TOKYO J. MATH.
VOL. 31, NO. 1, 2008

M-matrices of the Ternary Golay Code and the Mathieu Group M_{12}

Maro KIMIZUKA and Ryuji SASAKI

Nihon University

(Communicated by K. Shinoda)

Abstract. In this article, we define M-matrices of the ternary Golay code and build fundamental properties of the ternary Golay code on M-matrices. Moreover, using four M-matrices of the ternary Golay code, we give order three elements, in the Mathieu group M_{12}, which generate M_{11} and M_{12}.

1. Introduction

First of all we shall explain M-matrices. Let C be a code, which is *naturally* represented as a linear subspace of the space

$$V = \oplus_{1\le i\le k;1\le j\le n}\mathbf{F}_p e_{ij} .$$

For example, the binary Golay code and the ternary Golay code have this property. A $k \times n$ matrix

$$A = (a_{ij}e_{ij}) \quad a_{ij} \in \mathbf{F}_p^\times .$$

is called an M-matrix of C if a family of vectors of V derived from A in a *certain* way forms a basis of C. The existence of M-matrices of C characterizes some structure of the code C. The notion of M-matrices is, though implicitly, contained in many articles, e.g. [2] Ch.11, [3], [4] and [6].

At least in the case of the binary Golay code, T. Kondo [10] recognized the usefulness of M-matrices, gave an explicit definition of them and discussed their good properties. In his paper [12], M. Sawabe determines the conjugacy classes of the Mathieu group M_{24} by making use of M-matrices, of the binary Golay code. We also use M-matrices to give involutions which generate M_{22},M_{23} and M_{24} in [7].

In §2 and §3, we shall define M-matrices of the ternary Golay code and build fundamental properties of the ternary Golay code on M-matrices. We can define M-matrices of a somewhat general code, which contain the binary and the ternary Golay codes as special cases(cf. [11]).

Received August 17, 2006; revised November 6, 2006
Mathematics Subject Classification: 20D08, 20D06, 20B20, 90B05.
Key words: Golay code; Mathieu group.

In §4, we shall give an application of M-matrices to the Mathieu group M_{12}. In his paper [1], N. Chigira gives involutions in the symmetric group S_{12}, using four 3×3 boards, which generate M_{11} and M_{12}. In fact, his boards come naturally from four M-matrices of the ternary Golay code. Using these M-matrices, we shall give a set of order three elements in M_{12}, which generate M_{11} and M_{12}. R.T. Curtis ([4], [5]) also discusses generators of M_{12}, which are different from ours. We shall discuss similar results, for M_{12}, as the above Sawabe's in a separate paper.

2. Ternary Golay codes and their M-matrices

In this section, we recall shortly fundamental results about ternary Golay codes, for details we refer to [6] Ch.7, and introduce their M-matrices. Throughout this paper, we fix a set $\Omega = \{\omega_1, \omega_2, \dots, \omega_{12}\}$ and the vector space V generated by the elements of Ω over $\mathbf{F}_3$:

$$V = \sum_{i=1}^{12} \mathbf{F}_3 \omega_i \, .$$

For a vector $v = \sum_i a_i \omega_i$, we denote by supp(v) the support of v, i.e., the set of ω_i's with $a_i \neq 0$ and define the weight of v by the cardinality of supp(v).

A d-dimensional linear subspace C of V is said to be a ternary $[12, d, w]$ code if the minimum weight, which is called the minimum weight of C, of non-zero vectors of C is w.

We shall give proofs of known facts, for convenience of explanation, with some exceptions.

LEMMA 1 ([6] Ch.7 Lemma 7.14). *Let C be any ternary $[12, 6, w]$ code with $w \geq 6$. For each vector v of weight ≤ 3, let $R(v)$ be the coset $v + C \in V/C$. Then the following holds*:

1. *If the weight of u is 1 or 2 and $R(u) = R(v)$, then $u = v$.*
2. *If the weight of u is 3 and $R(u) = R(v)$, then the weight of v is also 3.*
3. *There are 2×12 cosets containing a vector of weight 1.*
4. *There are $4 \times \binom{12}{2}$ cosets containing a vector of weight 2.*
5. *There are $2 \times \binom{12}{3}$ cosets containing four vectors T_1, T_2, T_3, T_4 of weight 3. In this case they satisfy*

$$\Omega = \mathrm{supp}(T_1) \cup \mathrm{supp}(T_2) \cup \mathrm{supp}(T_3) \cup \mathrm{supp}(T_4) \, . \tag{1}$$

6. $w = 6$.

PROOF. If u is a vector of weight 1 or 2 and v is a vector of weight ≤ 3, then, by the condition $w \geq 6$, $u = v$. Assume a coset R has a vector T of weight 3. If a different vector T' from T of weight 3 is contained in R, then by $w \geq 6$ we see $\mathrm{supp}(T) \cap \mathrm{supp}(T') = \emptyset$. Therefore there are at most four vectors contained in R of weight 3. On the other hand, we have

$$3^6 = 1 + 2\binom{12}{1} + 4\binom{12}{2} + 2^3\frac{1}{4}\binom{12}{3}.$$

Hence for each vector T of weight 3 there must be exactly four vectors T' of weight 3 satisfying $T - T' \in C$, and then the weight of $T - T'$ is 6. □

Let C be a ternary $[12, 6, 6]$ code. A set of four vectors $\{T_1, T_2, T_3, T_4\}$ of weight 3 satisfying (1) is called a *foursome* of the code C. The support of a codeword of weight 6 is called a *hexad*.

LEMMA 2. *Let C be any ternary* $[12, 6, 6]$ *code. The the following hold*:

1. *The number of codewords of weight* 6 *is* 264.
2. *Let u and v be codewords of weight* 6. *If* $\mathrm{supp}(u) = \mathrm{supp}(v)$ *then* $u = \pm v$. *In particular there are* 132 *hexads.*
3. *Let $\mathcal{B}$ denote the set of all hexads. Then the pair $(\Omega, \mathcal{B})$ is a Steiner system* $S(12, 6, 5)$.

PROOF. Each foursome yields $\binom{4}{2}$ codewords of weight 6. On the other hand each codeword of weight 6 comes from $\frac{1}{2}\binom{6}{3}$ foursomes. There are $2 \times \binom{12}{3}$ foursomes by Lemma 1 (4) so that the number of codewords of weight 6 is

$$2 \times \binom{12}{3}\binom{4}{2}/\frac{1}{2}\binom{6}{3} = 264.$$

If two codewords u and v of weight 6 have the same support, then $u - v \in C$. Since the minimum weight of C is 6, it follows that $u = v$ or $u = -v$.

Now we shall show that $(\Omega, \mathcal{B})$ is a Steiner System $S(12, 6, 5)$. For any 5-set P of Ω, let A and B be two different hexads containing P, and let u and v be codewords with $\mathrm{supp}(u) = A$ and $\mathrm{supp}(v) = B$. Then $u + v$ and $u - v$ are of weight 6. If $u + v$ (resp. $u - v$) is of weight 6, then $u - v$ (resp. $u + v$) is of weight 3. This is absurd. Thus we see that any 5-set of Ω is contained in at most one hexad. Therefore, since the number of the hexads is 132, there are $132 \cdot 6$ 5-sets of Ω which are contained in hexads. On the other hand there are $\binom{12}{5} = 132 \cdot 6$ 5-sets. Therefore any 5-set of Ω is contained in a unique hexad. □

Now we shall define M-matrices of a ternary $[12, 6, 6]$ code C in the space $V = \sum_{i=1}^{12} \mathbf{F}_3\omega_i$. Let

$$\phi : \mathbf{F}_3 \times \{1, 2, 3, 4\} \to \Omega = \{\omega_1, \ldots, \omega_{12}\}$$

be a bijection, which is called a labelling. For each $(c, i) \in \mathbf{F}_3 \times \{1, 2, 3, 4\}$, take a non-zero vector $m(c, i)$ in the 1-dimensional subspace $\mathbf{F}_3\phi(c, i)$ of V. A matrix

$$M = (m(c, i)) = \begin{pmatrix} m(0, 1) & m(0, 2) & m(0, 3) & m(0, 4) \\ m(1, 1) & m(1, 2) & m(1, 3) & m(1, 4) \\ m(-1, 1) & m(-1, 2) & m(-1, 3) & m(-1, 4) \end{pmatrix}$$

is called an M-*matrix* of C with respect to a labeling ϕ if the following, which can be naturally considered as elements of V, are contained in C:

$$A_{i-1} = \begin{pmatrix} m(0,1) & -m(0,i) \\ m(1,1) & -m(1,i) \\ m(-1,1) & -m(-1,i) \end{pmatrix}, \quad 2 \le i \le 4, \tag{2}$$

$$A_4 = \begin{pmatrix} -m(0,1) & m(0,2) & m(0,3) & m(0,4) \\ m(1,1) & & & \\ m(-1,1) & & & \end{pmatrix},$$

$$A_5 = \begin{pmatrix} m(0,1) & m(0,2) & & \\ -m(1,1) & & m(1,3) & \\ m(-1,1) & & & m(-1,4) \end{pmatrix},$$

$$A_6 = \begin{pmatrix} m(0,1) & & m(0,3) & \\ -m(1,1) & & & m(1,4) \\ m(-1,1) & m(-1,2) & & \end{pmatrix}.$$

As is easily seen that these 6 are linearly independent over $\mathbf{F}_3$; hence they forms a basis of C. Notice that every inner product (A_i, A_j) in the space V is zero. In particular, a ternary [12, 6, 6] code is self-dual, if it has an M-matrix.

M-matrices of C owe their origin to the following observation. Define a linear map l_i by

$$l_i : \sum_{c \in \mathbf{F}_3} \mathbf{F}_3 m(c,i) \to \mathbf{F}_3, \quad \sum x(c,i) m(c,i) \mapsto \sum_{c \in \mathbf{F}_3} c x(c,i),$$

and their direct sum

$$L = \bigoplus_{i=1}^{4} l_i : V \to (\mathbf{F}_3)^4.$$

Let

$$s_i : \mathbf{F}_3 \to \sum_{c \in \mathbf{F}_3} \mathbf{F}_3 m(c,i), \quad c \mapsto m(c,i),$$

and define the map

$$S_0 = \bigoplus_{i=1}^{4} s_i : (\mathbf{F}_3)^4 \to V.$$

Then the map

$$S : (\mathbf{F}_3)^4 \to V, \quad S(t) = X + S_0(t),$$

where

$$X = m(0, 1) + m(1, 1) + m(-1, 1)\,,$$

is a set-theoretic section of the $\mathbf{F}_3$-linear map L. Finally we define the inclusion

$$\iota_i : \mathbf{F}_3 \to \sum_{c\in\mathbf{F}_3} \mathbf{F}_3 m(c, i)\,, \quad t \mapsto \sum_{c\in\mathbf{F}_3} tm(c, i)$$

and set

$$I = \bigoplus_{i=1}^{4} \iota_i : (\mathbf{F}_3)^4 \to V\,.$$

Let

$$e_1 = (1, 0, 0, 0)\,, \quad e_2 = (0, 1, 0, 0)\,, \quad e_3 = (0, 0, 1, 0)\,, \quad e_4 = (0, 0, 0, 1)$$

be the standard basis of $\mathbf{F}_3^4$.

Under these notation, we have the following:

1. The image of L is the tetracode $\mathcal{T}$:

$$\mathcal{T} := \{(a, b, \phi(1), \phi(-1)) \mid a, b \in \mathbf{F}_3 := \{0, \pm 1\},\ \phi(x) := ax + b\}\,.$$

2. $A_1 = I(e_1) - I(e_2)$, $A_2 = I(e_1) - I(e_3)$, $A_3 = I(e_1) - I(e_4)$.
3. $A_4 = S((0, 0, 0, 0))$, $A_5 = S((1, 0, 1, -1))$, $A_6 = S((1, -1, 0, 1))$, where $\{(1, 0, 1, -1), (1, -1, 0, 1)\}$ is a basis of the tetracode $\mathcal{T}$.

Generalizing this situation, we define codes of *type MOG* and introduce M-matrices of such a code (cf. [11]). Notice that *MOG* comes from the *Miracle Octad Generator* (cf. [2] Ch. 11).

The set of codewords of weight 6 is explicitly given if a code has an M-matrix:

LEMMA 3 ([6] (7.9)). *Let the notation be as above. Then the set of codewords of C of weght* 6 *is represented in the follwoing*:

$$\pm(I(e_i) - I(e_j)), \pm(I(e_i) + S(t)), S(t) - S(t'), \pm(I(e_i) + I(e_j) - S(t))$$

where $1 \le i < j \le 4$ *and* $t \ne t'$ *run over the tetracode* $\mathcal{T}$.

PROOF. The number of these codewords is

$$2 \cdot 6 + 2 \cdot 36 + 72 + 2 \cdot 54 = 264\,.$$

On the other hand, by Lemma 2 there are 264 codewords of weght 6. Thus we get the lemma. □

Now we shall prove the main theorem in this section. In the proof, a codeword $a_{i_1}\omega_{i_1} + \cdots + a_{i_k}\omega_{i_k}$ is sometimes denoted by $\{a_{i_1}\omega_{i_1}, \ldots, a_{i_k}\omega_{i_k}\}$.

THEOREM 4. *Let C be an arbitrary ternary* $[12, 6, 6]$ *code in V. For any codeword* $X := \{-x_1, x_2, x_3, x_4, y_1, z_1\} \in C$ *of weight* 6, *there exists uniquely an M-matrix of the form*:

$$\begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & & & \\ z_1 & & & \end{pmatrix}.$$

PROOF. Notice that, by Lemma 2, $(\Omega, \mathcal{B})$ is the Steiner system $S(12, 6, 5)$. For a vector x of weight 1, we denote by $\bar{x}$ the supprt of x. By Lemma 1, we see that the set $T_1 = \{x_1, y_1, z_1\}$ determines a foursome T_1, T_2, T_3, T_4 so that they satisfy

1. $\Omega = \cup_{i=1}^{4} \mathrm{supp}(T_i)$,
2. $T_i - T_j = \{x_i, y_i, z_i, -x_j, -y_j, -z_j\}$ is a codeword of weight 6.

There is a codeword $K (\neq X)$ of weight 6 such that $\mathrm{supp}(K) \supset \{\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{x}_2\}$ and $x_2 \in K$. Considering $K - (T_1 - T_2)$ and $K - X$, we see that K is equal to one of the following:

$$K_1 = \{x_1, -y_1, z_1, x_2, *, *\} \quad \text{or} \quad K_2 = \{x_1, y_1, -z_1, x_2, *, *\}.$$

We take K_1 as K and set $K \cap T_3 = \{y_3\}, K \cap T_4 = \{z_4\}$, renaming when necessary. And we can set $K_2 = \{x_1, y_1, -z_1, x_2, z_3, y_4\}$. Lastly let L be a codeword of weight 6 such that $\mathrm{supp}(L) \supset \{\bar{x}_1, \bar{y}_1, \bar{z}_1, \bar{x}_3, \bar{y}_4\}$. By a similar way, we have $L = \{x_1, -y_1, z_1, x_3, y_4, *\}$, and set $L \cap T_2 = \{z_2\}$. Thus we have determined the set $\{y_2, y_3, y_4, z_2, z_3, z_4\}$. Since the set $\{T_1 - T_i \ (i = 2, 3, 4), X, K, L\}$ is contained in C and $X = A_4, K = A_5, L = A_6$ in (2), it follows that the matrix

$$M = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix}$$

is an M-matrix of C by the definition. The uniqueness is obvious from the construction. □

COROLLARY 5. *Any two ternary* $[12, 6, 6]$ *codes are isomorphic.*

PROOF. Let C and C' be two ternary $[12, 6, 6]$ codes. Let $M = (m_{ij})$ and $M' = (m'_{ij})$ be M-matrices of C and C', respectively. Then the mapping defined by $m_{ij} \mapsto m'_{ij}$ gives an isomorphism of C to C'. □

From now on we call a ternary $[12, 6, 6]$ code *the ternary Golay code*.

3. Automorphisms of the ternary golay code and the mathieu group M_{12}

In this section, we shall survey automorphisms of the ternary Golay code and the Mathieu group M_{12}, for details and further discussion we refer to [6] Ch.5, Ch.7 and [11]. We shall begin with the monomial group with coefficient in the prime field $\mathbf{F}_p$. Let S_n be the symmetric group of degree n. Then S_n acts on the group $(\mathbf{F}_p^{\times})^n$ via

$$(a_1, \ldots, a_n)^{\sigma} = (a_{1^{\sigma^{-1}}}, \ldots, a_{n^{\sigma^{-1}}}).$$

By this action, we get the semi-direct product $S_n \ltimes (\mathbf{F}_p^\times)^n$, which is isomorphic to the group $\mathrm{Mon}(n, \mathbf{F}_p)$ of monomial matrices with coefficient in $\mathbf{F}_p^\times$:

$$\phi : S_n \ltimes (\mathbf{F}_p^\times)^n \to \mathrm{Mon}(n, \mathbf{F}_p)\,, \quad (\sigma, a) \mapsto A\,,$$

where

$$a = (a_1, \ldots, a_n)\,, \quad A_{ij} = \begin{cases} a_{i^\sigma} & \text{if } j = i^\sigma \\ 0 & \text{otherwise}\ . \end{cases}$$

For an element $A \in \mathrm{Mon}(n, \mathbf{F}_p)$, we set

$$\phi^{-1}(A) = (\sigma(A), a(A))\,.$$

For a linear code D in $\mathbf{F}_p^n$, we define, as usual, the group of automorphisms of D by

$$\mathrm{Aut}(D) = \{m \in \mathrm{Mon}(n, \mathbf{F}_p) \mid D^m = D\}\,.$$

The semi-direct product $N := \mathrm{Mon}(n, \mathbf{F}_p) \ltimes \mathbf{F}_p^n$ acts on the set $\Omega_0 = \{(c, i) \mid c \in \mathbf{F}_p, 1 \le i \le n\}$ via

$$(c, i)^{(m,v)} = (ca_{i^{\sigma(m)}} + v_{i^{\sigma(m)}}, i^{\sigma(m)})\,, \quad m \in \mathrm{Mon}(n, \mathbf{F}_p)\,, \quad v \in \mathbf{F}_p^n\,, \tag{3}$$

where

$$\sigma(m) = \sigma(A)\,, \quad (a_1, \ldots, a_n) = a(m) = a(A)\,.$$

Then N can be considered as a subgroup of the permutation group of the set Ω_0. Therefore the group N acts on $V = \sum_{(c,i)\in\Omega_0} \mathbf{F}_p(c, i) \simeq (\mathbf{F}_p)^{p\times n}$, which is the space of $p \times n$ matrices with coefficient in $\mathbf{F}_p$, under the following:

$$x^\nu = \sum_{(c,i)\in\Omega_0} x(c, i)(c, i)^\nu = \sum_{(c,i)\in\Omega_0} x((c, i)^{\nu^{-1}})(c, i)\,,$$

for

$$x = \sum_{(c,i)\in\Omega_0} x(c, i)(c, i),\ \nu \in N\,.$$

Thus we have an injective homomorphism

$$N = \mathrm{Mon}(n, \mathbf{F}_p) \ltimes (\mathbf{F}_p)^n \to \mathrm{Mon}(pn, \mathbf{F}_p)\,. \tag{4}$$

From now on we fix a ternary Golay code C in the ambient space $V = \sum_{i=1}^{12} \mathbf{F}_3\omega_i$ as in the previous section. Let $M = (m(c, i))$ $(c \in \mathbf{F}_3, 1 \le i \le 4)$ be an M-matrix of C. Then we defined, in §3, the following mappings:

$$L : V \simeq (\mathbf{F}_3)^{3\times 4} \to (\mathbf{F}_3)^4,\ S, I : (\mathbf{F}_3)^4 \to V \simeq (\mathbf{F}_3)^{3\times 4}\,.$$

Notice that the image of the linear map L is the tetracode $\mathcal{T}$.

Then, for $x \in V$, we have the following:

1. $L(x^{(m,0)}) = L(x)^m, \ (m,0) \in N$.
2. $L(x^{(1,v)}) = L(x) + (\dots, (I(e_i), x)v_i, \dots)$, where 1 stands for the identity matrix, $(1, v) \in N$ and (x, y) means the inner product of the space V over $\mathbf{F}_3$.

Using these, we obtain the following (cf. [6] (7.19) and [11]):

LEMMA 6. *Let the notation be as above. The restriction of the injective homomorphism* (4) *gives an isomorphism*

$$\mathrm{Aut}(\mathcal{T}) \ltimes \mathcal{T} \xrightarrow{\sim} (\mathrm{Mon}(4, \mathbf{F}_3) \ltimes (\mathbf{F}_3)^4) \cap \mathrm{Aut}(C) .$$

Finally we recall the relation between the group $\mathrm{Aut}(C)$ of automorphisms of the ternary Golay code C and the Mathieu group M_{12}. Every automorphism of C induces a permutation of Ω; hence we obtain a group homomorphism

$$f : \mathrm{Aut}(C) \to \mathrm{Sym}(\Omega) \simeq S_{12} ,$$

where $\mathrm{Sym}(\Omega)$ is the permutation group of the set Ω.

LEMMA 7. *The kernel of the homomorphism f is the group $\langle -1 \rangle$ generated by the involution -1 of C defined by $x \mapsto -x$.*

PROOF. Take a codeword X of weight 6. Let M be an M-matrix of C associated with X (cf. Theorem 4). If an automorphism m of C induces the identity permutation on Ω, then, by Lemma 2, we have $X^m = X$ or $X^m = -X$; hence, by the uniqueness of M-matrices in Theorem 4, $M^m = M$ or $M^m = -M$. Thus m must be the identity or -1. □

THEOREM 8. *The factor group $\mathrm{Aut}(C)/\langle -1 \rangle$ acts regularly on the set $\mathcal{S}$ of all ordered sequences of 5 elements from Ω. In particular,*

$$|\mathrm{Aut}(C)/\langle -1 \rangle| = |\mathcal{S}| = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 .$$

PROOF. Let $A = (a_1, \dots, a_5)$ and $B = (b_1, \dots, b_5)$ be two sequences in $\mathcal{S}$. Then by Lemma 2 there are two codewords $X = \{-x_1, x_2, \dots, x_6\}$ and $Y = \{-y_1, y_2, \dots, y_6\}$ of C such that

$$\mathrm{supp}(x_i) = a_i , \quad \mathrm{supp}(y_i) = b_i \ (1 \le i \le 5) .$$

Then by Theorem 4 there are two M-matrices of the form

$$(m(c,i)) = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & & & \\ x_6 & & & \end{pmatrix}, \quad (m(c,i)') = \begin{pmatrix} y_1 & y_2 & y_3 & y_4 \\ y_5 & & & \\ y_6 & & & \end{pmatrix}.$$

Then the mapping m defined by $m(c,i) \longmapsto m(c,i)'$ is an automorphism of C and m induces the permutation σ with $A^\sigma = B$. Thus we have the transitivity.

If the permutation induced by an automorphism m of C satisfies $A^\sigma = A$, then by Lemma 2 the above X^m must be X or $-X$. Hence for the M-matrix $(m(c,i))$ associated with X, we have $(m(c,i))^m = (m(c,i))$ or $(-m(c,i))$. Thus m must be 1 or -1, and we get the regularity. □

Since the ternary Golay code C is unique up to isomorphisms, it follows that the structure of $\mathrm{Aut}(C)$ is also unique up isomorphisms. The group $\mathrm{Aut}(C)/\langle -1 \rangle$ is called the *Mathieu group* of degree 12, which will be denoted by M_{12}. By Lemma 7, M_{12} is a permutation group on the set Ω. Let M_{11} denote the stabilizer subgroup of M_{12} at a point from Ω. By Lemma 8, M_{12} is 5-transitive on Ω, and the structure of M_{11} does not depend on a choice of a point. M_{11} is called the *Mathieu group* of degree 11.

4. Involutions and order three elements in M_{12}

In this section we denote by C the ternary Golay code in $V = \sum_{i=1}^{12} \mathbf{F}_3 \omega_i$. Let $M = (m(c,i))$ be an M-matrix of C. We shall use a traditional notation, so we rename by the following (cf. [2] Ch.10):

$$\Omega^* := (m(c,i)) = \begin{pmatrix} \infty & -8 & 0 & -1 \\ 7 & -X & 3 & -9 \\ 6 & -2 & 4 & -5 \end{pmatrix} \quad (X = 10)\,.$$

Now consider the following three M-matrices:

$$\begin{pmatrix} 7 & 0 & -3 & -1 \\ 5 & 8 & -X & -9 \\ \infty & 2 & -4 & -6 \end{pmatrix}, \quad \begin{pmatrix} 5 & -3 & X & -1 \\ 6 & -0 & 8 & -9 \\ 7 & -2 & 4 & -\infty \end{pmatrix}, \quad \begin{pmatrix} 6 & X & -8 & -1 \\ \infty & 3 & -0 & -9 \\ 5 & 2 & -4 & -7 \end{pmatrix}.$$

It is easily checked that these are in fact M-matrices by Lemma 3 or the property of MINIMOG ([2] Ch.11) or the table in the appendix. Removing the first column from these four M-matrices and forgetting the sign, we have the following 3×3 boards:

8	0	1
X	3	9
2	4	5

,

0	3	1
8	X	9
2	4	6

,

3	X	1
0	8	9
2	4	∞

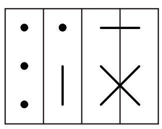
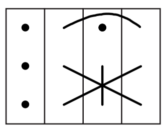
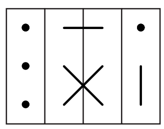
,

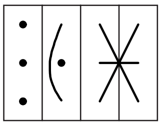
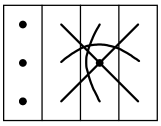
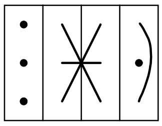
X	8	1
3	0	9
2	4	7

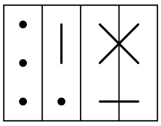
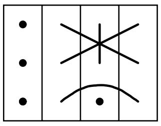
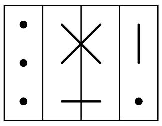
.

Chigira [1] gave these boards without using M-matrices, and gave involutions, by using these boards, in the symmetric group S_{12} which generate the Mathieu groups M_{11} and M_{12}. The involutions given by him are the following:

LEMMA 9. *For each M-matrix of C, we have nine involutions in* $\mathrm{Aut}(C)$:

 ,  ,  ,

 ,  ,  ,

 ,  ,  .

PROOF. These involutions are contained in $\mathrm{Aut}(\mathcal{T}) \ltimes \mathcal{T} \subset \mathrm{Aut}(C)$. For example, the fourth involution g in the above is defined by

$$\left(\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & & -1 \\ & & -1 & \end{pmatrix}, (0, -1, -1, -1)\right) \in \mathrm{Aut}(\mathcal{T}) \ltimes \mathcal{T}\,.$$

In fact, this corresponds to the element

$$(\sigma, a, v) := (((3\ 4), (1, -1, -1, -1)), (0, -1, -1, -1))$$

in the group

$$(S_4 \ltimes (\mathbf{F}_3^\times)^4) \ltimes \mathbf{F}_3^4 \simeq \mathrm{Mon}(4, \mathbf{F}_3) \ltimes \mathbf{F}_3^4\,.$$

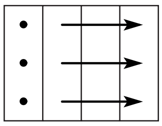
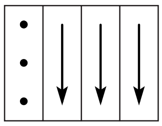
Then, by the action (3), we have the following:

$$\begin{array}{llll} (0,1) \mapsto (0,1)\,, & (0,2) \mapsto (-1,2)\,, & (0,3) \mapsto (-1,4)\,, & (0,4) \mapsto (-1,3)\,, \\ (1,1) \mapsto (1,1)\,, & (1,2) \mapsto (1,2)\,, & (1,3) \mapsto (1,4)\,, & (1,4) \mapsto (1,3)\,, \\ (-1,1) \mapsto (-1,1)\,, & (-1,2) \mapsto (0,2)\,, & (-1,3) \mapsto (0,4)\,, & (-1,4) \mapsto (0,3). \end{array}$$

Thus we get the fourth involution. □

As for order three elements, we have the following:

LEMMA 10. *For each M-matrix of C, we have two order three elements in* $\mathrm{Aut}(C)$:

 ,  .

PROOF. These order three elements are also contained in $\mathrm{Aut}(\mathcal{T}) \ltimes \mathcal{T} \subset \mathrm{Aut}(C)$. In fact, they are, respectively, defined by

$$\left(\begin{pmatrix} 1 & & & \\ & & 1 & \\ & & & 1 \\ & 1 & & \end{pmatrix}, (0,0,0,0)\right), \quad \left(\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, (0,1,1,1)\right) \in \mathrm{Aut}(\mathcal{T}) \ltimes \mathcal{T}\,.$$

□

We denote by I, J, K, L the above four M-matrices. For I, we denote by $I_3 = \{i_1, i_2\}$ the set of two order three elements in M_{12} given in the above lemma:

$$i_1 = (8\ 0\ 1)(X\ 3\ 9)(2\ 4\ 5)(6)(\infty)(7),\ \ i_2 = (8\ X\ 2)(0\ 3\ 4)(1\ 9\ 5)(6)(\infty)(7)\,.$$

Similarly we define J_3, K_3, L_3.

The following is an order three version of the theorem proved by Chigira ([1], Th.1.1).

THEOREM 11. *Under the above notation,*

(1) $\langle I_3, J_3\rangle \simeq \langle I_3, K_3\rangle \simeq \langle I_3, L_3\rangle \simeq \langle J_3, K_3\rangle \simeq \langle J_3, L_3\rangle \simeq \langle K_3, L_3\rangle \simeq L_2(9) \simeq A_6$,

(2) $\langle I_3, J_3, K_3\rangle \simeq \langle I_3, J_3, L_3\rangle \simeq \langle I_3, K_3, L_3\rangle \simeq \langle J_3, K_3, L_3\rangle \simeq M_{11}$ *and*

(3) $\langle I_3, J_3, K_3, L_3\rangle \simeq M_{12}$.

PROOF. These three are proved by tedious calculation, so we shall only give outline of proofs of (2) and (3).

By the definition, we have

$$\begin{aligned} i_1 &= (8\ 0\ 1)(X\ 3\ 9)(2\ 4\ 5)(6)(\infty 0)(7)\,, & i_2 &= (8\ X\ 2)(0\ 3\ 4)(1\ 9\ 5)(6)(\infty)(7)\\ j_1 &= (0\ 3\ 1)(8\ X\ 9)(2\ 4\ 6)(5)(\infty)(7)\,, & j_2 &= (0\ 8\ 2)(3\ X\ 4)(1\ 9\ 6)(5)(\infty)(7)\\ k_1 &= (3\ X\ 1)(0\ 8\ 9)(2\ 4\ \infty)(5)(6)(7)\,, & k_2 &= (3\ 0\ 2)(X\ 8\ 4)(1\ 9\ \infty)(5)(6)(7)\\ l_1 &= (X\ 8\ 1)(3\ 0\ 9)(2\ 4\ 7)(5)(6)(\infty)\,, & l_2 &= (X\ 3\ 2)(8\ 0\ 4)(1\ 9\ 7)(5)(6)(\infty)\,. \end{aligned}$$

Then we see that

$$\begin{aligned} x_1 &:= i_1 j_1 = (8\ 3)(1\ X)(2\ 6)(5\ 4)(0)(9)(\infty)(7)\\ b_1 = x_2 &:= i_1 j_2 = (8)(X)(\infty)(7)(0\ 9\ 4\ 5)(1\ 2\ 3\ 6)\\ x_3 &:= i_1 i_2 = (8\ 3\ 5)(0\ 9\ 2)(1\ X\ 4)(6)(\infty)(7)\\ x_4 &:= j_1 j_2 = (8\ 4\ 1)(0\ X\ 6)(3\ 9\ 2)(5)(\infty)(7)\\ x_5 &:= (x_4)^{x_3} = (3\ 1\ X)(9\ 4\ 6)(5\ 2\ 0)(8)(\infty)(7)\\ a_3 = x_6 &:= (x_1)^{x_5^2} = (8\ X)(2)(6)(3\ 1)(5\ 4)(0\ 9)(\infty)(7)\\ x_7 &:= i_2 j_2 = (8\ 4)(0\ X)(1\ 6)(3)(9\ 5)(2)(\infty)(7)\\ x_8 &:= j_1 k_2 = (8)(0)(1\ 2\ X\ \infty)(2\ 9\ 4\ 6)(5)(7)\\ x_9 &:= i_1 k_1 = (8)(0\ 3)(1\ 9)(X)(2\ \infty)(5\ 4)(6)(7)\\ y_1 &:= i_2 k_2 = (8)(0)(1\ \infty)(X\ 3)(9\ 5)(2\ 4)(6)(7)\\ a_4 = y_2 &:= (y_1)^{x_1} = (3)(0)(X\ \infty)(1\ 8)(9\ 4)(6\ 5)(2)(7)\\ y_3 &:= (x_8)^{x_9 x_5^{-1}} = (8)(X)(6\ \infty\ 1\ 5)(2\ 3\ 0\ 4)(9)(7)\\ y_4 &:= i_2 k_1 k_2 = (8\ 9\ 5\ 0)(1\ 2\ \infty\ 3)(X)(4)(6)(7)\\ y_5 &:= (y_4)^{x_6} = (X\ 0\ 4\ 9)(3\ 2\ \infty\ 1)(8)(5)(6)(7)\\ y_6 &:= (y_5)^{y_3^2} = (X\ 2\ 3\ 9)(4\ 0\ 5\ 6)(8)(\infty)(1)(7)\\ b_2 = y_7 &:= (y_6)^{x_5} = (3\ 0\ 1\ 4)(6\ 5\ 2\ 9)(8)(\infty)(X)(7)\\ y_8 &:= (x_8)^{x_2^2} = (8)(4)(3\ 6\ X\ \infty)(1\ 5\ 0\ 2)(9)(7) \end{aligned}$$

$$a_2 = y_9 := (x_7)^{y_8^{-1}} = (8\ 4)(5\ 6)(2\ 3)(\infty)(9\ 1)(0)(X)(7)$$
$$z_1 := j_2 l_2 = (8\ X)(\infty)(0)(1\ 7)(3)(9\ 6)(2\ 4)(5)$$
$$a_5 = z_2 := (z_1)^{x_8^{-1}} = (8\ 2)(X)(0)(\infty\ 7)(6)(3\ 4)(1\ 9)(5)\,.$$

Thus we have shown that $b_1, b_2, a_2, a_3, a_4, a_5$ are contained in $\langle I_3, J_3, K_3, L_3\rangle$. Then it is easily seen that $\langle I_3, J_3, K_3, L_3\rangle$ is 5-transitive; thus we have

$$|\langle I_3, J_3, K_3, L_3\rangle| \geq 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = |M_{12}|\,.$$

On the other hand, we have, by the definition, $\langle I_3, J_3, K_3, L_3\rangle \subset M_{12}$; hence we have $\langle I_3, J_3, K_3, L_3\rangle = M_{12}$.

Now we shall prove (2). By symmetry, we only show $\langle I_3, J_3, K_3\rangle = M_{11}$. By easy calculations we see that $\langle I_3, J_3, K_3\rangle$ contains

$$\{b_1, b_2, a_2, a_3, a_4\}\,.$$

Then by the same argument as above, we have $\langle I_3, J_3, K_3\rangle = M_{11}$. □

We notice that the same argument as above works for involutions, so we get another proof for Chigira's Theorem.

Moreover we have the following theorem. To prove it we need further tedious calculation, so we shall omit its proof. For details, we refer to [9].

THEOREM 12. *Under the the same notation as above, we have*

1. $\langle i_1, j_1\rangle \simeq \langle i_2, j_2\rangle \simeq \langle i_1, k_1\rangle \simeq \langle i_2, k_2\rangle \simeq A_4$,
2. $\langle i_1, j_2\rangle \simeq \langle i_2, j_1\rangle \simeq A_6 \simeq L_2(9)$,
3. $\langle i_1, k_2\rangle \simeq \langle i_2, k_1\rangle \simeq A_5$,
4. $\langle i_1, j_1, k_1\rangle \simeq \langle i_2, j_2, k_2\rangle \simeq A_5$,
5. $\langle i_1, j_1, k_1, l_1\rangle \simeq \langle i_2, j_2, k_2, l_2\rangle \simeq A_6 \simeq L_2(9)$,
6. $\langle i_s, j_s, k_t\rangle \simeq \langle i_s, j_t, k_s\rangle \simeq \langle i_t, j_s, k_s\rangle \simeq M_{11}\quad (\{1,2\} = \{t, s\})$,
7. $\langle i_s, j_s, k_s, l_t\rangle \simeq \langle i_s, j_s, k_t, l_s\rangle \simeq \langle i_s, j_t, k_s, l_s\rangle \simeq \langle i_t, j_s, k_t, l_t\rangle \simeq \langle i_s, j_t, k_s, l_t\rangle \simeq \langle i_s, j_t, k_t, l_s\rangle \simeq M_{12}\quad (\{1,2\} = \{t, s\})$.

Appendix

The table of cordwords of weight 6 of the ternary Golay code generated by (cf. Page 4 (2)) with basis Ω^* (cf. Page 10). In the bable $\bar{\alpha}$ stands for $-\alpha$, and the codewords corespond to those of the ternary Golay code in Conway-Sloane [2] Ch. 10.

Support	Element	Element	Support	Element	Element
$\infty 01239$	$\bar{\infty}0\bar{1}23\bar{9}$	$\infty\bar{0}1\bar{2}\bar{3}9$	$\infty 12459$	$\bar{\infty}12\bar{4}\bar{5}9$	$\infty\bar{1}\bar{2}45\bar{9}$
$\infty 01247$	$\infty 01247$	$\bar{\infty}\bar{0}\bar{1}\bar{2}4\bar{7}$	$\infty 12468$	$\bar{\infty}\bar{1}24\bar{6}8$	$\infty 1\bar{2}\bar{4}6\bar{8}$
$\infty 01256$	$\infty\bar{0}\bar{1}2\bar{5}6$	$\bar{\infty}01\bar{2}5\bar{6}$	$\infty 1257X$	$\infty 1\bar{2}\bar{5}\bar{7}X$	$\bar{\infty}\bar{1}257\bar{X}$
$\infty 0128X$	$\bar{\infty}\bar{0}12\bar{8}X$	$\infty 0\bar{1}\bar{2}8\bar{X}$	$\infty 1269X$	$\bar{\infty}\bar{1}\bar{2}69X$	$\infty 12\bar{6}\bar{9}\bar{X}$
$\infty 01348$	$\bar{\infty}01\bar{3}\bar{4}8$	$\infty\bar{0}\bar{1}34\bar{8}$	$\infty 12789$	$\infty\bar{1}2\bar{7}\bar{8}9$	$\bar{\infty}1\bar{2}78\bar{9}$
$\infty 01357$	$\infty 0\bar{1}\bar{3}5\bar{7}$	$\bar{\infty}\bar{0}13\bar{5}7$	$\infty 13456$	$\infty 1\bar{3}4\bar{5}\bar{6}$	$\bar{\infty}\bar{1}3\bar{4}56$
$\infty 0136X$	$\bar{\infty}\bar{0}\bar{1}\bar{3}\bar{6}\bar{X}$	$\infty 0136X$	$\infty 13479$	$\bar{\infty}\bar{1}\bar{3}479$	$\infty 13\bar{4}\bar{7}\bar{9}$
$\infty 0145X$	$\infty\bar{0}1\bar{4}5\bar{X}$	$\bar{\infty}0\bar{1}4\bar{5}X$	$\infty 1359X$	$\infty\bar{1}3\bar{5}9\bar{X}$	$\bar{\infty}1\bar{3}5\bar{9}X$
$\infty 01469$	$\bar{\infty}\bar{0}146\bar{9}$	$\infty 0\bar{1}\bar{4}\bar{6}9$	$\infty 13689$	$\infty\bar{1}\bar{3}68\bar{9}$	$\bar{\infty}13\bar{6}\bar{8}9$
$\infty 01589$	$\bar{\infty}\bar{0}\bar{1}589$	$\infty 01\bar{5}\bar{8}\bar{9}$	$\infty 1378X$	$\infty 1\bar{3}7\bar{8}\bar{X}$	$\bar{\infty}\bar{1}3\bar{7}8X$
$\infty 01678$	$\bar{\infty}0\bar{1}67\bar{8}$	$\infty\bar{0}1\bar{6}\bar{7}8$	$\infty 14578$	$\infty\bar{1}\bar{4}\bar{5}78$	$\bar{\infty}145\bar{7}\bar{8}$
$\infty 0179X$	$\bar{\infty}01\bar{7}9\bar{X}$	$\infty\bar{0}\bar{1}7\bar{9}X$	$\infty 1467X$	$\infty\bar{1}46\bar{7}\bar{X}$	$\bar{\infty}1\bar{4}\bar{6}7X$
$\infty 02345$	$\bar{\infty}\bar{0}2\bar{3}45$	$\infty 0\bar{2}3\bar{4}\bar{5}$	$\infty 1489X$	$\bar{\infty}\bar{1}\bar{4}\bar{8}\bar{9}\bar{X}$	$\infty 1489X$
$\infty 02368$	$\infty 02\bar{3}\bar{6}\bar{8}$	$\bar{\infty}\bar{0}\bar{2}368$	$\infty 15679$	$\bar{\infty}\bar{1}\bar{5}\bar{6}\bar{7}\bar{9}$	$\infty 15679$
$\infty 0237X$	$\bar{\infty}0\bar{2}\bar{3}7X$	$\infty\bar{0}23\bar{7}\bar{X}$	$\infty 1568X$	$\bar{\infty}1\bar{5}68\bar{X}$	$\infty\bar{1}5\bar{6}\bar{8}X$
$\infty 0246X$	$\infty\bar{0}\bar{2}4\bar{6}X$	$\bar{\infty}02\bar{4}6\bar{X}$	$\infty 23469$	$\infty 23469$	$\bar{\infty}\bar{2}\bar{3}\bar{4}\bar{6}\bar{9}$
$\infty 02489$	$\infty\bar{0}2\bar{4}8\bar{9}$	$\bar{\infty}0\bar{2}4\bar{8}9$	$\infty 23478$	$\bar{\infty}23\bar{4}7\bar{8}$	$\infty\bar{2}\bar{3}4\bar{7}8$
$\infty 02578$	$\bar{\infty}02\bar{5}\bar{7}8$	$\infty\bar{0}\bar{2}57\bar{8}$	$\infty 2356X$	$\bar{\infty}23\bar{5}\bar{6}X$	$\infty\bar{2}\bar{3}56\bar{X}$
$\infty 0259X$	$\bar{\infty}\bar{0}\bar{2}\bar{5}\bar{9}\bar{X}$	$\infty 0259X$	$\infty 23579$	$\infty 2\bar{3}\bar{5}7\bar{9}$	$\bar{\infty}\bar{2}35\bar{7}9$
$\infty 02679$	$\bar{\infty}\bar{0}2\bar{6}79$	$\infty 0\bar{2}6\bar{7}\bar{9}$	$\infty 2389X$	$\infty\bar{2}3\bar{8}\bar{9}X$	$\bar{\infty}2\bar{3}89\bar{X}$
$\infty 03467$	$\infty\bar{0}\bar{3}\bar{4}67$	$\bar{\infty}034\bar{6}\bar{7}$	$\infty 24567$	$\infty 2\bar{4}5\bar{6}\bar{7}$	$\bar{\infty}\bar{2}4\bar{5}67$
$\infty 0349X$	$\infty 0\bar{3}4\bar{9}\bar{X}$	$\bar{\infty}\bar{0}3\bar{4}9X$	$\infty 2458X$	$\bar{\infty}\bar{2}\bar{4}58X$	$\infty 24\bar{5}\bar{8}\bar{X}$
$\infty 03569$	$\bar{\infty}0\bar{3}\bar{5}69$	$\infty\bar{0}35\bar{6}\bar{9}$	$\infty 2479X$	$\bar{\infty}24\bar{7}\bar{9}X$	$\infty\bar{2}\bar{4}79\bar{X}$
$\infty 0358X$	$\bar{\infty}035\bar{8}\bar{X}$	$\infty\bar{0}\bar{3}\bar{5}8X$	$\infty 25689$	$\bar{\infty}256\bar{8}\bar{9}$	$\infty\bar{2}\bar{5}\bar{6}89$
$\infty 03789$	$\bar{\infty}\bar{0}\bar{3}\bar{7}\bar{8}\bar{9}$	$\infty 03789$	$\infty 2678X$	$\infty 2678X$	$\bar{\infty}\bar{2}\bar{6}\bar{7}\bar{8}\bar{X}$
$\infty 04568$	$\bar{\infty}\bar{0}\bar{4}\bar{5}\bar{6}\bar{8}$	$\infty 04568$	$\infty 3457X$	$\infty 3457X$	$\bar{\infty}\bar{3}\bar{4}\bar{5}\bar{7}\bar{X}$
$\infty 04579$	$\infty\bar{0}4\bar{5}\bar{7}9$	$\bar{\infty}0\bar{4}57\bar{9}$	$\infty 34589$	$\infty\bar{3}\bar{4}5\bar{8}9$	$\bar{\infty}34\bar{5}8\bar{9}$
$\infty 0478X$	$\bar{\infty}\bar{0}478\bar{X}$	$\infty 0\bar{4}\bar{7}\bar{8}X$	$\infty 3468X$	$\infty 3\bar{4}\bar{6}8\bar{X}$	$\bar{\infty}\bar{3}46\bar{8}X$
$\infty 0567X$	$\infty 0\bar{5}\bar{6}7\bar{X}$	$\bar{\infty}\bar{0}56\bar{7}X$	$\infty 35678$	$\infty 3\bar{5}6\bar{7}\bar{8}$	$\bar{\infty}\bar{3}5\bar{6}78$
$\infty 0689X$	$\infty\bar{0}6\bar{8}9\bar{X}$	$\bar{\infty}0\bar{6}8\bar{9}X$	$\infty 3679X$	$\bar{\infty}367\bar{9}\bar{X}$	$\infty\bar{3}\bar{6}\bar{7}9X$
$\infty 1234X$	$\bar{\infty}1\bar{2}34\bar{X}$	$\infty\bar{1}2\bar{3}\bar{4}X$	$\infty 4569X$	$\bar{\infty}45\bar{6}9\bar{X}$	$\infty\bar{4}\bar{5}6\bar{9}X$
$\infty 12358$	$\infty 12358$	$\bar{\infty}\bar{1}\bar{2}\bar{3}\bar{5}\bar{8}$	$\infty 46789$	$\infty 4\bar{6}7\bar{8}\bar{9}$	$\bar{\infty}\bar{4}6\bar{7}89$
$\infty 12367$	$\bar{\infty}12\bar{3}6\bar{7}$	$\infty\bar{1}\bar{2}3\bar{6}7$	$\infty 5789X$	$\bar{\infty}\bar{5}7\bar{8}9X$	$\infty 5\bar{7}8\bar{9}\bar{X}$
012346	$\bar{0}123\bar{4}\bar{6}$	$0\bar{1}\bar{2}\bar{3}46$	$04589X$	$\bar{0}45\bar{8}\bar{9}X$	$0\bar{4}\bar{5}89\bar{X}$
$01235X$	$\bar{0}\bar{1}\bar{2}35X$	$012\bar{3}\bar{5}\bar{X}$	$04679X$	$04679X$	$\bar{0}\bar{4}\bar{6}\bar{7}\bar{9}\bar{X}$
012378	$\bar{0}\bar{1}2\bar{3}78$	$01\bar{2}3\bar{7}\bar{8}$	056789	$\bar{0}\bar{5}678\bar{9}$	$05\bar{6}\bar{7}\bar{8}9$
012458	$0\bar{1}2\bar{4}5\bar{8}$	$\bar{0}1\bar{2}4\bar{5}8$	123457	$1\bar{2}\bar{3}\bar{4}57$	$\bar{1}234\bar{5}\bar{7}$
$01249X$	$01\bar{2}\bar{4}\bar{9}X$	$\bar{0}\bar{1}249\bar{X}$	123489	$\bar{1}\bar{2}3\bar{4}89$	$12\bar{3}4\bar{8}\bar{9}$
012579	$\bar{0}125\bar{7}\bar{9}$	$0\bar{1}\bar{2}\bar{5}79$	123569	$\bar{1}2\bar{3}5\bar{6}9$	$1\bar{2}3\bar{5}6\bar{9}$
$01267X$	$\bar{0}1\bar{2}67\bar{X}$	$0\bar{1}2\bar{6}\bar{7}X$	$12368X$	$1\bar{2}\bar{3}\bar{6}8X$	$\bar{1}236\bar{8}\bar{X}$

Support	Element	Element	Support	Element	Element
0 1 2 6 8 9	$0\,1\,2\,6\,8\,9$	$\bar{0}\,\bar{1}\,\bar{2}\,\bar{6}\,\bar{8}\,\bar{9}$	1 2 3 7 9 X	$1\,2\,3\,7\,9\,X$	$\bar{1}\,\bar{2}\,\bar{3}\,\bar{7}\,\bar{9}\,\bar{X}$
0 1 3 4 5 9	$\bar{0}\,\bar{1}\,\bar{3}\,\bar{4}\,\bar{5}\,\bar{9}$	$0\,1\,3\,4\,5\,9$	1 2 4 5 6 X	$\bar{1}\,\bar{2}\,\bar{4}\,\bar{5}\,\bar{6}\,\bar{X}$	$1\,2\,4\,5\,6\,X$
0 1 3 4 7 X	$0\,\bar{1}\,3\,\bar{4}\,7\,\bar{X}$	$\bar{0}\,1\,\bar{3}\,4\,\bar{7}\,X$	1 2 4 6 7 9	$1\,\bar{2}\,4\,\bar{6}\,\bar{7}\,9$	$\bar{1}\,2\,\bar{4}\,6\,7\,\bar{9}$
0 1 3 5 6 8	$0\,\bar{1}\,3\,\bar{5}\,\bar{6}\,8$	$\bar{0}\,1\,\bar{3}\,5\,6\,\bar{8}$	1 2 4 7 8 X	$\bar{1}\,\bar{2}\,4\,7\,\bar{8}\,X$	$1\,2\,\bar{4}\,\bar{7}\,8\,\bar{X}$
0 1 3 6 7 9	$\bar{0}\,\bar{1}\,3\,6\,\bar{7}\,9$	$0\,1\,\bar{3}\,\bar{6}\,7\,\bar{9}$	1 2 5 6 7 8	$\bar{1}\,\bar{2}\,5\,6\,\bar{7}\,8$	$1\,2\,\bar{5}\,\bar{6}\,7\,\bar{8}$
0 1 3 8 9 X	$0\,\bar{1}\,\bar{3}\,\bar{8}\,9\,X$	$\bar{0}\,1\,3\,8\,\bar{9}\,\bar{X}$	1 2 5 8 9 X	$\bar{1}\,2\,\bar{5}\,8\,\bar{9}\,X$	$1\,\bar{2}\,5\,\bar{8}\,9\,\bar{X}$
0 1 4 5 6 7	$\bar{0}\,\bar{1}\,4\,5\,\bar{6}\,7$	$0\,1\,\bar{4}\,\bar{5}\,6\,\bar{7}$	1 3 4 5 8 X	$\bar{1}\,\bar{3}\,4\,5\,8\,\bar{X}$	$1\,3\,\bar{4}\,\bar{5}\,\bar{8}\,X$
0 1 4 6 8 X	$0\,1\,4\,\bar{6}\,\bar{8}\,\bar{X}$	$\bar{0}\,\bar{1}\,\bar{4}\,6\,8\,X$	1 3 4 6 7 8	$1\,3\,4\,6\,7\,8$	$\bar{1}\,\bar{3}\,\bar{4}\,\bar{6}\,\bar{7}\,\bar{8}$
0 1 4 7 8 9	$\bar{0}\,1\,\bar{4}\,7\,\bar{8}\,9$	$0\,\bar{1}\,4\,\bar{7}\,8\,\bar{9}$	1 3 4 6 9 X	$\bar{1}\,3\,4\,\bar{6}\,\bar{9}\,X$	$1\,\bar{3}\,\bar{4}\,6\,9\,\bar{X}$
0 1 5 6 9 X	$0\,\bar{1}\,5\,6\,\bar{9}\,\bar{X}$	$\bar{0}\,1\,\bar{5}\,\bar{6}\,9\,X$	1 3 5 6 7 X	$1\,3\,5\,\bar{6}\,\bar{7}\,\bar{X}$	$\bar{1}\,\bar{3}\,\bar{5}\,6\,7\,X$
0 1 5 7 8 X	$\bar{0}\,\bar{1}\,\bar{5}\,\bar{7}\,\bar{8}\,\bar{X}$	$0\,1\,5\,7\,8\,X$	1 3 5 7 8 9	$1\,\bar{3}\,\bar{5}\,\bar{7}\,8\,9$	$\bar{1}\,3\,5\,7\,\bar{8}\,\bar{9}$
0 2 3 4 7 9	$\bar{0}\,\bar{2}\,3\,4\,7\,\bar{9}$	$0\,2\,\bar{3}\,\bar{4}\,\bar{7}\,9$	1 4 5 6 8 9	$\bar{1}\,4\,\bar{5}\,6\,\bar{8}\,9$	$1\,\bar{4}\,5\,\bar{6}\,8\,\bar{9}$
0 2 3 4 8 X	$\bar{0}\,\bar{2}\,\bar{3}\,\bar{4}\,\bar{8}\,\bar{X}$	$0\,2\,3\,4\,8\,X$	1 4 5 7 9 X	$\bar{1}\,\bar{4}\,5\,\bar{7}\,9\,X$	$1\,4\,\bar{5}\,7\,\bar{9}\,\bar{X}$
0 2 3 5 6 7	$0\,2\,3\,5\,6\,7$	$\bar{0}\,\bar{2}\,\bar{3}\,\bar{5}\,\bar{6}\,\bar{7}$	1 6 7 8 9 X	$1\,6\,\bar{7}\,\bar{8}\,\bar{9}\,X$	$\bar{1}\,\bar{6}\,7\,8\,9\,\bar{X}$
0 2 3 5 8 9	$0\,\bar{2}\,\bar{3}\,5\,8\,\bar{9}$	$\bar{0}\,2\,3\,\bar{5}\,\bar{8}\,9$	2 3 4 5 6 8	$2\,\bar{3}\,\bar{4}\,\bar{5}\,6\,8$	$\bar{2}\,3\,4\,5\,\bar{6}\,\bar{8}$
0 2 3 6 9 X	$0\,\bar{2}\,3\,\bar{6}\,9\,\bar{X}$	$\bar{0}\,2\,\bar{3}\,6\,\bar{9}\,X$	2 3 4 5 9 X	$2\,3\,\bar{4}\,5\,\bar{9}\,\bar{X}$	$\bar{2}\,\bar{3}\,4\,\bar{5}\,9\,X$
0 2 4 5 6 9	$0\,2\,4\,\bar{5}\,\bar{6}\,\bar{9}$	$\bar{0}\,\bar{2}\,\bar{4}\,5\,6\,9$	2 3 4 6 7 X	$2\,\bar{3}\,4\,\bar{6}\,7\,\bar{X}$	$\bar{2}\,3\,\bar{4}\,6\,\bar{7}\,X$
0 2 4 5 7 X	$\bar{0}\,2\,\bar{4}\,\bar{5}\,7\,X$	$0\,\bar{2}\,4\,5\,\bar{7}\,\bar{X}$	2 3 5 7 8 X	$\bar{2}\,3\,\bar{5}\,7\,8\,\bar{X}$	$2\,\bar{3}\,5\,\bar{7}\,\bar{8}\,X$
0 2 4 6 7 8	$\bar{0}\,2\,4\,6\,\bar{7}\,\bar{8}$	$0\,\bar{2}\,\bar{4}\,\bar{6}\,7\,8$	2 3 6 7 8 9	$2\,3\,\bar{6}\,\bar{7}\,8\,\bar{9}$	$\bar{2}\,\bar{3}\,6\,7\,8\,9$
0 2 5 6 8 X	$0\,\bar{2}\,\bar{5}\,6\,\bar{8}\,X$	$\bar{0}\,2\,5\,\bar{6}\,8\,\bar{X}$	2 4 5 7 8 9	$\bar{2}\,\bar{4}\,\bar{5}\,\bar{7}\,\bar{8}\,\bar{9}$	$2\,4\,5\,7\,8\,9$
0 2 7 8 9 X	$\bar{0}\,\bar{2}\,\bar{7}\,8\,9\,X$	$0\,2\,7\,\bar{8}\,\bar{9}\,\bar{X}$	2 4 6 8 9 X	$2\,\bar{4}\,\bar{6}\,\bar{8}\,9\,X$	$\bar{2}\,4\,6\,8\,\bar{9}\,\bar{X}$
0 3 4 5 6 X	$0\,\bar{3}\,\bar{4}\,5\,\bar{6}\,X$	$\bar{0}\,3\,4\,\bar{5}\,6\,\bar{X}$	2 5 6 7 9 X	$\bar{2}\,5\,\bar{6}\,7\,\bar{9}\,X$	$2\,\bar{5}\,6\,\bar{7}\,9\,\bar{X}$
0 3 4 5 7 8	$\bar{0}\,3\,\bar{4}\,5\,\bar{7}\,8$	$0\,\bar{3}\,4\,\bar{5}\,7\,\bar{8}$	3 4 5 6 7 9	$3\,\bar{4}\,\bar{5}\,\bar{6}\,7\,9$	$\bar{3}\,4\,5\,6\,\bar{7}\,\bar{9}$
0 3 4 6 8 9	$0\,3\,\bar{4}\,6\,\bar{8}\,\bar{9}$	$\bar{0}\,\bar{3}\,4\,\bar{6}\,8\,9$	3 4 7 8 9 X	$\bar{3}\,\bar{4}\,7\,8\,\bar{9}\,X$	$3\,4\,\bar{7}\,\bar{8}\,9\,\bar{X}$
0 3 5 7 9 X	$\bar{0}\,\bar{3}\,5\,7\,9\,\bar{X}$	$0\,3\,\bar{5}\,\bar{7}\,\bar{9}\,X$	3 5 6 8 9 X	$3\,5\,6\,8\,9\,X$	$\bar{3}\,\bar{5}\,\bar{6}\,\bar{8}\,\bar{9}\,\bar{X}$
0 3 6 7 8 X	$0\,\bar{3}\,6\,\bar{7}\,8\,\bar{X}$	$\bar{0}\,3\,\bar{6}\,7\,\bar{8}\,X$	4 5 6 7 8 X	$4\,\bar{5}\,\bar{6}\,\bar{7}\,8\,X$	$\bar{4}\,5\,6\,7\,\bar{8}\,\bar{X}$

References

[1] N. CHIGIRA, On Involutions Which Generate Mathieu Groups M_{11} and M_{12}, Comm. Alg. **32** (2004), 3179–3188.

[2] J. H. CONWAY and N. J. A. SLOANE, *Sphere Packings, Lattices and Groups*, Springer-Verlag, 1988.

[3] R. T. CURTIS, A new combinatorial approach to M_{24}, Math. Proc. Camb. Phil. Soc. **79**, 1976.

[4] R. T. CURTIS, The Steiner system $S(5, 6, 12)$, the Mathieu group M_{12} and the "kitten", *Computational Group Theory*, Ac. Press, NY, 1984, 352–358.

[5] R. T. CURTIS, Geometric interpretations of the 'natural' generators of the Mathieu groups, Math. Proc. Cambridge Phil. Soc. (1990) 107, 19–26.

[6] R. L. GRIESS, *Twelve Sporadic Groups*, Springer-Verlag, 1998.

[7] M. KIMIZUKA and R. SASAKI, Involutions of the Mathieu group M_{24}, to appear in Hokkaido Math. J.

[8] M. KIMIZUKA and R. SASAKI, Codes of split type, preprint.

[9] M. KIMIZUKA, R. SASAKI and T. SATOH, Order three elements which generate the Mathieu groups (in Japanese), J. Res. Inst. Sci. Technol. Nihon Univ. No. 108 (2006), 9–13.

[10] T. KONDO, Mathieu Groups and Conway Groups (in Japanese), Lecture note at University of Tokyo, 1996.

[11] R. SASAKI, Codes of type MOG, submitted.

[12] M. SAWABE, A combinatorial approach to the conjugacy classes of the Mathieu simple groups, M_{24}, M_{23}, M_{22}, J. Math. Soc. Japan **51** (1999), 661–678.

Present Addresses:
MARO KIMIZUKA
NIHON FUKUSHI UNIVERSITY AFFILIATED HIGH SCHOOL
e-mail: maro_kimizuka@yahoo.co.jp

RYUJI SASAKI
DEPARTMENT OF MATHEMATICS, COLLEGE OF SCIENCE AND TECHNOLOGY,
NIHON UNIVERSITY,
KANDA SURUGADAI, CHIYODA-KU, TOKYO, 101–8308, JAPAN.
e-mail: ryu@math.cst.nihon-u.ac.jp