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Geometric Morita Equivalence for Twisted Poisson Manifolds

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Abstract. We introduce notions of Morita equivalence for both twisted symplectic groupoids and integrable twisted Poisson manifolds without terms of groupoids. We show that two integrable twisted Poisson manifolds are Morita equivalent if and only if their associated groupoids are Morita equivalent as twisted symplectic groupoids.

0. Introduction

Morita equivalence was introduced to Poisson Geometry by P. Xu [17], [18]. If two Poisson manifolds are Morita equivalent, their representation categories are equivalent like algebraic Morita theory. Subsequently, he defined the notion of Morita equivalence for quasi-symplectic groupoids to discuss the momentum map theory [16].

Our purpose in this paper is to introduce Morita equivalence for twisted Poisson manifolds which are integrable. A twisted Poisson manifold is a smooth manifold M equipped with a bivector $\pi \in \Gamma(\wedge^2 TM)$ and a closed 3-form $\phi \in \Omega^3(M)$ which satisfy $\frac{1}{2}[\pi, \pi]_{SN} = \wedge^3 \pi^\sharp(\phi)$, where $[\cdot, \cdot]_{SN}$ denotes the Schouten-Nijenhuis bracket and $\pi^\sharp$ means a contraction, namely, $\pi^\sharp : T^*M \to TM$. This concept arose from physics and was called WZW-Poisson manifold [14]. The integrability means that there exists a twisted symplectic groupoid whose source fiber is simply-connected over a twisted Poisson manifold. Here, a twisted symplectic groupoid is a quasi-groupoid $(G \rightrightarrows G_0, \omega, \psi)$ such that the 2-form ω is non-degenerate. The notion of Morita equivalence for twisted symplectic groupoids can be defined by adding the non-degeneracy condition to the one for quasi-symplectic groupoids. Therefore, one of the way to define Morita equivalence for twisted Poisson manifolds is to use the one for twisted symplectic groupoids. However, we define Morita equivalence for them without the notion of groupoids and show that this definition without using the terms of groupoids is equivalent to the one which we define in terms of twisted symplectic groupoids, namely

MAIN THEOREM. *Two integrable twisted Poisson manifolds P and Q are Morita equivalent if and only if their twisted symplectic groupoids $\Gamma(P)$ and $\Gamma(Q)$ whose respective source fibers are simply-connected are Morita equivalent as twisted symplectic groupoids.*

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1. Preliminaries

1.1. Twisted Poisson structures. Let M be a smooth manifold, and ϕ a closed 3-form on M. A bivector $\pi \in \Gamma(\wedge^2 TM)$ is called a ϕ-twisted Poisson bivector if it satisfies the following condition:

$$\frac{1}{2}[\pi, \pi]_{SN} = \wedge^3 \pi^\sharp(\phi)\,. \tag{1}$$

A ϕ-twisted Poisson manifold is a smooth manifold M equipped with a ϕ-twisted Poisson bivector π. This can be understood by means of Dirac structure. We define the following two operations $\langle \cdot, \cdot \rangle_+$, $[\![\cdot, \cdot]\!]$ on $\Gamma(TM \oplus T^*M)$:

1. $\langle (X, \xi), (Y, \eta) \rangle_+ = \xi(Y) + \eta(X) \ \in C^\infty(M)$;
2. $[\![(X, \xi), (Y, \eta)]\!] = ([X, Y],\ \mathcal{L}_X \eta + i_Y d\xi + i_{X \wedge Y}\phi) \ \in \Gamma(TM \oplus T^*M)$.

A subbundle $L \subset TM \oplus T^*M$ is called a ϕ-twisted Dirac structure if the following conditions are satisfied:

1. L is maximal isotropic with regard to the first pairing $\langle \cdot, \cdot \rangle_+$;
2. L is closed with regard to the second bracket $[\![\cdot, \cdot]\!]$.

Here, maximal isotropic means that the rank of L is equal to dim M and $\langle \cdot, \cdot \rangle_+$ is identically 0 on $\Gamma(L)$. A bivector π becomes a ϕ-twisted Poisson bivector if and only if the graph $L_\pi \subset TM \oplus T^*M$ of π has a ϕ-twisted Dirac structure.

Similarly, the graph induced by a non-degenerate 2-form ω which satisfies $d\omega + \phi = 0$ is a ϕ-twisted Dirac structure. Then ω is called a ϕ-twisted symplectic form.

One of the remarkable properties for ϕ-twisted Poisson manifold is that the Jacobi identity does not necessarily hold. We define brackets and Hamiltonian vector fields by using a ϕ-twisted Poisson bivector π;

$$\{f, g\} = \pi(df, dg)\,, \quad H_f = \pi^\sharp(df)\,.$$

Then, (1) is equivalent to the condition

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} + \phi(H_f, H_g, H_h) = 0 \tag{2}$$

for all smooth functions f, g, h. Here, we give some examples.

EXAMPLE 1.1. Let A be a set of elements $(x_1, x_2, x_3, x_4) \in \mathbf{R}^4$ that satisfy $x_1 = 0$ or $x_3 = 0$. For a closed 3-form $\phi = ((1/{x_3}^2)dx_2 - (1/{x_1}^2)dx_4) \wedge dx_1 \wedge dx_3$ on $\mathbf{R}^4 \setminus A$, a bivector $\pi = x_3(\partial/\partial x_1) \wedge (\partial/\partial x_2) + x_1(\partial/\partial x_3) \wedge (\partial/\partial x_4)$ satisfies the condition (1) . Namely, $(\mathbf{R}^4 \setminus A, \pi, \phi)$ is a ϕ-twisted Poisson manifold.

EXAMPLE 1.2. The set of 2×2 matrices with real entries $M_2(\mathbf{R})$ has a twisted Poisson structure. Let $E_i (i = 1, 2, 3, 4)$ be a standard basis of $M_2(\mathbf{R})$ and we denote the dual basis of E_i by ${E_i}^* (i = 1, 2, 3, 4)$. By giving a bivector $\pi = E_1 \wedge E_4 + E_2 \wedge E_3$ for a closed 3-form $\phi = -({E_1}^* + {E_4}^*) \wedge {E_2}^* \wedge {E_3}^*$, π is a ϕ-twisted Poisson bivector on $M_2(\mathbf{R})$.

1.2. Dirac maps. Let $(Q,\ L_Q)$ and $(P,\ L_P)$ be ϕ_Q-twisted and ϕ_P-twisted Dirac structures, respectively. A smooth map $J : Q \to P$ is said to be a forward Dirac map when the following condition holds for all $x \in Q$:

$$(L_P)_{J(x)} = \{\, (J_*(V),\ \eta) \mid V \in T_x Q,\ \eta \in T^*_{J(x)} P \text{ and } (V,\ \eta \circ J_*) \in (L_Q)_x \}\,.$$

If both Q and P are twisted Poisson manifolds, we call the forward-Dirac map $J : Q \to P$ $(\phi_Q,\ \phi_P)$-twisted Poisson map. A $(\phi_Q,\ \phi_P)$-twisted Poisson map $J : Q \to P$ implies the following condition:

$$\pi_P(\alpha, \beta) = \pi_Q(J^*\alpha, J^*\beta) \tag{3}$$

for any 1-form $\alpha, \beta \in \Omega^1(P)$. Then the next lemma follows from (1) and (3) .

LEMMA 1.3. *Let (Q, π_Q, ϕ_Q) and (P, π_P, ϕ_P) be twisted Poisson manifold. Suppose that $J : Q \to P$ is a (ϕ_Q, ϕ_P)-twisted Poisson map, then ϕ_Q and ϕ_P satisfy the following condition*:

$$\wedge^3(\pi_Q^\sharp \circ J^*)(\phi_Q) = \wedge^3 \pi_P^\sharp(\phi_P)$$

Here, we define a analogue of a symplectic realization for a subsequent discussion.

DEFINITION 1.4. Let (P, π, ϕ) be a ϕ-twisted Poisson manifold. A twisted symplectic realization of (P, π, ϕ) is an forward Dirac map $J : Q \to P$, where Q is a $J^*\phi$-twisted symplectic manifold.

2. Morita equivalence for twisted symplectic groupoids

We need some preparations before discussing Morita equivalence of twisted Poisson manifolds. Suppose that there is a left (or right) action of a Lie groupoid G over X on a smooth manifold M. The groupoid action is called principal with regard to a smooth map $J : M \to Y$ between M and any smooth manifolds Y if J is surjective submersion and G acts on freely and transitively on each fiber of J.

If Lie groupoids G and H act on a smooth manifold X from the left and right, respectively, and the actions commute, we call X a (G, H)-bibundle. A (G, H)-bibundle is left principal when the left G-action is principal with regard to the momentum map for the right H-action. Similarly, it is called right principal when the right H-action is principal with regard to the momentum map for the left G-action, and biprincipal when it is principal with regard to both left and right actions.

P. Xu introduced Morita equivalence for quasi-symplectic groupoids in [16]. We can define Morita equivalence for twisted symplectic groupoids easily by deforming his definition.

A ϕ-twisted symplectic groupoid Γ is a Lie groupoid equipped with a non-degenerate 2-form ω such that $\phi + \omega$ is 3-cocycle in the bar-de Rham complex obtained from Γ. That is to say, a Lie groupoid $\Gamma_1 \rightrightarrows \Gamma_0$ is called ϕ-twisted symplectic groupoid if and only if it has a non-degenerate 2-form $\omega \in \Omega^2(\Gamma_1)$ and a closed 3-form $\phi \in \Omega^3(\Gamma_0)$ which satisfies the following conditions:

1. $d\omega = s^*\phi - t^*\phi$;

2. A 2-form $\omega \oplus \omega \oplus (-\omega)$ vanishes on the graph of the groupoid multiplication $\mathrm{graph}(\Gamma_1^{(2)} \to \Gamma_1) \subset \Gamma_1 \times \Gamma_1 \times \Gamma_1$,

where s and t are the source and target maps of $\Gamma_1 \rightrightarrows \Gamma_0$ respectively, and $\Gamma_1^{(2)}$ denotes a fiber product with regard to source and target maps. Namely, $\Gamma_1^{(2)} = \{ (g,h) \mid g,h \in \Gamma_1, s(g) = t(h) \}$.

EXAMPLE 2.1. Let (P, ω, ϕ) be a ϕ-twisted symplectic manifold. Then the pair groupoid $(P \times P \rightrightarrows P, \omega \oplus (-\omega), \phi \oplus (-\phi))$ is a ϕ-twisted symplectic groupoid.

DEFINITION 2.2. Twisted symplectic groupoids $(G \rightrightarrows G_0, \omega_G, \phi_G)$ and $(H \rightrightarrows H_0, \omega_H, \phi_H)$ are said to be Morita equivalent if there exists a biprincipal (G, H)-bibundle $G_0 \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} H_0$ together with a non-degenerate 2-form $\omega_X \in \Omega^2(X)$ such that

1. $d\omega_X + \rho^*\phi_G - \sigma^*\phi_H = 0$;
2. A 2-form $\omega_G \oplus (-\omega_H) \oplus \omega_X \oplus (-\omega_X)$ vanishes on the graph of the $(G \times H)$-action $\Lambda \subset G \times H \times X \times X$, where the action is given by $(g,h) \cdot x = gxh^{-1}$ for any $g \in G, h \in H, x \in X$ such that $s(g) = \rho(x)$ and $s(h) = \sigma(x)$.

Note that the following proposition holds:

PROPOSITION 2.3. *Let $(G \rightrightarrows G_0, \omega_G, \phi_G)$ and $(H \rightrightarrows H_0, \omega_H, \phi_H)$ be twisted symplectic groupoids, and X a biprincipal (G, H)-bibundle equipped with a non-degenerate 2-form. Then, the following are equivalent*:

1. *$\omega_G \oplus \omega_X \oplus (-\omega_X)$ vanishes on the graph of G-action, and $\omega_X \oplus \omega_H \oplus (-\omega_X)$ also does on the graph of H-action*;
2. *A 2-form $\omega_G \oplus (-\omega_H) \oplus \omega_X \oplus (-\omega_X)$ vanishes on the graph of the $(G \times H)$-action $\Lambda \subset G \times H \times X \times X$, where the action is given by $(g,h) \cdot x = gxh^{-1}$ for any $g \in G, h \in H, x \in X$ such that $s(g) = \rho(x)$ and $s(h) = \sigma(x)$.*

PROOF. Suppose that p is any point in Λ and the mappings

$$t \mapsto (g_i(t),\ h_i(t),\ x_i(t),\ u_i(t)) \in \Lambda \quad (u_i(t) = g_i(t) \cdot x_i(t) \cdot h_i(t)^{-1},\ i = 1, 2) \tag{4}$$

are any smooth path through p at $t = 0$.
We consider two new paths in the graph of the left action of G on X denoted by $\mathrm{graph}(G*X \to X)$ by using (4)

$$t \mapsto (g_i(t),\ x_i(t),\ v_i(t))$$

where $v_i = g_i(t) \cdot x(t)$. Then we obtain

$$\omega_G(g_1'(0), g_2'(0)) + \omega_X(x_1'(0), x_2'(0)) - \omega_X(v_1'(0), v_2'(0)) = 0 \tag{5}$$

by the assumption.

Similarly, by considering the following two paths in the graph of the right action of H on X denoted by $\mathrm{graph}(X * H \to X)$

$$t \mapsto (u_i(t),\ h_i(t),\ v_i(t))\,.$$

We obtain

$$\omega_X(u_1'(0), u_2'(0)) + \omega_H(h_1'(0), h_2'(0)) - \omega_X(v_1'(0), v_2'(0)) = 0\,. \tag{6}$$

By (5) and (6) , we can verify

$$\omega_G(g_1'(0), g_2'(0)) - \omega_H(h_1'(0), h_2'(0)) + \omega_X(x_1'(0), x_2'(0)) - \omega_X(u_1'(0), u_2'(0)) = 0\,.$$

Conversely, suppose that the condition *2.* in Proposition 2.3 holds, and we define two path in graph$(G * X \to X)$

$$t \mapsto (\alpha_i(t),\ \beta_i(t),\ \gamma_i(t)) \quad (\gamma_i(t) = \alpha_i(t) \cdot \beta_i(t),\ i = 1, 2) \tag{7}$$

We construct new paths as follows by using (7)

$$t \mapsto (\alpha_i(t),\ \kappa_i(t),\ \gamma_i(t))\,, \quad (i = 1, 2)$$

where $\kappa_i(t) = \varepsilon_H \circ \sigma \circ \beta_i(t)$. Here ε_H is the identity section of H. By the assumption, we then obtain

$$\omega_G(\alpha_1'(0), \alpha_2'(0)) + \omega_X(\beta_1'(0), \beta_2'(0)) - \omega_H(\kappa_1'(0), \kappa_2'(0)) - \omega_X(\gamma_1'(0), \gamma_2'(0)) = 0\,.$$

Here, we notice $\omega_H(\kappa_1'(0), \kappa_2'(0)) = 0$. because H is twisted symplectic groupoid. Therefore, it follows that

$$\omega_G(\alpha_1'(0), \alpha_2'(0)) + \omega_X(\beta_1'(0), \beta_2'(0)) - \omega_X(\gamma_1'(0), \gamma_2'(0)) = 0\,.$$

Similarly, we obtain

$$\omega_X(\lambda_1'(0), \lambda_2'(0)) + \omega_H(\mu_1'(0), \mu_2'(0)) - \omega_X(\nu_1'(0), \nu_2'(0)) = 0$$

from the following two paths in graph$(X * H \to X)$.

$$t \mapsto (\lambda_i(t),\ \mu_i(t),\ \nu_i(t)) \quad (\nu_i(t) := \lambda_i(t) \cdot \mu_i(t),\ i = 1, 2)$$

□

We observe that twisted symplectic groupoids $(G \rightrightarrows G_0, \omega_G, \phi_G)$ and $(H \rightrightarrows H_0, \omega_H, \phi_H)$ are Morita equivalent if and only if there exists a biprincipal (G, H)-bibundle $G_0 \xleftarrow{\rho} X \xrightarrow{\sigma} H_0$ together with a non-degenerate 2-form ω_X such that $d\omega_X + \rho^*\phi_G - \sigma^*\phi_H = 0$ and they satisfy either of the conditions in Proposition 2.3.

3. Morita equivalence for integrable twisted Poisson manifolds

Given a Lie groupoid $G \rightrightarrows X$, we can construct Lie algebroid whose fiber on $x \in X$ is $\mathrm{Ker}(ds)_x$, where s is a source map of the Lie groupoid. We denote by $\mathcal{A}(G)$ the Lie algebroid associated with $G \rightrightarrows X$. A Lie algebroid A is said to be integrable if there exists a Lie groupoid G such that $\mathcal{A}(G)$ is isomorphic to A.

If (P, π, ϕ) is a ϕ-twisted Poisson manifold, then the cotangent bundle T^*P has Lie algebroid structure with bracket

$$[\alpha, \beta] = \mathcal{L}_{\pi^\sharp\alpha}\beta - \mathcal{L}_{\pi^\sharp\beta}\alpha + d(\pi(\alpha, \beta)) + \phi(\pi^\sharp\alpha, \pi^\sharp\beta, \cdot)$$

for all $\alpha, \beta \in \Omega^1(P)$. A ϕ-twisted Poisson manifold P is called integrable when the Lie algebroid T^*P is integrable. As is explained in [6], there exists twisted symplectic groupoid $\Gamma(P)$ over P whose fibers are source-simply connected if and only if P is integrable.

Let us recall how to construct $\Gamma(P)$ from T^*P. A C^1-curve $a : [0, 1] \to T^*P$ is called cotangent path if

$$(\pi^\sharp \circ a)(t) = \frac{d}{dt}(\varpi \circ a)$$

where $\varpi : T^*P \to P$ is a bundle projection, and $\varpi \circ a$ is of class C^2. We denote the space of cotangent paths with the topology of uniform convergence by $P(T^*P)$.

Given a TM-connection $\widetilde{\nabla}$ on T^*P, we define a connection ∇ on T^*P by $\nabla_\alpha\beta := \widetilde{\nabla}_{\pi^\sharp\alpha}\beta$. Then, the torsion of ∇ is defined as usual:

$$T_\nabla(\alpha, \beta) = \nabla_\alpha\beta - \nabla_\beta\alpha - [\alpha, \beta]$$

.

Let us fix a connection ∇ with torsion T_∇ and a family of cotangent path a_ε which is of class C^2 on ε and the base paths $\varpi \circ a_\varepsilon$ have fixed end points. Then the differential equation

$$\partial_t b - \partial_\varepsilon a_\varepsilon = T_\nabla(a_\varepsilon, b_\varepsilon)\,, \quad b(\varepsilon, 0) = 0$$

has a unique solution $b(\varepsilon, t)$. Moreover the solution does not depend on ∇ (see [8]).

The cotangent paths a_1, a_2 are said to be homotopic and write $a_1 \sim a_2$, if there exists a family $a_\varepsilon(t)$ of cotangent paths with the property that the solution b of the above differential equation satisfies $b(\varepsilon, 1) = 0$ for any $\varepsilon \in [0, 1]$. We denote by $\Gamma(P)$ the space of cotangent homotopy classes of cotangent paths, that is $\Gamma(P) = P(T^*P)/\sim$. It is known that $\Gamma(P) \rightrightarrows P$ becomes a twisted symplectic groupoid with source-simply connected fibers (see [2], [6] and [8]).

DEFINITION 3.1. Let P and Q be integrable ϕ_P-twisted and ϕ_Q-twisted Poisson manifolds, respectively. P and Q are said to be Morita equivalent when there exists a smooth manifold X together with a non-degenerate 2-form ω_X and two surjective submersions $J_1 : X \to P$ and $J_2 : X \to Q$ which satisfy the following conditions:

1. J_1 and J_2 are complete twisted symplectic realization and complete anti-twisted symplectic realization, respectively;
2. (X, ω_X) is a $(J_1{}^*\phi_P - J_2{}^*\phi_Q)$-twisted symplectic manifold;
3. J_1 and J_2 have connected, and simply-connected fibers;
4. $\{J_1{}^*C^\infty(P),\ J_2{}^*C^\infty(Q)\}_X = 0$

where $\{\cdot,\ \cdot\}_X$ is the Poisson bracket induced from non-degenerate 2-form ω_X.

EXAMPLE 3.2. In the case of $\phi_P = \phi_Q = 0$, our definition of Morita equivalence corresponds to the one for usual Poisson manifolds (see [18]).

EXAMPLE 3.3. Let S be an ordinary symplectic manifold which is connected. We denote by $\overline{S}$ the universal cover of S with base point p. Then $\overline{S}$ is a biprincipal $(\Gamma(S), \pi(S, p))$-bibundle. Note that $\pi(S, p) \rightrightarrows \{p\}$ is a symplectic groupoid over the zero-dimensional Poisson manifold p (see [3] and [4]).

If two integrable twisted Poisson manifolds P and Q are Morita equivalent, the twisted symplectic groupoids which are associated with them respectively must be Morita equivalent in the meaning of Definition 2.2.

In order to prove Main Theorem, we prepare the following proposition.

PROPOSITION 3.4. *Suppose that integrable twisted Poisson manifolds P and Q are Morita equivalent $P \overset{\rho}{\leftarrow} X \overset{\sigma}{\rightarrow} Q$. Then it follows that $[H_{\rho^* f}, H_{\sigma^* g}] = 0$ for any smooth functions $f \in C^\infty(P)$ and $g \in C^\infty(Q)$.*

PROOF. We note that the following claim $(*)$ holds for any $h \in C^\infty(X)$:

$$([H_{\rho^* f}, H_{\sigma^* g}] + H_{\{\rho^* f, \sigma^* g\}})h = (\rho^* \phi_P - \sigma^* \phi_Q)(H_{\rho^* f}, H_{\sigma^* g}, H_h) . \qquad (*)$$

Assuming this claim for the moment, we complete the proof. From the assumption that X is ϕ-twisted symplectic manifold and $\{\rho^* C^\infty(P), \ \sigma^* C^\infty(Q)\}_X = 0$, we have

$$\begin{aligned}
[H_{\rho^* f}, H_{\sigma^* g}]h &= -d\omega_X(H_{\rho^* f}, H_{\sigma^* g}, H_h) \\
&= (i_{H_{\sigma^* g}} d i_{H_{\rho^* f}} \omega_X - i_{H_{\sigma^* g}} \mathcal{L}_{H_{\rho^* f}} \omega_X)(H_h) \\
&= (\mathcal{L}_{H_{\sigma^* g}} i_{H_{\rho^* f}} \omega_X)(H_h) - (i_{H_{\sigma^* g}} \mathcal{L}_{H_{\rho^* f}} \omega_X)(H_h) .
\end{aligned}$$

The first term in the right hand side is

$$\begin{aligned}
\mathcal{L}_{H_{\sigma^* g}} i_{H_{\rho^* f}} \omega_X(H_h) &= (\mathcal{L}_{H_{\sigma^* g}} d(\rho^* f))(H_h) = H_h(H_{\sigma^* g}(\rho^* f)) \\
&= H_h(\omega_X(H_{\rho^* f}, H_{\sigma^* g})) = 0 .
\end{aligned}$$

Similarly, the second term is

$$\begin{aligned}
(i_{H_{\sigma^* g}} \mathcal{L}_{H_{\rho^* f}} \omega_X)(H_h) &= \mathcal{L}_{H_{\rho^* f}} i_{H_{\sigma^* g}} \omega_X(H_h) - i_{[H_{\rho^* f}, H_{\sigma^* g}]} \omega_X(H_h) \\
&= -i_{[H_{\rho^* f}, H_{\sigma^* g}]} \omega_X(H_h) .
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
[H_{\rho^* f}, H_{\sigma^* g}]h &= i_{[H_{\rho^* f}, H_{\sigma^* g}]} \omega_X(H_h) = -i_{H_h} \omega_X([H_{\rho^* f}, H_{\sigma^* g}]) \\
&= -dh([H_{\rho^* f}, H_{\sigma^* g}]) = -[H_{\rho^* f}, H_{\sigma^* g}]h ,
\end{aligned}$$

which implies $[H_{\rho^* f}, H_{\sigma^* g}]h = 0$, □

It remains to show the claim $(*)$ in Proposition 3.4

CLAIM. Let P be a ϕ-twisted Poisson manifold. Then the following equation holds

$$([H_f, H_g] + H_{\{f,g\}})h = \phi(H_f, H_g, H_h) \quad (f, g, h \in C^\infty(P)) \tag{8}$$

PROOF. Note that the left hand side is equivalent to $\{h, \{f, g\}\}+\{f, \{g, h\}\}+\{g, \{h, f\}\}$. Then (8) is easily obtained from (2). □

Let us assume that $J : Q \to P$ is a complete (ϕ_Q, ϕ_P)-twisted Poisson map which satisfies $\phi_Q = J^*\phi_P$, where completeness means that the Hamiltonian vector field H_{J^*f} is complete for any complete Hamiltonian vector field H_f on P. Then due to $\phi_Q = J^*\phi_P$, we can verify that $J : Q \to P$ induces a Lie algebroid action of T^*P on Q by

$$\Omega^1(P) \to \mathfrak{X}(Q), \quad \alpha \mapsto \pi_Q^\sharp(J^*\alpha)$$

We can see that this Lie algebroid action raises a groupoid action of $\Gamma(P)$ on Q when P is integrable(see [1], [7] and [12]). We prove Main Theorem on the basis of this fact.

PROOF. Let us assume that two integrable twisted Poisson manifold P and Q are Morita equivalent. From Proposition 3.4, we obtain that $\theta_s^{J_1} \circ \theta_t^{J_2} = \theta_t^{J_2} \circ \theta_s^{J_1}$ whenever either side is defined. Here we denote by $\theta_t^{J_i}$ $(i = 1, 2)$ the Hamiltonian flow induced by a smooth function $J_i^* f$. The commutativity for Hamiltonian flows shows that two groupoid actions commute with each other (see [9], [11] and [18]). The proof for that X is biprincipal $(\Gamma(P), \Gamma(Q))$-bibundle is the same as that for Theorem 3.2 in [18]. According to the fact mentioned above, the left groupoid action of $\Gamma(P)$ on $J_1 : X \to P$ and right groupoid action of $\Gamma(Q)$ on $J_2 : X \to Q^-$ arise respectively. We consider the groupoid action of $\Gamma(P) \times \Gamma(Q)$ on $J_1 \times J_2 : X \to P \times Q^-$ obtained from their two actions, where $\Gamma(P) \times \Gamma(Q)$-action is given by $(g, h) \cdot x := gxh^{-1}$ for any $x \in X$, $g \in G$, $h \in H$ such that $s(g) = J_1(x)$ and $s(h) = J_2(x)$. The map $J_1 \times J_2 : X \to P \times Q^-$ is complete, and X is $({J_1}^*\phi_P - {J_2}^*\phi_Q)$-twisted symplectic manifold. Therefore we can see that a non-degenerate 2-form $\omega_{\Gamma(P)} \oplus (-\omega_{\Gamma(Q)}) \oplus \omega_X \oplus (-\omega_X)$ vanishes on the graph of $\Gamma(P) \times \Gamma(Q)$-action by using Corollary 7.4 in [2]. It is shown that the condition *2* in Definition 2.2 holds.

Conversely, suppose that $\Gamma(P) \rightrightarrows P$ and $\Gamma(Q) \rightrightarrows Q$ are Morita equivalent as twisted symplectic groupoids. It is clear that biprincipal $(\Gamma(P), \Gamma(Q))$-bibundle X has a $(J_1^*\phi_P - J_2^*\phi_Q)$-twisted symplectic structure. The groupoid $\Gamma(P) \rightrightarrows P$ acts on X from the left such that $\omega_{\Gamma(P)} \oplus \omega_X \oplus (-\omega_X)$ vanishes on the graph of $\Gamma(P)$-action. Therefore, $J_1 : X \to P$ is complete Poisson map (see [11] and [18]). We can prove that $J_2 : X \to Q$ is complete anti-Poisson map in the same way. □

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