^*, \tag{3.5}$$

where

$$\begin{aligned}
\Theta_0^* &= 1,\\
\Theta_1^* &= \Big(t\frac{\partial}{\partial t} - \lambda_1^*(\theta)\Big),\\
\Theta_2^* &= \Big(t\frac{\partial}{\partial t} - \lambda_2^*(\theta)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_1^*(\theta)\Big),\\
&\quad \cdots\cdots\cdots\\
\Theta_m^* &= \Big(t\frac{\partial}{\partial t} - \lambda_m^*(\theta)\Big)\Big(t\frac{\partial}{\partial t} - \lambda_{m-1}^*(\theta)\Big)\cdots\Big(t\frac{\partial}{\partial t} - \lambda_1^*(\theta)\Big)
\end{aligned}$$

and $\lambda_i^*(\theta) = \lambda_i(\theta) + \delta\theta$ $(i = 1, 2, \dots, m)$.

It is easy to see by Lemma 3 that $\lambda_i^*(\theta) \in S_1$ $(i = 1, \dots, m)$, $a_j(t, t^\delta x;\theta) \in S_{m-j}([0,T], X_{R_1})$ $(j < m)$, and $b_{q,j}(t, t^\delta x;\theta) \in S_{m-q-j}([0,T], X_{R_1})$ $(q + j \le m, q > 0)$.

Since c_1) is assumed, we have $b - \mathrm{Re}\lambda_i^*(l) = b - \mathrm{Re}\lambda_i(l) - \delta l \geq cl - \delta l = (c-\delta)l$ $(l = 0, 1, \ldots)$; therefore, if $0 < \delta < c$ holds we have $b - \mathrm{Re}\lambda_i^*(l) \geq c^* l$ $(l = 0, 1, \ldots)$ with $c^* = c - \delta > 0$. By Lemma 3 we have $|a_j(t, t^\delta x; l)|_{R_1} \leq |a_j(t; l)|_R$; therefore we easily see:

$$\sup_{\substack{0 \leq t \leq T_0 \\ l \geq 0}} \frac{|a_j(t, x^\delta x; l)|_{R_0}}{(1+l)^{m-j}} = o(1) \quad (\text{as } T_0 \longrightarrow 0 \text{ and } R_0 \longrightarrow 0)\,.$$

Since $\delta > 0$ is sufficiently small, we may assume that $\mu - \delta > 0$ holds. Thus, the reduced equation (3.5) satisfies all the assumption except (3.2) in Proposition 1. Hence, if we know the condition

$$\left| \left(t \frac{\partial}{\partial t} \right)^j u^*(t) \right|_{R_1} = O(t^a) \quad (\text{as } t \longrightarrow 0) \;\; \text{for } j = 0, 1, \ldots, m-1 \tag{3.6}$$

by applying Proposition 1 to (3.5) we can obtain the conclusion of Proposition 2, and the proof of Proposition 2 is completed.

Thus, lastly let us prove (3.6). If $s \geq a$ holds, (3.6) follows from the assumption (3.3). Therefore, from now we assume the condition $0 < s < a$.

Let $\delta > 0$ be as above, and take an $N \in \mathbf{N}^*$ sufficiently large so that $\delta N + s \geq a$ holds. We express $u(t, x)$ in the form

$$u(t, x) = \sum_{l=0}^{N-1} \phi_l(t) x^l + w(t, x) x^N.$$

Then, by (3.3), (3.4) and $0 < s < a$ we see that

$$\left(t \frac{\partial}{\partial t} \right)^j \phi_l(t) = O(t^a) \;\; (\text{as } t \longrightarrow 0) \quad \text{for } 0 \leq l < N, \quad \text{and} \tag{3.7}$$

$$\left| \left(t \frac{\partial}{\partial t} \right)^j w(t) \right|_R = O(t^s) \;\; (\text{as } t \longrightarrow 0) \tag{3.8}$$

hold for $j = 0, 1, \ldots, m-1$; hence, if we take $0 < R_0 < R$ we have

$$\left| \left(\frac{\partial}{\partial x} \right)^l \left(t \frac{\partial}{\partial t} \right)^j w(t) \right|_{R_0} = O(t^s) \;\; (\text{as } t \longrightarrow 0) \;\; \text{for any } l \in \mathbf{N} \tag{3.9}$$

for $j = 0, 1, \ldots, m-1$.

Set $w^*(t, x) = w(t, t^\delta x)$. By the definition we have

$$u^*(t, x) = \sum_{l=0}^{N-1} \phi_l(t) t^{\delta l} x^l + w^*(t, x) t^{\delta N} x^N$$

and by Lemma 2 we have

$$\left(t \frac{\partial}{\partial t} \right)^j w^*(t, x) = \left[\left(t \frac{\partial}{\partial t} + \delta x \frac{\partial}{\partial x} \right)^j w \right] (t, t^\delta x)\,.$$

Since $0 < R_1 < R/T^\delta$ is assumed, by (3.9) we have

$$\left|\left(t\frac{\partial}{\partial t}\right)^j w^*(t)\right|_{R_1} \leq \left|\left(t\frac{\partial}{\partial t}+\delta x\frac{\partial}{\partial x}\right)^j w(t)\right|_{T^\delta R_1} = O(t^s) \quad (\text{as } t \longrightarrow 0) \tag{3.10}$$

for $j = 0, 1, \ldots, m-1$. Thus, by (3.7) and (3.10) we obtain

$$\begin{aligned}\left|\left(t\frac{\partial}{\partial t}\right)^j u^*(t)\right|_{R_1} &\leq \sum_{l=0}^{N-1}\left|\left(t\frac{\partial}{\partial t}\right)^j (\phi_l(t)t^{\delta l})\right| {R_1}^l + \left|\left(t\frac{\partial}{\partial t}\right)^j (w^*(t)t^{\delta N})\right|_{R_1} {R_1}^N \\ &\leq \sum_{l=0}^{N-1} O(t^{a+\delta l}) + O(t^{s+\delta N}) = O(t^a) \quad (\text{as } t \longrightarrow 0)\end{aligned}$$

for $j = 0, 1, \ldots, m-1$. This proves (3.6). □

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