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The Addition Law Attached to a Stratification of a Hyperelliptic Jacobian Variety

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Abstract. This article shows explicit relations between fractional expressions of Schottky-Klein type for hyperelliptic σ-functions and a product of differences of the algebraic coordinates on each stratum of a natural stratification in a hyperelliptic Jacobian.

1. Introduction

In this paper we shall consider the addition law on the Jacobian for any hyperelliptic curve. To motivate our investigation, we start from the genus one case, and recall that Weierstrass showed that principal relations in the theory of elliptic functions can be derived from the well-known addition formula

$$\frac{\sigma(v+u)\sigma(v-u)}{\sigma(v)^2\sigma(u)^2} = \wp(v) - \wp(u)\,, \tag{1.1}$$

where $\wp(u) = -(d^2/du^2)\log\sigma(u)$.

We consider an elliptic curve C_1 defined by $y^2 = x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$ with unique point ∞ at infinity. Let $(x(u), y(u))$ be the inverse function of

$$(x, y) \mapsto u = \int_\infty^{(x,y)} \frac{dx}{2y} \quad \text{modulo the periods}\,.$$

Then $x(u)$ is equal to the function $\wp(u)$ attached to C_1 up to an additive constant. Hence we have

$$\frac{\sigma(v+u)\sigma(v-u)}{\sigma(v)^2\sigma(u)^2} = x(v) - x(u)\,. \tag{1.2}$$

We recall some generalizations of (1.1) or (1.2). For the sake of simplicity we restrict ourselves to the genus two curve C_2 defined by $y^2 = x^5 + \lambda_4 x^4 + \lambda_3 x^3 + \lambda_2 x^2 + \lambda_1 x + \lambda_0$, but in many cases such generalizations can be proved for any hyperelliptic curve. Before

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describing our generalizations, we note that there is a nice generalization of the Weierstrass elliptic σ-function (see 3.1 below for the definition), which is a theta function on $\mathbf{C}^2$ (now we are assuming the genus is two). Indeed if we define $\wp_{ij}(u) = -(\partial^2/\partial u_i \partial u_j)\log\sigma(u)$, then we have a classical formula[1]

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} = \wp_{11}(v) - \wp_{11}(u) - \wp_{12}(u)\wp_{22}(v) + \wp_{12}(v)\wp_{22}(u)\,, \tag{1.3}$$

which is a natural generalization of (1.1) based on the idea of Jacobi. This formula is expressed in terms of the algebraic coordinates as

$$\begin{aligned}\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma(v)^2} &= \frac{f(x_1,x_2)-2y_1y_2}{(x_1-x_2)^2} - \frac{f(z_1,z_2)-2w_1w_2}{(w_1-w_2)^2}\\ &\quad + x_1x_2(z_1+z_2) - z_1z_2(x_1+x_2)\,,\end{aligned} \tag{1.4}$$

where $u = \big(\int_\infty^{(x_1,y_1)} + \int_\infty^{(x_2,y_2)}\big)\big(\frac{dx}{2y}, \frac{xdx}{2y}\big)$, $v = \big(\int_\infty^{(z_1,w_1)} + \int_\infty^{(z_2,w_2)}\big)\big(\frac{dx}{2y}, \frac{xdx}{2y}\big)$, and $f(x,z)$ is a rather complicated polynomial of x and z such that $f(x,z) = f(z,x)$ (see [B1], p. 211 for this).

On the other hand, the following is another generalization of (1.2) given by the third author. Let u and v vary on the canonical universal Abelian covering of the curve C_2 presented in $\mathbf{C}^2$ (this is $\kappa^{-1}(\Theta^{[1]})$ in the notation of Section 2 below), and $x(u)$ be a function such that $u = \int_\infty^{(x(u),y(u))}\big(\frac{dx}{2y}, \frac{xdx}{2y}\big)$. Then we have

$$-\frac{\sigma(v+u)\sigma(v-u)}{\sigma_2(v)^2\sigma_2(u)^2} = x(v) - x(u)\,, \tag{1.5}$$

where $\sigma_2(u) = \partial\sigma(u)/\partial u_2$.

It is natural to seek a unified understanding of all the formulae (1.3), (1.4), and (1.5) above. There are several hints. The first hint is the fact that the right hand side of (1.3) has a determinantal expression [EEP], which is given by using the genus two case of results in [Ô]. The second hint is the following formula:

$$\begin{aligned}&-\frac{\sigma(u^{(1)}+u^{(2)}+v)\sigma(u^{(1)}+u^{(2)}-v)}{\sigma(u^{(1)}+u^{(2)})^2\sigma_2(v)^2}\\ &\qquad = x(v)^2 - \wp_{22}(u^{(1)}+u^{(2)})x(v) - \wp_{12}(u^{(1)}+u^{(2)})\end{aligned} \tag{1.6}$$

for $u^{(1)}$, $u^{(2)}$, and v varying on the canonical universal Abelian covering of C_2 in $\mathbf{C}^2$. This appeared in [G] for the first time, and a generalization of this for any hyperelliptic curve was reported in [BES], (3.21), without proof.

The main result (Theorem 4.2 below) of this paper is a unification of (1.5) and (1.6). There should exist a unification of all the formulae above, and we hope to give such a formulation in the near future.

[1] A generalization of this formula to the case $g = 3$ is given by the Baker pffafian built from Kleinian $\wp$-functions.

NOTATION. The symbol $(d^\circ(z) \geq n)$ denotes terms of total degree at least n with respect to a variable z.

2. Stratification of the Jacobian of a Hyperelliptic Curve

Throughout this article we deal with a hyperelliptic (or elliptic) curve C_g of genus $g > 0$ given by the affine equation

$$y^2 = f(x),$$

where we are assuming that $f(x)$ is of the form

$$f(x) = x^{2g+1} + \lambda_{2g}x^{2g} + \cdots + \lambda_2 x^2 + \lambda_1 x + \lambda_0$$

with the λ_j being complex numbers. Then a canonical basis of the space of the differentials of the first kind on C_g is given by

$$\omega_1 := \frac{dx}{2y}, \quad \omega_2 := \frac{xdx}{2y}, \ldots, \omega_g := \frac{x^{g-1}dx}{2y}$$

using the algebraic coordinate (x, y) of C_g. Let α_j and β_j $(j = 1, \ldots, g)$ be a standard homology basis on C_g. Namely, they give

$$\mathrm{H}_1(C_g, \mathbf{Z}) = \bigoplus_{j=1}^{g} \mathbf{Z}\alpha_j \oplus \bigoplus_{j=1}^{g} \mathbf{Z}\beta_j,$$

and their intersection products are given by $[\alpha_i, \alpha_j] = 0, [\beta_i, \beta_j] = 0$, $[\alpha_i, \beta_j] = -[\beta_i, \alpha_j]\delta_{i,j}$. We denote matrices of the half-periods with respect to the differentials ω_i and the homology basis α_j, β_j by

$$\omega' := \frac{1}{2}\left[\int_{\alpha_j} \omega_i\right], \quad \omega'' := \frac{1}{2}\left[\int_{\beta_j} \omega_i\right].$$

We introduce differentials of the second kind

$$dr_j := \frac{1}{2y}\sum_{k=j}^{2g-j}(k+1-j)\lambda_{k+1+j}x^k dx, \quad (j = 1, \ldots, g)$$

and the matrices of half-periods

$$\eta' := \frac{1}{2}\left[\int_{\alpha_j} dr_i\right], \quad \eta'' := \frac{1}{2}\left[\int_{\beta_j} dr_i\right]$$

of this differentials with respect to α_j and β_j. These $2g$ meromorphic differentials u_i and r_i $(i = 1, \ldots, g)$ are chosen in such the way that the half-periods matrices ω', ω'', η', η'' satisfy the generalized Legendre relation

$$\mathfrak{M}\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix} {}^t\mathfrak{M} = \frac{\sqrt{-1}\pi}{2}\begin{bmatrix} 0 & -1_g \\ 1_g & 0 \end{bmatrix},$$

where $\mathfrak{M} = \begin{bmatrix} \omega' & \omega'' \\ \eta' & \eta'' \end{bmatrix}$. Let $\Lambda = 2\big(\mathbf{Z}^g\omega' \oplus \mathbf{Z}^g\omega''\big)$. Then Λ is a lattice in $\mathbf{C}^g$ and the Jacobi variety $\mathcal{J}(C_g)$ of C_g is given by

$$\mathcal{J}(C_g) := \mathbf{C}^g/\Lambda\,.$$

We use the modulus $\mathbf{T} := \omega'^{-1}\omega''$ to define the σ-function of C_g later.

For $k = 0, \dots, g$, the *Abel map* ϕ_k from the k-th symmetric product $\mathrm{Sym}^k(C_g)$ of the curve C_g to $\mathcal{J}(C_g)$ is the map

$$\phi_k : \mathrm{Sym}^k(C_g) \to \mathcal{J}(C_g) \ \text{ given by } \ (Q_1, \dots, Q_k) \mapsto \sum_{i=1}^{k} \int_{\infty}^{Q_i} (\omega_1, \dots, \omega_g) \mod \Lambda\,.$$

We denote the natural quotient map by κ:

$$\kappa : \mathbf{C}^g \to \mathcal{J}(C_g) = \mathbf{C}^g/\Lambda\,.$$

We denote by $\Theta^{[k]}$ the image $\phi_k(\mathrm{Sym}^k(C_g))$ of the Abel map ϕ_k above. Now we have the following stratification:

$$\{O\} = \Theta^{[0]} \subset \Theta^{[1]} \subset \Theta^{[2]} \subset \cdots \subset \Theta^{[g-1]} \subset \Theta^{[g]} = \mathcal{J}(C_g)\,,$$

where O is the origin of $\mathcal{J}(C_g)$. It is known that each $\Theta^{[k]}$ is a subvariety of $\mathcal{J}(C_g)$. We shall refer to the subvariety $\Theta^{[k]}$ as the k-th stratum of $\mathcal{J}(C_g)$.

The following Lemma follows from a straightforward calculation of the Abelian integral

$$u_i = \int_{\infty}^{(x,y)} \frac{x^{i-1}dx}{2y}$$

by using a power series expansion and integrating term by term.

LEMMA 2.1. *Let $u = (u_1, \dots, u_g) \in \kappa^{-1}(\Theta^{[1]})$. We denote by $\big(x(u), y(u)\big) \in C_g$ the algebraic coordinate whose image under the Abel map ϕ_k is u modulo Λ. Then we have the following properties.*

(1) *The variable u_g is a local parameter around* $(0, \dots, 0)$ *on $\kappa^{-1}(\Theta^{[1]})$, and $u_1, \dots, u_g$ are functions of u_g on $\kappa^{-1}(\Theta^{[1]})$ near* $(0, \dots, 0)$.

(2) *The functions $x(u), y(u)$ have the following Laurent expansions around* $(0, 0, \dots, 0)$ *on $\kappa^{-1}(\Theta^{[1]})$:*

$$x(u) = {u_g}^{-2} + (d^\circ(u_g) \geq 0)\,,$$

$$y(u) = {u_g}^{-2g-1} + (d^\circ(u_g) \geq -2g+1)\,.$$

For a proof of the above, we refer the reader to Lemmas 3.8 and 3.9 in [Ô], for instance. □

3. The Sigma Function and Its Derivatives

In this section, we will introduce the hyperelliptic θ-function and σ-function. The later is a natural generalization of the Weierstrass σ-function.

Let a and b be two vectors in $\mathbf{R}^g$. We recall the theta function with respect to the lattice of periods generated by 1_g and $\mathbf{T} = \omega'^{-1}\omega''$ with characteristic ${}^t[a\ b]$, which is a function of $z \in \mathbf{C}^g$ defined by

$$\theta\begin{bmatrix} a \\ b \end{bmatrix}(z) = \theta\begin{bmatrix} a \\ b \end{bmatrix}(z;\mathbf{T}) = \sum_{n\in\mathbf{Z}^g} \exp\left[2\pi\sqrt{-1}\left\{\frac{1}{2}\,{}^t(n+a)\mathbf{T}(n+a) + {}^t(n+a)(z+b)\right\}\right],$$

as usual. Let $\delta' = {}^t\left[\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2}\right]$ and $\delta'' = {}^t\left[\frac{g}{2}, \frac{g-1}{2}, \ldots, \frac{1}{2}\right]$. Then the half-period vector $\delta'\omega' + \delta''\omega''$ is the so-called Riemann constant for $\mathcal{J}(C_g)$.

DEFINITION 3.1. The σ-function (see, for example, [B1], p. 336) is given by

$$\sigma(u) = \gamma_0 \exp\left(-\frac{1}{2}\,{}^t u\eta'\omega'^{-1}u\right)\vartheta\begin{bmatrix} \delta'' \\ \delta' \end{bmatrix}\left(\frac{1}{2}\omega'^{-1}u;\mathbf{T}\right),$$

where γ_0 is a certain non-zero constant depending of C_g, which is explained in [BEL] p. 32 and [Ô], Lemma 4.2. We regard the domain $\mathbf{C}^g$ where this function is defined as $\kappa^{-1}(\mathcal{J}(C_g))$.

Following the paper [Ô], we introduce multi-indices $\natural^n$ and their associated derivatives $\sigma_{\natural^n}(u)$ of $\sigma(u)$ as follows:

DEFINITION 3.2. We define

$$\natural^n = \begin{cases} \{n+1, n+3, \ldots, g-1\} & \text{if } g-n \equiv 0 \mod 2, \\ \{n+1, n+3, \ldots, g\} & \text{if } g-n \equiv 1 \mod 2. \end{cases}$$

By using this notation we have partial derivatives of $\sigma(u)$ associated with these multi-indices, namely,

$$\sigma_{\natural^n}(u) = \left(\prod_{i\in\natural^n}\frac{\partial}{\partial u_i}\right)\sigma(u).$$

Moreover we write $\sharp := \natural^1$ and $\flat := \natural^2$, so that $\sigma_\sharp(u) = \sigma_{\natural^1}(u)$ and $\sigma_\flat(u) = \sigma_{\natural^2}(u)$.

Several examples of $\sigma_{\natural^n}(u)$ are given in the following table[2].

genus	$\sigma_\sharp \equiv \sigma_{\natural^1}$	$\sigma_\flat \equiv \sigma_{\natural^2}$	$\sigma_{\natural^3}$	$\sigma_{\natural^4}$	$\sigma_{\natural^5}$	$\sigma_{\natural^6}$	$\sigma_{\natural^7}$	$\sigma_{\natural^8}$	$\cdots$
1	σ	σ	σ	σ	σ	σ	σ	σ	$\cdots$
2	σ_2	σ	σ	σ	σ	σ	σ	σ	$\cdots$
3	σ_2	σ_3	σ	σ	σ	σ	σ	σ	$\cdots$
4	σ_{24}	σ_3	σ_4	σ	σ	σ	σ	σ	$\cdots$
5	σ_{24}	σ_{35}	σ_4	σ_5	σ	σ	σ	σ	$\cdots$
6	σ_{246}	σ_{35}	σ_{46}	σ_5	σ_6	σ	σ	σ	$\cdots$
7	σ_{246}	σ_{357}	σ_{46}	σ_{57}	σ_6	σ_7	σ	σ	$\cdots$
8	σ_{2468}	σ_{357}	σ_{468}	σ_{57}	σ_{68}	σ_7	σ_8	σ	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

For $u \in \kappa^{-1}\mathcal{J}(C_g)$, we denote by u' and u'' the unique elements in $\mathbf{R}^g$ such that $u = 2(u'\omega' + u''\omega'')$. We introduce a $\mathbf{C}$-valued $\mathbf{R}$-bilinear form $L(\ ,\ )$ defined by

$$L(u, v) = {}^t u(\eta' u' + \eta'' u'')$$

for $u, v \in \kappa^{-1}\mathcal{J}(C_g)(= \mathbf{C}^g)$. Let

$$\chi(\ell) = \exp[2\pi\sqrt{-1}({}^t\ell'\delta'' - {}^t\ell''\delta') - \pi\sqrt{-1}\,{}^t\ell'\ell''] \,.$$

The following facts are essential for our main result.

PROPOSITION 3.3. (1) *For $u \in \kappa^{-1}(\Theta^{[n]})$ and $\ell \in \Lambda$, we have*

$$\sigma_{\natural^n}(u + \ell) = \chi(\ell)\sigma_{\natural^n}(u) \exp L(u + \tfrac{1}{2}\ell, \ell) \,.$$

(2) *Let n be a positive integer $n \le g$. Let v, $u^{(1)}$, $u^{(2)}, \dots, u^{(n)}$ be elements in $\kappa^{-1}(\Theta^{[1]})$. If $u^{(1)}+\cdots+u^{(n)} \notin \kappa^{-1}(\Theta^{[n-1]})$, then the function $v \mapsto \sigma_{\natural^{n+1}}(u^{(1)}+\cdots+u^{(n)}+v)$ has zeros only at $v = (0, \dots, 0)$ modulo Λ of order $g - n$ and at $-u^{(1)}$ modulo Λ of order 1. Around $(0, 0, \dots, 0)$ following expansion with respect to v_g:*

$$\sigma_{\natural^{n+1}}(u^{(1)} + \cdots + u^{(n)} + v) = (-1)^{(g-n)(g-n-1)/2}\sigma_{\natural^n}(u){v_g}^{g-n} + (d^\circ(v_g) \ge g - n + 1) \,.$$

(3) *For $v \in \kappa^{-1}(\Theta^{[1]})$,*

$$\sigma_{\natural^1}(v) = -(-1)^{g(g-1)/2}{v_g}^g + (d^\circ(v_g) \ge g + 2) \,.$$

PROOF. The assertion (1) is proved by Lemma 7.3 in [Ô]. The assertions (2) and (3) are proved by Proposition 7.5 in [Ô]. □

[2] One can see that numbers appearing in those multi-index $\natural^n$ are naturally related to the Weierstrass gap sequence 1, 3, 5, ..., $2g - 1$ at the Weierstrass point ∞ at infinity.

REMARK 3.4. (1) If $n = g$, the assertion 3.3(1) is the classical relation for $\sigma(u)$ with respect to a translation by any period $\ell \in \Lambda$ (see [B1], p. 286).

Namely, we have the same relations for translations by any period $\ell \in \Lambda$ for the special partial-derivatives $\sigma_{\natural^n}(u)$ on $\kappa^{-1}(\Theta^{[n]})$ as that for σ itself. The essence of the proof in [Ô] of this fact is that the derivative $\sigma_{\check{\natural}^n}(u)$ for any proper subset $\check{\natural}^n$ of $\natural^n$ vanishes on the stratum $\Theta^{[n]}$, which is proved by investigating the Riemann singularity theorem explicitly.

(2) We see by considering the unification of 3.3(2) and 3.3(3) above that it would be natural to define $\sigma_{\natural^0}(u) = -1$ (a constant function).

(3) The statements 3.3(2) and (3) complement the Riemann singularity theorem ([R]) (see also [ACGH], p. 226–227) by exhibiting the orders of vanishing on each stratum $\Theta^{[n]}$ in terms of the σ function.

4. Main Result

We start by recalling the following formula (Lemma 8.1 in [Ô]) without proof.

LEMMA 4.1. *Suppose u and v are variables on $\kappa^{-1}(\Theta^{[1]})$. Then we have*

$$(-1)^g \frac{\sigma_\flat(u+v)\sigma_\flat(u-v)}{\sigma_\sharp(u)^2\sigma_\sharp(v)^2} = x(u) - x(v)\,.$$

The following relation is our main theorem and is an extension of both (3.21) in [BES] and the formula above.

LEMMA 4.2. *Let m and n be positive integers such that $m+n \le g+1$. Let*

$$u = \sum_{i=1}^{m} \int_{\infty}^{(x_i, y_i)} (\omega_1, \ldots, \omega_g) \ \in \kappa^{-1}(\Theta^{[m]})\,, \quad v = \sum_{j=1}^{n} \int_{\infty}^{(x'_j, y'_j)} (\omega_1, \ldots, \omega_g) \ \in \kappa^{-1}(\Theta^{[n]})$$

Then the following relation holds:

$$\frac{\sigma_{\natural^{m+n}}(u+v)\sigma_{\natural^{m+n}}(u-v)}{\sigma_{\natural^m}(u)^2\sigma_{\natural^n}(v)^2} = (-1)^{\delta(g,n)} \prod_{i=1}^{m}\prod_{j=1}^{n}(x_i - x'_j)\,,$$

where $\delta(g,n) = \frac{1}{2}n(n-1) + gn$.

PROOF. We prove the desired formula by induction with respect to m and n. First we suppose that the $2g$ points $u^{(1)}, \ldots, u^{(g)}$ and $v^{(1)}, \ldots, v^{(g)}$ are given. Then by 3.3 we see that both sides of the desired formula are functions on $\Theta^{[1]}$ with respect to each variable in the $u^{(i)}$ and $v^{(j)}$. We let $u^{[i]} = u^{(1)} + \cdots + u^{(i)}$ and $v^{[j]} = v^{(1)} + \cdots + v^{(j)}$ for $0 \le i \le g$ and $0 \le j \le g$. If $m = n = 1$, the assertion is just Lemma 4.1. Therefore, the assertion is proved by reducing

$$\begin{aligned} &\frac{\sigma_{\natural^{m+n+1}}(u^{[m]}+v^{[n+1]})\sigma_{\natural^{m+n+1}}(u^{[m]}-v^{[n+1]})}{\sigma_{\natural^{m}}(u^{[m]})^2\sigma_{\natural^{n+1}}(u^{[n+1]})^2} \\ &= (-1)^{\delta(g,n+1)}\prod_{i=1}^{m}\prod_{j=1}^{n+1}(x_i-x'_j) \end{aligned} \tag{4.3a}$$

and

$$\begin{aligned} &\frac{\sigma_{\natural^{m+n+1}}(u^{[m+1]}+v^{[n]})\sigma_{\natural^{m+n+1}}(u^{[m+1]}-v^{[n]})}{\sigma_{\natural^{m+1}}(u^{[m+1]})^2\sigma_{\natural^{n}}(u^{[n]})^2} \\ &= (-1)^{\delta(g,n)}\prod_{i=1}^{m+1}\prod_{j=1}^{n}(x_i-x'_j)\,, \end{aligned} \tag{4.3b}$$

to the formula

$$\frac{\sigma_{\natural^{m+n}}(u^{[m]}+v^{[n]})\sigma_{\natural^{m+n}}(u^{[m]}-v^{[n]})}{\sigma_{\natural^{m}}(u^{[m]})^2\sigma_{\natural^{n}}(v^{[n]})^2} = (-1)^{\delta(g,n)}\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-x'_j)\,. \tag{4.4}$$

We denote the index of the sign in 3.3(2) by $\varepsilon(g,n)$, that is $\varepsilon(g,n)=(g-n)(g-n-1)/2$. Then the left hand side of (4.3a) is

$$\begin{aligned} &\frac{\sigma_{\natural^{m+n+1}}(u^{[m]}+v^{[n+1]})\sigma_{\natural^{m+n+1}}(u^{[m]}-v^{[n+1]})}{\sigma_{\natural^{m}}(u^{[m]})^2\sigma_{\natural^{n+1}}(v^{[n+1]})^2} \\ &= \Big[\sigma_{\natural^{m+n}}(u^{[m]}+v^{[n]})\big\{(-1)^{\varepsilon(g,m+n)}\big(v_g^{(n+1)}\big)^{g-m-n}+\cdots\big\} \\ &\qquad \times\sigma_{\natural^{m+n}}(u^{[m]}-v^{[n]})\big\{(-1)^{\varepsilon(g,m+n)}\big(-v_g^{(n+1)}\big)^{g-m-n}+\cdots\big\}\Big] \\ &\qquad\qquad \Big/\ \sigma_{\natural^{m}}(u^{[m]})^2\sigma_{\natural^{n}}(v^{[n]})^2\big\{(-1)^{\varepsilon(g,n)}\big(v_g^{(n+1)}\big)^{g-n}+\cdots\big\}^2 \\ &= \frac{\sigma_{\natural^{m+n}}(u^{[m]}+v^{[n]})\sigma_{\natural^{m+n}}(u^{[m]}-v^{[n]})}{\sigma_{\natural^{m}}(u^{[m]})^2\sigma_{\natural^{n}}(u^{[n]})^2}\left\{(-1)^{g-m-n}\frac{1}{\big(v_g^{(n+1)}\big)^{2m}}+\cdots\right\} \end{aligned} \tag{4.3a$'$}$$

by Lemma 2.1. The right hand side of (4.3a) is

$$\begin{aligned} &(-1)^{\delta(g,n+1)}\prod_{i=1}^{m}\prod_{j=1}^{n+1}(x_i-x'_j) \\ &= (-1)^{\delta(g,n+1)}(-x'_{n+1})^m\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-x'_j)+\big(d^\circ(x_{n+1})\le m-1\big) \\ &= \frac{(-1)^{\delta(g,n+1)+m}}{\big(v^{(n+1)}\big)^{2m}}\prod_{i=1}^{m}\prod_{j=1}^{n}(x_i-x'_j)+\big(d^\circ(v^{(n+1)})\ge -2m+1\big) \end{aligned} \tag{4.3a$''$}$$

The index of the sign of the last expression in (4.3a$''$) is

$$\begin{aligned}\delta(g,n+1)+m &= \tfrac{1}{2}(n+1)n+g(n+1)-1+m\\ &= \tfrac{1}{2}n(n-1)+n+gn+g-1+m\\ &\equiv \tfrac{1}{2}n(n-1)+gn-1+(g-n-m) \quad \bmod 2\\ &= \delta(g,n)+(g-n-m)\,.\end{aligned}$$

This is equal to the sum of the indices of the signs in (4.4) and one of the last factors in (4.3a$'$). Thus, the leading terms of the expansions with respect to $v_g^{(n+1)}$ of the two sides completely coincide. Until this point, the assumption $m+n \le g+1$ was not essential. Now we check the divisors of the two sides regarded as functions of $v^{(n+1)}$ modulo Λ. Using the assumption $m+n \le g+1$, we can determine the divisors of the two sides exactly by Proposition 3.4. For the left hand side of (4.3a), the numerator has zeros at $v^{(n+1)} = (0,0,\dots,0)$ of order $2(g-m-n)$ modulo Λ, at $\pm u^{(1)}, \dots, \pm u^{(m)}, -v^{(1)}, \dots, -v^{(n)}$ of order 1 modulo Λ. The denominator has zeros at $v^{(n+1)} = (0,0,\dots,0)$ of order $2(g-n)$ modulo Λ, at $-v^{(1)}$, $\dots, -v^{(n)}$ of order 1 modulo Λ. Therefore the left hand side of (4.3a) has only poles at $v^{(n+1)} = (0,0,\dots,0)$ of order $2m$ modulo Λ, and has zeros at $\pm u^{(1)}, \dots, \pm u^{(m)}$ of order 1 modulo Λ. These pole and zeros coincide with those of the left hand side and have the same order, because, for u and $v \in \kappa^{-1}(\Theta^{[1]})$, we have $x(u)-x(v)$ if and only if $u = \pm v$. Thus, we have reduced the formula (4.3a) to the equality (4.4). The formula (4.3b) is similarly reduced to (4.4). Hence, we have proved the assertion. □

Baker ([B1] and [B2]) defined

$$\wp_{ij}(u) = -\frac{\partial^2}{\partial u_i \partial u_j}\log\sigma(u)$$

for $0 \le i \le g$, $0 \le j \le g$, and $u = (u_1,\dots,u_g) \in \mathbf{C}^g$, which is the natural generalizations of the Weierstrass $\wp$ function. As we mentioned earlier, the following special case of $(m,n) = (g,1)$ in 4.2 appeared in [BES], (3.21), which was the motivation of this paper.

COROLLARY 4.5 ([BES], (3.21)). *We suppose that* $(x,y), (x_1,y_1), \dots, (x_g,y_g)$ *are* $g+1$ *points on* C_g. *Let*

$$\begin{aligned}F_g(x) &= (x-x_1)(x-x_2)\cdots(x-x_g)\\ u &= \sum_{i=1}^{g}\int_\infty^{(x_i,y_i)}(\omega_1,\dots,\omega_g) \in \kappa^{-1}(\Theta^{[g]})(=\boldsymbol{C}^g)\,,\\ v &= \int_\infty^{(x,y)}(\omega_1,\dots,\omega_g) \in \kappa^{-1}(\Theta^{[1]})\,.\end{aligned}$$

Then we have the relation

$$\frac{\sigma(u+v)\sigma(u-v)}{\sigma(u)^2\sigma_\sharp(v)^2} = (-1)^{\frac{1}{2}g(g+1)}F_g(x)$$
$$= x^g - \wp_{gg}(u)x^{g-1} - \wp_{g,g-1}(u)x^{g-2} - \cdots - \wp_{g1}(u)\,.$$

PROOF. The first equality is obvious from 4.2. The second equality follow from the fact that

$$\wp_{g,g-k+1}(u) = (-1)^{k+1} \sum_{1\le i_1<i_2<\cdots<i_k\le g} x_{i_1}x_{i_2}\cdots x_{i_k}$$

for $1 \le k \le g$, which was given by Baker (see [B3], for example). □

5. Some Remarks

We remark finally on related work and on future possibilities for our result. The polynomial F_g plays the role of the master polynomial in the theory of hyperelliptic functions. Bolza showed how to express the polynomial F_g in terms of Kleinian $\wp$-functions; in this context we shall call the master polynomial F_g the Bolza polynomial. Its zeros give the solution to the Jacobi inversion problem. In the 2×2 Lax representation of a dynamical system associated with a hyperelliptic curve, it plays the role of the U polynomial among Jacobi's U, V, W-triple ([Mu]). Vanhaecke studied the properties of $\Theta^{[k]}$ using U, V, W-polynomials which are constructed on the basis of the master polynomial F_g ([V2]). In [BES], the authors applied a particular case of the addition theorem mentioned above with a Bolza polynomial to compute the norm of a wave function to the Schrödinger equation with a finite-gap potential.

A similar polynomial $F_k(z) \equiv U(z) := (z-x_1)\cdots(z-x_k)$ over $\Theta^{[k]}$ plays an essential role in the studies of the structures of the subvarieties ([AF] and [Ma]), namely,

$$\frac{\sigma_{\natural^{k+1}}(u-v)\sigma_{\natural^{k+1}}(u+v)}{\sigma_{\natural^1}(v)^2\sigma_{\natural^k}(u)^2} = (-1)^{g-k}F_k(x)\,.$$

For the case $k = 1$, it appeared in [Ô] as a Frobenius-Stickelberger type relation of higher genus, which determines an algebraic structure of the curve.

We emphasize that studies of the θ-divisor are currently of interest. In particular, the inversion of higher genera hyperelliptic integrals with respect to the restriction to the θ-divisor was recently used in the problem of the analytic description of a conformal map of a domain in the half-plane to its complement, by means of a reduction of the Benney system [BG1], [BG2]. The same method of inversion of an ultraelliptic integral was used in [EPR] to describe motion of a double pendulum.

In was shown in a series of publications by Vanhaecke [V1], [V2], Abenda and Fedorov [AF] and one of the current authors [Ma] and others that the θ-divisor can serve as a carrier of integrability. Grant [G], and Cantor [C] found algebraic structures on $\Theta^{[1]}$ which are related to division polynomials whose zeros determine n-time points. Recently one of the authors and

Eilbeck and Previato used the Grant-Jorgenson formula ([G] and [J]) to derive an analogue of the Frobenius-Stickelberger addition formula for three variables in the case of a genus two hyperelliptic curve ([EEP]).

Thus we hope that our main theorem will have some influence on this type of study based on the Riemann singularity theorem.

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