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Symplectic Volumes of Certain Symplectic Quotients Associated with the Special Unitary Group of Degree Three

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Abstract. We consider the symplectic quotient for a direct product of several integral coadjoint orbits of $SU(3)$ and investigate its symplectic volume. According to a fundamental theorem for symplectic quotients, it is equivalent to studying the dimension of the trivial part in a tensor product of several irreducible representations for $SU(3)$, and its asymptotic behavior. We assume that either all of coadjoint orbits are flag manifolds of $SU(3)$, or all are complex projective planes. As main results, we obtain an explicit formula for the symplectic volume in each case.

1. Introduction

Let G be a compact Lie group, $\mathfrak{g}$ the Lie algebra of G, and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Under the left coadjoint action of G on $\mathfrak{g}^*$, let $\mathcal{O}_\lambda$ be the orbit through $\lambda \in \mathfrak{g}^*$, which has a natural symplectic (in fact, Kähler) structure. For $\lambda_1, \dots, \lambda_n \in \mathfrak{g}^*$, let us consider the quotient space

$$\mathcal{M} := \{(x_1, \dots, x_n) \in \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} \mid x_1 + \cdots + x_n = 0\}/G ,$$

where G acts diagonally on the direct product of coadjoint orbits. We assume that $\mathcal{M}$ is not empty and is a smooth manifold. The space $\mathcal{M}$ has the associated symplectic (or Kähler) structure as the symplectic (or Kähler) quotient of the direct product of coadjoint orbits. The topology and the symplectic geometry of $\mathcal{M}$ are quite interesting.

For example, in the case $G = SU(2)$, $\mathcal{M}$ is identified with the moduli space of polygons in $\mathbf{R}^3$ with fixed lengths of edges. Many results have been obtained from various points of view (see, e.g. [9], [14], [27] and references cited therein). In particular, explicit formulas for the symplectic volume $\mathrm{vol}(\mathcal{M})$ of $\mathcal{M}$, and for the generating function of the cohomology intersection pairings, which is closely related to $\mathrm{vol}(\mathcal{M})$, are given in [26].

In this paper, we consider the case $G = SU(3)$. Our aim is to express the symplectic volume $\mathrm{vol}(\mathcal{M})$ of $\mathcal{M}$ in an explicit form. As in the case $G = SU(2)$, it might contain much information of the cohomology intersection pairing of $\mathcal{M}$. Except for the orbit consisting only of the origin, each coadjoint orbit of $SU(3)$ is diffeomorphic to either the flag manifold

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$SU(3)/T$ or the complex projective plane $\mathbf{P}^2(\mathbf{C})$, where T denotes the standard maximal torus of $SU(3)$. We will restrict ourselves to the following two cases.

Case 1. $\mathcal{O}_{\lambda_i} \cong SU(3)/T$ for all $i = 1, \dots, n$.

Case 2. $\mathcal{O}_{\lambda_i} \cong \mathbf{P}^2(\mathbf{C})$ for all $i = 1, \dots, n$.

Furthermore, we assume that the symplectic form of each coadjoint orbit represents an integral cohomology class. More precisely, we assume that $\lambda_i \in \Lambda_+$ for all $i = 1, \dots, n$, where Λ_+ denotes the set of dominant integral weights of G (see Section 2, for the details).

As we will discuss in Section 2, under certain conditions on $\lambda_1, \dots, \lambda_n \in \Lambda_+$, we can express $\mathrm{vol}(\mathcal{M})$ in terms of representations of G. Namely, $\mathrm{vol}(\mathcal{M})$ is equal to

$$\mathcal{V}(\lambda_1, \dots, \lambda_n) := \lim_{k\to\infty} \frac{1}{k^d} \dim_{\mathbf{C}}(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G ,$$

where V_λ denotes the irreducible representation of G with the highest weight $\lambda \in \Lambda_+$, and k runs over positive integers while k goes to infinity. The number d corresponds to the complex dimension of $\mathcal{M}$, hence $d = 3n - 8$ (resp. $d = 2n - 8$) in Case 1 (resp. in Case 2). Besides, in general, for a representation V of G, V^G denotes the subspace of V consisting of all G-invariant elements. Here, the theorem of Guillemin-Sternberg (and its generalization) on the characteristic numbers of symplectic quotients (see, e.g. [6], [21]) plays the key role, as well as the Borel-Weil theorem and the Hirzebruch-Riemann-Roch theorem.

Main results in this paper are the explicit formulas for $\mathcal{V}(\lambda_1, \dots, \lambda_n)$ and hence for $\mathrm{vol}(\mathcal{M})$. They are given in Theorem 4.5 and Corollary 4.9 (resp. in Theorem 5.6 and Corollary 5.9) for Case 1 (resp. for Case 2). The results are rather complicated alternating sums. Here in this Introduction, we state only the main theorem for Case 1. The details of the notations will be given in Section 2.

THEOREM 4.5. *Let $n \geq 3$ be an integer and let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ with $l_i > m_i > 0$ $(i = 1, \dots, n)$ satisfy the assumptions* (A1) *and* (A2) *in Section* 2.5. *For $\lambda = (\lambda_1, \dots, \lambda_n)$, denote by $\mathcal{I}_\lambda$ the set of* 6*-partition $(I_1, \dots, I_6)$ of $\{1, \dots, n\}$ such that*

$$l_{I_1,I_2} + m_{I_4,I_5} < \frac{1}{3}(L + M), \quad l_{I_3,I_4} + m_{I_6,I_1} < \frac{1}{3}(L + M),$$

and denote by $\mathcal{J}_\lambda$ the set of $(I_1, \dots, I_6)$ such that

$$l_{I_3,I_4} + m_{I_6,I_1} > \frac{1}{3}(L + M), \quad l_{I_5,I_6} + m_{I_2,I_3} > \frac{1}{3}(L + M),$$

where $L = l_1 + \cdots + l_n$ and $M = m_1 + \cdots + m_n$. Define the functions A_λ on $\mathcal{I}_\lambda$ and B_λ on $\mathcal{J}_\lambda$ as follows.

$$A_\lambda(I_1, \dots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3n-8)!} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3}$$
$$\left(\frac{L+M}{3} - l_{I_3,I_4} - m_{I_6,I_1}\right)^c \left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right)^{3n-8-c} ,$$

$$B_\lambda(I_1,\dots,I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3n-8)!}\sum_{c=0}^{n-3}\binom{3n-8}{c}\binom{2n-6-c}{n-3}$$
$$\left(l_{I_3,I_4}+m_{I_6,I_1}-\frac{L+M}{3}\right)^c\left(l_{I_5,I_6}+m_{I_2,I_3}-\frac{L+M}{3}\right)^{3n-8-c}.$$

Then we have

$$\mathcal{V}(\lambda_1,\dots,\lambda_n) = \sum_{\mathcal{I}_\lambda} A_\lambda(I_1,\dots,I_6) + \sum_{\mathcal{J}_\lambda} B_\lambda(I_1,\dots,I_6)\,.$$

Let us mention that a 6-partition $(I_1,\dots,I_6)$ of $\{1,\dots,n\}$ corresponds to an n-tuple $(w_1,\dots,w_n)$ in the Weyl group, which in turn corresponds to a fixed point for the action of the maximal torus T of $G = SU(3)$ on $\mathcal{O}_{\lambda_1}\times\cdots\times\mathcal{O}_{\lambda_n}$. It might be interesting that each term in the formula above is a certain value of the hypergeometric function. Corresponding to the fact that $\mathbf{P}^2(\mathbf{C})$ is a degenerate coadjoint orbit, the proof for Case 2 is technically more complicated than that for Case 1. We indicate that a special case of Corollary 5.9 is obtained also in [20], although the method is completely different from ours.

In [24], we will study the symplectic volume $\mathrm{vol}(\mathcal{M})$ of $\mathcal{M}$ in the more general setting that G is any connected, simply connected, compact simple Lie group. The results for $G = SU(3)$ in this paper, as well as the previous results for $G = SU(2)$, will provide important examples. On the other hand, it is shown in [12] that $\mathcal{M}$ is identified with the moduli space of flat G-connections over the punctured sphere, with fixed conjugacy classes for the holonomies around the punctures. Hence we would be able to express $\mathrm{vol}(\mathcal{M})$ by the so-called Witten's volume formula (see, e.g. [29], [19], [22]), which is in the form of an infinite series. It is quite different from the formula given in this paper (see the example in Section 4.2). The details will be studied in [24], too. (We refer to [15], [27] for the case $G = SU(2)$).

This paper is organized as follows. In Section 2, after reviewing some generalities on coadjoint orbits, we prove the identity $\mathrm{vol}(\mathcal{M}) = \mathcal{V}(\lambda_1,\dots,\lambda_n)$, which allows us to reduce the study on $\mathrm{vol}(\mathcal{M})$ to that on asymptotic behaviors for tensor products of irreducible representations of G. Then, we translate the problem into a combinatorial form via the representation theory of $SU(3)$, such as the Weyl character formula and the Weyl integration formula. These arguments above are essentially the same with those in [25], [26] for $SU(2)$, and available also for a general compact Lie group G. At the end of Section 2, we clarify our assumptions on $\lambda_1,\dots,\lambda_n$.

In Section 3, we prepare a lemma on the asymptotic behavior of a certain sum consisting of products of binomial coefficients. It is here that the hypergeometric integrals appear. In Section 4 and 5, we state and prove our main theorems. We also consider several examples and write down the volume formulas for them more explicitly.

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2. Formulation of the problem

2.1. Preliminaries. We refer to [3] for the generalities on compact Lie groups and their representations.

Let $G = SU(3)$, $\mathfrak{g} = \mathfrak{su}(3)$, T the standard maximal torus of G consisting of diagonal matrices in G, and $\mathfrak{t}$ its Lie algebra. Let $\mathfrak{g}^*$ and $\mathfrak{t}^*$ be the duals of $\mathfrak{g}$ and $\mathfrak{t}$, respectively. We denote by $\langle\ ,\ \rangle$ the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$, or between $\mathfrak{t}^*$ and $\mathfrak{t}$. Let $W \cong \mathfrak{S}_3$ be the Weyl group of $G = SU(3)$ with respect to T. We define the AdG-invariant positive definite inner product $(\ ,\ )$ on $\mathfrak{g}$ by

$$(X, Y) := -\frac{1}{4\pi^2}\mathrm{Tr}XY \quad (X, Y \in \mathfrak{g}) .$$

If we identify $\mathfrak{t}^*$ with $\mathfrak{t}$ by means of $(\ ,\ )$, the action of the Weyl group W on $\mathfrak{t}^* \cong \mathfrak{t}$ is given by permutations of diagonal entries. The elements

$$H_1 = 2\pi\sqrt{-1}\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} , \quad H_2 = 2\pi\sqrt{-1}\begin{pmatrix}0 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} ,$$

in $\mathfrak{t}$ are generators of the integral lattice $\mathrm{Ker}(\exp : \mathfrak{t} \to T)$ and form a basis of $\mathfrak{t}$. Define $\omega_1, \omega_2 \in \mathfrak{t}^*$ by $\langle \omega_i, H_j\rangle = \delta_{ij}$. Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, ω_1, ω_2 corresponds the elements

$$\Omega_1 = \frac{2\pi\sqrt{-1}}{3}\begin{pmatrix}2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -1\end{pmatrix} , \quad \Omega_2 = \frac{2\pi\sqrt{-1}}{3}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2\end{pmatrix} ,$$

in $\mathfrak{t}$, respectively. Define

$$\mathfrak{t}^*_+ := \mathbf{R}_{\geq 0}\omega_1 + \mathbf{R}_{\geq 0}\omega_2 , \quad \Lambda_+ := \mathbf{Z}_{\geq 0}\omega_1 + \mathbf{Z}_{\geq 0}\omega_2 ,$$

then $\mathfrak{t}^*_+$ is a positive Weyl chamber and Λ_+ is the associated set of dominant integral weights. We write an element λ in $\mathfrak{t}^*_+$ or Λ_+ in the following form:

$$\lambda = (l - m)\omega_1 + m\omega_2 \quad (l \geq m \geq 0) .$$

Under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, it corresponds to the element

$$\frac{2\pi\sqrt{-1}}{3}\begin{pmatrix}2l - m & 0 & 0\\ 0 & -l + 2m & 0\\ 0 & 0 & -l - m\end{pmatrix} = 2\pi\sqrt{-1}\begin{pmatrix}l - \frac{l+m}{3} & 0 & 0\\ 0 & m - \frac{l+m}{3} & 0\\ 0 & 0 & 0 - \frac{l+m}{3}\end{pmatrix}$$

in $\mathfrak{t}$.

Irreducible representations of G are, by assigning their highest weights, in one-to-one correspondence with elements in Λ_+. Denote by V_λ the irreducible representation of G with the highest weight $\lambda \in \Lambda_+$ and by $\chi_\lambda : G \to \mathbf{C}$ the character of V_λ. When we write an element of T as $t = \mathrm{diag}(x, y, z)$ ($|x| = |y| = |z| = 1, xyz = 1$), the Weyl character formula tells us that

$$\chi_\lambda(t) = \frac{1}{(x-y)(x-z)(y-z)} \begin{vmatrix} x^{l+2} & x^{m+1} & 1 \\ y^{l+2} & y^{m+1} & 1 \\ z^{l+2} & z^{m+1} & 1 \end{vmatrix}$$

for $\lambda = (l-m)\omega_1 + m\omega_2 \in \Lambda_+$.

2.2. Coadjoint orbits. Although we mainly consider the case $G = SU(3)$, most of the followings still hold when G is a general compact Lie group. For further details on coajoint orbits, see, e.g. [18], [16]. We also refer to [2], [11] for the Borel-Weil Theorem.

The left coadjoint action of G on $\mathfrak{g}^*$ is defined by $g \cdot f := \mathrm{Ad}^*(g^{-1})f$ for $g \in G$ and $f \in \mathfrak{g}^*$, where

$$\langle \mathrm{Ad}^*(g^{-1})f, X\rangle = \langle f, \mathrm{Ad}(g^{-1})X\rangle\,,$$

for $X \in \mathfrak{g}$.

If we identity $\mathfrak{g}^*$ with $\mathfrak{g}$ by the inner product $(\ ,\ )$, the coadjoint action corresponds to the adjoint action. We regard $\mathfrak{t}^*$ as a subspace of $\mathfrak{g}^*$ by the identification

$$\mathfrak{t}^* = \{f \in \mathfrak{g}^* \mid t \cdot f = f \ (\forall t \in T)\}\,.$$

Hence $\mathfrak{t}^*_+$ and Λ_+ also can be regarded as subsets in $\mathfrak{g}^*$. We denote by $\mathcal{O}_\lambda$ the coadjoint orbit through $\lambda \in \mathfrak{t}^*_+$. The intersection $\mathcal{O}_\lambda \cap \mathfrak{t}^*$ is the W-orbit through λ, and $\mathcal{O}_\lambda \cap \mathfrak{t}^*_+$ consists of the single point λ.

Let G_λ be the isotropy subgroup at $\lambda = (l-m)\omega_1 + m\omega_2 \in \mathfrak{t}^*_+$ ($l, m \in \mathbf{R}, l \geq m \geq 0$) for the coadjoint action of $G = SU(3)$ on $\mathfrak{g}^*$.

(1) If $l > m > 0$, then $G_\lambda = T$ and $\mathcal{O}_\lambda \cong G/T$.

(2) If $l > 0, m = 0$, then $G_\lambda = \left\{ \begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \in SU(3) \right\}$ and $\mathcal{O}_\lambda \cong \mathbf{P}^2(\mathbf{C})$.

(3) If $l = m > 0$, then $\mathcal{O}_\lambda \cong \mathbf{P}^2(\mathbf{C})$ likewise.

(4) If $l = m = 0$, then $G_\lambda = G$ and $\mathcal{O}_\lambda = \{0\}$, of course.

On each coadjoint orbit $\mathcal{O}_\lambda$, there exists a natural G-invariant symplectic structure ω_λ, called the Kirillov-Kostant-Souriau symplectic form, defined by

$$(\omega_\lambda)_x(\tilde{X}, \tilde{Y}) = \langle x, [X, Y]\rangle \quad (x \in \mathcal{O}_\lambda,\ X, Y \in \mathfrak{g})\,,$$

where $\tilde{X}$ is the vector field on $\mathcal{O}_\lambda$ given by

$$\tilde{X}_x := \frac{d}{dt}(\exp tX) \cdot x \bigg|_{t=0}\,.$$

The action of G on $\mathcal{O}_\lambda$ is Hamiltonian and the associated moment map is given by the inclusion $\iota : \mathcal{O}_\lambda \hookrightarrow \mathfrak{g}^*$, that is, we have $d\langle \iota, X\rangle(\cdot) = \omega_\lambda(\tilde{X}, \cdot)$.

In addition, there exists a G-invariant complex structure J_λ on $\mathcal{O}_\lambda$, which is compatible with the symplectic structure ω_λ, that is, $\omega_\lambda(\cdot, J_\lambda \cdot)$ becomes a Riemann metric, and makes $\mathcal{O}_\lambda$ into a Kähler manifold. Moreover, in the case that $\lambda \in \Lambda_+$, there exists a G-equivariant holomorphic line bundle L_λ over $\mathcal{O}_\lambda$ such that $c_1(L_\lambda) = [\omega_\lambda]$. The Borel-Weil theorem asserts that

$$H^0(\mathcal{O}_\lambda, L_\lambda) = V_\lambda\,, \quad H^i(\mathcal{O}_\lambda, L_\lambda) = 0 \ (i > 0)$$

as representations of G, where $H^i(\mathcal{O}_\lambda, L_\lambda)$ stands for the i-th cohomology group of $\mathcal{O}_\lambda$ with coefficients in the sheaf of germs of holomorphic sections of L_λ.

REMARK 2.1. In other words, the Kähler structure on $\mathcal{O}_\lambda \cong G/G_\lambda$ and the holomorphic line bundle L_λ are characterized as follows (see [4], [7], [11]).

The contragredient representation V_λ^* of V_λ has the lowest weight $-\lambda$. Let $v \in V_\lambda^*$ be a lowest weight vector and denote by $[v]$ the corresponding element in $\mathbf{P}(V_\lambda^*)$. Then, $\mathcal{O}_\lambda \cong G/G_\lambda$ is identified with the G-orbit X through $[v]$ in $\mathbf{P}(V_\lambda^*)$. Since the complexification $G_{\mathbf{C}}$ of G acts on X, X turns out to be a complex submanifold, hence a Kähler submanifold of $\mathbf{P}(V_\lambda^*)$. The Kähler form ω_λ and the holomorphic line bundle L_λ over $\mathcal{O}_\lambda$ are identified with the restriction to X of the Fubini-Study Kähler form and the hyperplane bundle over $\mathbf{P}(V_\lambda^*)$, respectively.

REMARK 2.2. (1) For $k \in \mathbf{R}_{>0}$ and $\lambda \in \mathfrak{t}_+^*$, $\mathcal{O}_{k\lambda}$ and $\mathcal{O}_\lambda$ are the same as complex manifolds. If we compare the symplectic forms under this identification, we have $\omega_{k\lambda} = k\omega_\lambda$. In the case $k \in \mathbf{Z}_{>0}$ and $\lambda \in \Lambda_+$, we have $L_{k\lambda} = L_\lambda^{\otimes k}$.

(2) For $\lambda \in \Lambda_+$, the action on $\mathcal{O}_\lambda$ of the center $Z(G) \cong \mathbf{Z}/3\mathbf{Z}$ of $G = SU(3)$ is trivial, while those on L_λ and V_λ is not trivial in general. However, if we replace λ with 3λ, then these actions become trivial, too.

2.3. Symplectic quotient of a direct product of coadjoint orbits.

See, e.g. [1], [7], [17], [23], for general properties of symplectic and Kähler quotients. The following still hold for a general compact Lie group G.

Let $\lambda_1, \ldots, \lambda_n \in \mathfrak{t}_+^*$. The diagonal action of G on the direct product $\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}$ is also Hamiltonian and the moment map $\Phi : \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} \to \mathfrak{g}^*$ is given by $\Phi(x_1, \ldots, x_n) = x_1 + \cdots + x_n$. Consider the symplectic (or Kähler) quotient

$$\begin{aligned}\mathcal{M}(\lambda_1, \ldots, \lambda_n) &:= \Phi^{-1}(0)/G \\ &= \{(x_1, \ldots, x_n) \in \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} \mid x_1 + \cdots + x_n = 0\}/G\,.\end{aligned}$$

We often set $\lambda = (\lambda_1, \ldots, \lambda_n)$ for brevity, and write $\mathcal{M}(\lambda_1, \ldots, \lambda_n)$ as $\mathcal{M}(\lambda)$ or simply as $\mathcal{M}$. We assume that

(a0) $\Phi^{-1}(0) \neq \emptyset$,

(a1) 0 is a regular value of the moment map Φ, and $\mathcal{M}(\lambda)$ is a smooth manifold.

Then there exist a natural symplectic structure $\omega_{\mathcal{M}(\lambda)}$ and a compatible complex structure on $\mathcal{M}(\lambda)$, induced from $\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}$, which make $\mathcal{M}(\lambda)$ a Kähler manifold.

In the following, we suppose $\lambda_1, \dots, \lambda_n \in \Lambda_+$. Let L_{λ_i} be the G-equivariant holomorphic line bundle over $\mathcal{O}_{\lambda_i}$ as in Section 2.2 and let

$$\mathcal{L}(\lambda_i) := (\mathrm{pr}_i^* L_{\lambda_i}|_{\Phi^{-1}(0)})/G\,, \quad \mathcal{L}(\lambda) := (L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_n}|_{\Phi^{-1}(0)})/G\,,$$

where $\mathrm{pr}_i : \mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n} \to \mathcal{O}_{\lambda_i}$ is the i-th projection and $L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_n} = \bigotimes_{i=1}^n \mathrm{pr}_i^* L_{\lambda_i}$.

By (a1), the isotropy subgroup at each point in $\Phi^{-1}(0)$ is a finite group. Since its action on $\mathrm{pr}_i^* L_{\lambda_i}$, $L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_n}$ may not be trivial, $\mathcal{L}(\lambda_i)$ and $\mathcal{L}(\lambda)$ are orbifold holomorphic line bundles over $\mathcal{M}(\lambda)$, in general. We assume that

(a2) $\mathcal{L}(\lambda_i)$ is a genuine holomorphic line bundle over $\mathcal{M}(\lambda)$ for all $i = 1, \dots, n$.

Then we have $\mathcal{L}(\lambda) = \mathcal{L}(\lambda_1) \otimes \cdots \otimes \mathcal{L}(\lambda_n)$ and $c_1(\mathcal{L}(\lambda)) = c_1(\mathcal{L}(\lambda_1)) + \cdots + c_1(\mathcal{L}(\lambda_n)) = [\omega_{\mathcal{M}(\lambda)}]$.

REMARK 2.3. (1) It seems to be interesting to describe completely in terms of $\lambda = (\lambda_1, \dots, \lambda_n)$ the necessary and sufficient condition in order that the assumptions (a0)–(a2) for $\mathcal{M}(\lambda)$ and $\mathcal{L}(\lambda)$ hold. Although we do not pursue this problem in this paper, we will give certain conditions on λ in Section 2.5, which are closely related to (a1) and (a2).

(2) It would be also an interesting problem, to see how the topology of $(\mathcal{M}(\lambda), \omega_{\mathcal{M}(\lambda)})$ changes as λ varies. We will not discuss it in this paper, either. But in connection with it, we mention that the Lemma 2.1 in [26] by the second author is incorrect, where the change of $[\omega_{\mathcal{M}(\lambda)}]$ under small deformation of λ is considered in the case $G = SU(2)$. He would like to thank J-C. Hausmann for noticing it to him. We refer to [9] for a correct argument.

Now, in general, define $\chi(M, L) := \sum (-1)^i H^i(M, L)$ as a virtual vector space for a compact complex manifold M and a holomorphic line bundle L over M. Our aim is to study the characteristic number $\dim_{\mathbf{C}} \chi(\mathcal{M}(\lambda), \mathcal{L}(\lambda))$ and the symplectic volume $\mathrm{vol}(\mathcal{M}(\lambda)) = \int_{\mathcal{M}(\lambda)} \frac{(\omega_{\mathcal{M}(\lambda)})^d}{d!}$ of $\mathcal{M}(\lambda)$, where $d = d(\lambda)$ is the complex dimension of $\mathcal{M}(\lambda)$, that is, $d(\lambda) = \frac{1}{2} \sum_{i=1}^n \dim_{\mathbf{R}} G/G_{\lambda_i} - \dim_{\mathbf{R}} G$. The volume $\mathrm{vol}(\mathcal{M}(\lambda))$ is particularly interesting, since it contains much information on the cohomology intersection pairings $\int_M c_1(\mathcal{L}(\lambda_1))^{d_1} \cdots c_1(\mathcal{L}(\lambda_n))^{d_n}$ $(d_1 + \cdots + d_n = d)$.

REMARK 2.4. (1) As in Remark 2.2, $\mathcal{M}(k\lambda) = \mathcal{M}(\lambda)$ as manifolds and $\omega_{\mathcal{M}(k\lambda)} = k\omega_{\mathcal{M}(\lambda)}$ for $k \in \mathbf{R}_{>0}$ and $\lambda \in \mathfrak{t}_+^*$. In particular, it follows that $\mathrm{vol}(\mathcal{M}(k\lambda)) = k^d \cdot \mathrm{vol}(\mathcal{M}(\lambda))$. In the case $k \in \mathbf{Z}_{>0}$ and $\lambda \in \Lambda_+$, we have $\mathcal{L}(k\lambda) \cong \mathcal{L}(\lambda)^{\otimes k}$.

(2) Even if $\lambda \in (\Lambda_+)^n$ does not satisfy (a2), $c\lambda$ does satisfy (a2) for some positive integer c. Hence, as far as the symplectic volume $\mathrm{vol}(\mathcal{M}(\lambda))$ is concerned, we can assume (a2) without loss of generality.

PROPOSITION 2.5. *Suppose that* $\lambda = (\lambda_1, \dots, \lambda_n) \in (\Lambda_+)^n$ *satisfies* (a0), (a1), *and* (a2). *Then we have*

(1) $\chi(\mathcal{M}(\lambda), \mathcal{L}(\lambda)) = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G$,

(2) $\mathrm{vol}(\mathcal{M}(\lambda)) = \lim_{k \to \infty} \frac{1}{k^d} \cdot \dim_{\mathbf{C}} (V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G$,

where for a representation V of G define $V^G := \{v \in V \mid g \cdot v = v \ (\forall g \in G)\}$. In (2) k runs over positive integers while going to infinity.

PROOF. (1) In general, if a group G acts holomorphically on X and L is G-equivariant, we can regard $\chi(X, L)$ as a virtual representation of G. The theorem of Guillemin-Sternberg and its generalization (see, e.g. [6], [21]) tells us in our situation that

$$\chi(\mathcal{M}(\lambda), \mathcal{L}(\lambda)) = \chi(\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}, L_{\lambda_1} \boxtimes \cdots \boxtimes L_{\lambda_n})^G.$$

By the multiplicative property of χ (see the appendix in [10]) and the Borel-Weil theorem, we have

$$\mathrm{RHS} = (\chi(\mathcal{O}_{\lambda_1}, L_{\lambda_1}) \otimes \cdots \otimes \chi(\mathcal{O}_{\lambda_n}, L_{\lambda_n}))^G = (V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^G .$$

(2) By the Hirzebruch-Riemann-Roch theorem, we have

$$\dim_{\mathbf{C}} \chi(M, L) = \int_M \mathrm{ch}(L)\mathrm{td}(M) = \int_M e^{c_1(L)}\mathrm{td}(M)$$

for a compact complex manifold M and a holomorphic line bundle L over M, where $\mathrm{ch}(L)$ is the Chern character of L and $\mathrm{td}(M)$ is the Todd class of M. It follows that

$$\lim_{k\to\infty} \frac{1}{k^d} \cdot \dim_{\mathbf{C}} \chi(M, L^{\otimes k}) = \lim_{k\to\infty} \int_M \frac{e^{kc_1(L)}}{k^d}\mathrm{td}(M) = \int_M \frac{c_1(L)^d}{d!} .$$

Hence we have

$$\mathrm{vol}(\mathcal{M}(\lambda)) = \lim_{k\to\infty} \frac{1}{k^d} \cdot \dim_{\mathbf{C}} \chi(\mathcal{M}(\lambda), \mathcal{L}(\lambda)^{\otimes k}) .$$

Now, it follows from Remark 2.3 and (1) that

$$\chi(\mathcal{M}(\lambda), \mathcal{L}(\lambda)^{\otimes k}) = \chi(\mathcal{M}(k\lambda), \mathcal{L}(k\lambda)) = (V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G .$$

This completes the proof of (2). $\square$

REMARK 2.6. More generally, for a Lie subgroup U of G such that $T \subset U \subset G$, we may consider the corresponding invariants for the symplectic quotient of $\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}$ by the action of U. In this case, we should study $\dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^U$ and its asymptotic behavior.

2.4. Combinatorial Interpretation

DEFINITION 2.7. For $\lambda_1, \ldots, \lambda_n \in \Lambda_+$, define

$$\mathcal{Q}(\lambda_1, \ldots, \lambda_n) := \dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^G ,$$

$$\mathcal{V}(\lambda_1, \ldots, \lambda_n) := \lim_{k\to\infty} \frac{1}{k^d} \cdot \dim_{\mathbf{C}}(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^G ,$$

where $d = d(\lambda_1, \ldots, \lambda_n) := \frac{1}{2} \sum_{i=1}^n \dim_{\mathbf{R}} G/G_{\lambda_i} - \dim_{\mathbf{R}} G$.

Our purpose hereafter is to express these quantities as concrete as possible, and in particular to give an explicit formula for $\mathcal{V}(\lambda_1, \ldots, \lambda_n)$. The assumptions on $\lambda_1, \ldots, \lambda_n$ will be discussed in Section 2.5. Here, we prove a proposition that describes $\mathcal{Q}(\lambda_1, \ldots, \lambda_n) = \dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^{SU(3)}$ in a combinatorial fashion.

For $\lambda = (l-m)\omega_1 + m\omega_2 \in \Lambda_+$, define

$$D_\lambda(x,y,z) := \begin{vmatrix} x^{l+2} & x^{m+1} & 1 \\ y^{l+2} & y^{m+1} & 1 \\ z^{l+2} & z^{m+1} & 1 \end{vmatrix}, \quad D_0(x,y,z) := (x-y)(x-z)(y-z)\,.$$

Recall that the character χ_λ of the irreducible representation V_λ of $G = SU(3)$ is given by

$$\chi_\lambda(t) = \frac{D_\lambda(x,y,z)}{D_0(x,y,z)}\,,$$

for $t = \mathrm{diag}(x,y,z) \in T$.

PROPOSITION 2.8. *Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in (\Lambda_+)^n$. We write $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2$ $(i = 1, \ldots, n)$ and let $L = l_1 + \cdots + l_n$ and $M = m_1 + \cdots + m_n$. Then $\mathcal{Q}(\lambda) = \dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^{SU(3)}$ is equal to the coefficient of $x^{\frac{L+M+6}{3}} y^{\frac{L+M+6}{3}} z^{\frac{L+M+6}{3}}$ in the polynomial*

$$F_\lambda(x,y,z) := -\frac{1}{6} D_{\lambda_1}(x,y,z) \cdots D_{\lambda_n}(x,y,z) \cdot D_0(x,y,z)^{-n+2}\,.$$

PROOF. Denote by $d\mu_G, d\mu_T$ the normalized invariant measures on G, T, respectively. By the Weyl integration formula, we obtain

$$\begin{aligned} \dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^G &= \int_G \chi_{\lambda_1}(g) \cdots \chi_{\lambda_n}(g) d\mu_G \\ &= \frac{1}{6} \int_T \chi_{\lambda_1}(t) \cdots \chi_{\lambda_n}(t) |D_0(t)|^2 d\mu_T \\ &= -\frac{1}{6} \int_T \chi_{\lambda_1}(t) \cdots \chi_{\lambda_n}(t) D_0(t)^2 d\mu_T\,. \end{aligned}$$

Now if we write an element of T as $t = \mathrm{diag}(t_1, t_1^{-1}t_2, t_2^{-1})$, then $d\mu_T = \frac{dt_1}{2\pi\sqrt{-1}t_1} \frac{dt_2}{2\pi\sqrt{-1}t_2}$. Hence the above integral equals the coefficient of $t_1^0 t_2^0$, that is, the constant term in $F_\lambda(t_1, t_1^{-1}t_2, t_2^{-1})$.

On the other hand, $F_\lambda(x,y,z)$ is a homogeneous polynomial of x, y, z of degree $L + M + 6$. By substituting $x = t_1$, $y = t_1^{-1}t_2$, $z = t_2^{-1}$ into $x^a y^b z^c$, we have

$$x^a y^b z^c = t_1^{a-b} t_2^{b-c}\,,$$

which coincides with $t_1^0 t_2^0$ if and only if $a = b = c = \frac{L+M+6}{3}$. □

REMARK 2.9. (1) If $L+M \notin 3\mathbf{Z}$, then $\dim_{\mathbf{C}} \left(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n}\right)^{SU(3)} = 0$.

(2) Similarly, $\dim_{\mathbf{C}}(V_{\lambda_1} \otimes \cdots \otimes V_{\lambda_n})^T$ is equal to the coefficient of $x^{\frac{L+M+6}{3}} y^{\frac{L+M+6}{3}} z^{\frac{L+M+6}{3}}$ in $D_{\lambda_1}(x,y,z) \cdots D_{\lambda_n}(x,y,z) \cdot D_0(x,y,z)^{-n}$.

See Sections 4 and 5, for the more explicit description of $\mathcal{Q}(\lambda_1,\dots,\lambda_n)$, which is a complicated alternating sum of products of several binomial coefficients.

2.5. Assumptions on the weights. Let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ $(i = 1,\dots,n)$. For further investigations of $\mathcal{Q}(\lambda_1,\dots,\lambda_n)$ and $\mathcal{V}(\lambda_1,\dots,\lambda_n)$, we restrict ourselves to the following two cases.

Case 1. $n \geq 3$ and $l_i > m_i > 0$ for all $i = 1,\dots,n$.

Case 2. $n \geq 5$ and $l_i > m_i = 0$ for all $i = 1,\dots,n$.

In Case 1 we have $\mathcal{O}_{\lambda_i} \cong G/T$ and $d(\lambda_1,\dots,\lambda_n) = 3n-8$, while in case Case 2 we have $\mathcal{O}_{\lambda_i} \cong \mathbf{P}^2(\mathbf{C})$ and $d(\lambda_1,\dots,\lambda_n) = 2n-8$.

Moreover, we introduce the following assumptions (A1) and (A2) on $\lambda_1,\dots,\lambda_n \in \Lambda_+$, which are closely related to (a1) and (a2) in Seciton 2.3. Recall that we set

$$\Omega_1 = \frac{2\pi\sqrt{-1}}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

(A1) $\langle w_1\lambda_1 + \cdots + w_n\lambda_n, \Omega_1\rangle \neq 0$ for any $w_1,\dots,w_n \in W$.

(A2) $\lambda_1,\dots,\lambda_n \in 3\Lambda_+ = 3\mathbf{Z}_{\geq 0}\omega_1 + 3\mathbf{Z}_{\geq 0}\omega_2$.

REMARK 2.10. (1) In this paper, we do not discuss the condition (a0) that guarantees that $\mathcal{M}(\lambda) \neq \emptyset$. Hence in the following, $\mathcal{Q}(\lambda_1,\dots,\lambda_n)$ and $\mathcal{V}(\lambda_1,\dots,\lambda_n)$ might become 0.

(2) Actually, we could do without assuming (A1), to obtain the final formula for $\mathcal{V}(\lambda_1,\dots,\lambda_n)$ (see Remark 4.7 and 5.8).

(3) The condition that

$$w_1\lambda_1 + \cdots + w_n\lambda_n \neq 0 \text{ for any } w_1,\dots,w_n \in W,$$

which is weaker than (A1), follows from (a1). In fact, if $w_1\lambda_1 + \cdots + w_n\lambda_n = 0$, then the isotropy subgroup at $(w_1\lambda_1,\dots,w_n\lambda_n) \in \Phi^{-1}(0)$ contains T and is not a finite group. Thus 0 is not a regular value of the moment map Φ.

(4) If the isotropy subgroup at each point in $\Phi^{-1}(0)$ is the center $Z(G) \cong \mathbf{Z}/3\mathbf{Z}$ of $G = SU(3)$, (A2) implies (a2). As we noted in Remark 2.4(3), as far as the symplectic volume $\mathrm{vol}(\mathcal{M}(\lambda))$ is concerned, we may assume (A2) without loss of generality.

(5) Even if we replace (A2) with a weaker assumption such as

- $l_i + m_i \in 3\mathbf{Z}_{>0}$ for all $i = 1,\dots,n$, or
- $L + M \in 3\mathbf{Z}_{>0}$,

the arguments in Section 4 and 5 will work as well.

Before closing this section, we translate the condition (A1) into a more concrete form. Let $L = l_1 + \cdots + l_n$, $M = m_1 + \cdots + m_n$. For a subset I of $\{1, \ldots, n\}$, let us denote

$$l_I = \sum_{i \in I} l_i, \quad m_I = \sum_{i \in I} m_i .$$

Moreover, for two disjoint subsets I, J of $\{1, \ldots, n\}$, we write

$$l_{I,J} = l_I + l_J = \sum_{i \in I \cup J} l_i, \quad m_{I,J} = m_I + m_J = \sum_{i \in I \cup J} m_i ,$$

for brevity.

DEFINITION 2.11. A sequence $(I_1, \ldots, I_p)$ of p-subsets $I_1, \ldots, I_p$ in $\{1, \ldots, n\}$ is called a *p-partition* of $\{1, \ldots, n\}$, if $I_1 \cup \cdots \cup I_p = \{1, \ldots, n\}$ and $I_j \cap I_k = \emptyset$ $(j \neq k)$.

LEMMA 2.12. *Let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ $(i = 1, \ldots, n)$. The condition* (A1) *means that*

$$l_{J_1} + m_{J_2} \neq \frac{L+M}{3}$$

for any pair (J_1, J_2) of two disjoint subsets of $\{1, \ldots, n\}$. It is also equivalent to the condition that

$$l_{I_1,I_2} + m_{I_4,I_5} \neq \frac{L+M}{3} \quad \text{and} \quad l_{I_3,I_4} + m_{I_6,I_1} \neq \frac{L+M}{3} \quad \text{and} \quad l_{I_5,I_6} + m_{I_2,I_3} \neq \frac{L+M}{3}$$

for every 6-partition $(I_1, \ldots, I_6)$ of $\{1, \ldots, n\}$.

PROOF. Recall that under the identification $\mathfrak{t}^* \cong \mathfrak{t}$, $\lambda = (l - m)\omega_1 + m\omega_2$ becomes

$$\lambda = 2\pi\sqrt{-1}\begin{pmatrix} l - \frac{l+m}{3} & 0 & 0 \\ 0 & m - \frac{l+m}{3} & 0 \\ 0 & 0 & 0 - \frac{l+m}{3} \end{pmatrix}$$

and the action of the Weyl group W is given by the permutation of the diagonal entries. Let s_1 (resp. s_2) be the transposition between $(1, 1)$ and $(2, 2)$ entries (resp. $(2, 2)$ and $(3, 3)$ entries). We enumerate all elements in W as follows:

$$\sigma_1 = \mathrm{id}, \quad \sigma_2 = s_2, \quad \sigma_3 = s_1 \circ s_2, \quad \sigma_4 = s_1, \quad \sigma_5 = s_2 \circ s_1, \quad \sigma_6 = s_1 \circ s_2 \circ s_1 .$$

Given $w_1, \ldots, w_n \in W$, let $I_j = \{i \in \{1, \ldots, n\} \mid w_i = \sigma_j\}$ for $j = 1, \ldots, 6$. Then $(I_1, \ldots, I_6)$ is a 6-partition of $\{1, \ldots, n\}$ and we observe

$$\begin{aligned} &w_1\lambda_1 + \cdots + w_n\lambda_n \\ &= 2\pi\sqrt{-1}\begin{pmatrix} l_{I_1,I_2} + m_{I_4,I_5} - \frac{L+M}{3} & 0 & 0 \\ 0 & l_{I_3,I_4} + m_{I_6,I_1} - \frac{L+M}{3} & 0 \\ 0 & 0 & l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3} \end{pmatrix} , \end{aligned}$$

$$\langle w_1\lambda_1 + \cdots + w_n\lambda_n, \Omega_1\rangle = l_{I_1,I_2} + m_{I_4,I_5} - \frac{L+M}{3}.$$

It follows from (A1) that $l_{I_1,I_2} + m_{I_4,I_5} \neq \frac{L+M}{3}$. Since $(I_1 \cup I_2, I_4 \cup I_5)$ represents any pair of disjoint two subsets of $\{1,\ldots,n\}$, we obtain the first statement. The second one is obvious. □

REMARK 2.13. It follows from the proof that a 6-partition $(I_1,\ldots,I_6)$ of $\{1,\ldots,n\}$ corresponds to an n-tuple $(w_1,\ldots,w_n)$ in the Weyl group W, which in turn corresponds to a fixed point for the action of T on $\mathcal{O}_{\lambda_1} \times \cdots \times \mathcal{O}_{\lambda_n}$ (see Section 2.2).

3. A lemma

In this section, we prove a lemma on the asymptotic behavior of a certain sum consisting of products of three binomial coefficients.

Let $p, q, r \in \mathbf{Z}_{>0}$, $u, v, w \in \mathbf{Z}$, and $\alpha, \beta \in \mathbf{Z}_{\geq 0}$. If $\beta \geq \alpha$, define

$$A := \lim_{k\to\infty} \frac{1}{k^{p+q+r+1}} \sum_{j=0}^{k\alpha+v} \binom{j+u}{p}\binom{k\alpha-j+v}{q}\binom{k\beta-j+w}{r}$$

and if $\alpha \geq \beta$, define

$$B := \lim_{k\to\infty} \frac{1}{k^{p+q+r+1}} \sum_{j=0}^{k\beta+w} \binom{j+u}{p}\binom{k\alpha-j+v}{q}\binom{k\beta-j+w}{r}.$$

LEMMA 3.1. (1) *If $\beta > \alpha > 0$, then we have*

$$\begin{aligned} A &= \frac{1}{p!q!r!}\int_0^\alpha x^p(\alpha-x)^q(\beta-x)^r\,dx \\ &= \frac{1}{(p+q+r+1)!}\sum_{c=0}^{r}\binom{p+q+r+1}{c}\binom{q+r-c}{q}\alpha^{p+q+r+1-c}(\beta-\alpha)^c. \end{aligned}$$

(2) *If $\alpha > \beta > 0$, then we have*

$$\begin{aligned} B &= \frac{1}{p!q!r!}\int_0^\beta x^p(\alpha-x)^q(\beta-x)^r\,dx \\ &= \frac{1}{(p+q+r+1)!}\sum_{c=0}^{q}\binom{p+q+r+1}{c}\binom{q+r-c}{r}(\alpha-\beta)^c\beta^{p+q+r+1-c}. \end{aligned}$$

PROOF. (1) For polynomials $f(k), g(k)$ on k, let us write as $f(k) \sim g(k)$ if the top degree terms of them are equal. It follows that

$$\sum_{j=0}^{\alpha k+v}\binom{j+u}{p}\binom{k\alpha-j+v}{q}\binom{k\beta-j+w}{r} \sim \sum_{j=0}^{\alpha k+v}\frac{j^p}{p!}\frac{(k\alpha-j)^q}{q!}\frac{(k\beta-j)^r}{r!}$$

$$= \frac{k^{p+q+r+1}}{p!q!r!} \sum_{j=0}^{\alpha k+v} \left(\frac{j}{k}\right)^p \left(\alpha - \frac{j}{k}\right)^q \left(\beta - \frac{j}{k}\right)^r \cdot \frac{1}{k},$$

which implies that

$$A = \frac{1}{p!q!r!} \int_0^\alpha x^p (\alpha - x)^q (\beta - x)^r dx .$$

After expanding the third factor in the integrand as

$$(\beta - x)^r = (\beta - \alpha + \alpha - x)^r = \sum_{c=0}^{r} \binom{r}{c} (\beta - \alpha)^c (\alpha - x)^{r-c},$$

the second equality in (1) follows from the following.

$$\int_0^\alpha x^p (\alpha - x)^{q+r-c} dx = \alpha^{p+q+r+1-c} \frac{p!(q + r - c)!}{(p + q + r + 1 - c)!} .$$

The proof of (2) is similar. □

REMARK 3.2. Let $F(a, b; c; z)$ be the hypergeometric function. Recall that

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - tz)^{-a} dt ,$$

if $\operatorname{Re} c > \operatorname{Re} b > 0$ (see, e.g. [28]). We can express A and B as certain values of the hypergeometric function as follows.

$$A = \frac{\alpha^{p+q+1} \beta^r}{(p + q + 1)!r!} \cdot F\left(-r, p + 1; p + q + 2; \frac{\alpha}{\beta}\right)$$
$$B = \frac{\alpha^q \beta^{p+r+1}}{q!(p + r + 1)!} \cdot F\left(-q, p + 1; p + r + 2; \frac{\beta}{\alpha}\right)$$

Note that $F(-r, p + 1; p + q + 2; z)$ and $F(-q, p + 1; p + r + 2; z)$ are polynomials of z, called the Jacobi polynomials.

REMARK 3.3. In (1) let us denote

$$I = \frac{1}{p!q!r!} \int_0^\alpha x^p (\alpha - x)^q (\beta - x)^r dx ,$$
$$J = \frac{1}{(p + q + r + 1)!} \sum_{c=0}^{r} \binom{p + q + r + 1}{c} \binom{q + r - c}{q} \alpha^{p+q+r+1-c} (\beta - \alpha)^c .$$

If $\alpha = 0$, then $A = I = J = 0$. If $\alpha = \beta$, then

$$A = I = J = \frac{\alpha^{p+q+r+1}}{(p + q + r + 1)!} \binom{q + r}{q},$$

where it is supposed that $0^0 = 1$ in J. Thus, the identities in (1) still hold when $\beta \geq \alpha \geq 0$. Similarly, the identities in (2) still hold when $\alpha \geq \beta \geq 0$.

4. Quotient of product of flag manifolds of $SU(3)$

4.1. Main Theorem 1. In this section we consider Case 1. Namely, let $n \geq 3$ be an integer and let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ with $l_i > m_i > 0$ $(i = 1, \ldots, n)$ such that the assumption (A1) and (A2) in Section 2.5 hold. The aim is to obtain an explicit formula for

$$\mathcal{V}(\lambda_1, \ldots, \lambda_n) = \lim_{k\to\infty} \frac{1}{k^{3n-8}} \dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)} .$$

In fact, as we will see in Remark 4.7, we could do without assuming (A1) to obtain the final formula for $\mathcal{V}(\lambda_1, \ldots, \lambda_n)$. However, in order to avoid redundant arguments we keep the assumption (A1) unless otherwise stated.

Let us begin with the presentation of $\mathcal{Q}(k\lambda_1, \ldots, k\lambda_n) = \dim_{\mathbf{C}}(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n})^{SU(3)}$.

LEMMA 4.1. *Let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ $(i = 1, \ldots, n)$ be as above and let $L = l_1 + \cdots + l_n$, $M = m_1 + \cdots + m_n$. Then we have*

$$\begin{aligned}
&\dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)} \\
&\quad = -\frac{1}{6} \sum_{\substack{I_1,\ldots,I_6 \\ j_1,j_2,j_3}} (-1)^{|I_2|+|I_4|+|I_6|+j_1+j_2+j_3} \binom{-n+2}{j_1}\binom{-n+2}{j_2}\binom{-n+2}{j_3},
\end{aligned}$$

where the sum is taken over all 6-partitions $(I_1, \ldots, I_6)$ of $\{1, \ldots, n\}$ and all $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$ such that

$$\begin{cases}
k\left(l_{I_1,I_2} + m_{I_4,I_5} - \frac{L+M}{3}\right) + 2|I_1| + 2|I_2| + |I_4| + |I_5| + j_1 + j_2 - 2 = 0\,, \\
k\left(l_{I_3,I_4} + m_{I_6,I_1} - \frac{L+M}{3}\right) + 2|I_3| + 2|I_4| + |I_6| + |I_1| - j_1 + j_3 - n = 0\,, \\
k\left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) + 2|I_5| + 2|I_6| + |I_2| + |I_3| - j_2 - j_3 - 2n + 2 = 0\,.
\end{cases} \tag{4.2}$$

Here, for a subset I of $\{1, \ldots, n\}$, $|I|$ denotes the cardinality of I.

PROOF. According to Proposition 2.8, $\dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}$ is equal to the coefficient of $x^{\frac{k(L+M)+6}{3}} y^{\frac{k(L+M)+6}{3}} z^{\frac{k(L+M)+6}{3}}$ in

$$F_{k\lambda}(x, y, z) = -\frac{1}{6} D_{k\lambda_1}(x, y, z) \cdots D_{k\lambda_n}(x, y, z) \cdot D_0(x, y, z)^{-n+2}\,.$$

Although $F_{k\lambda}(x, y, z)$ is a polynomial on x, y, z, we expand it to a power series on the domain $|x| < |y| < |z|$. Since

$$\begin{aligned}
D_{k\lambda_i}(x, y, z) &= x^{kl_i+2}y^{km_i+1} - z^{km_i+1}x^{kl_i+2} + y^{kl_i+2}z^{km_i+1} \\
&\quad - x^{km_i+1}y^{kl_i+2} + z^{kl_i+2}x^{km_i+1} - y^{km_i+1}z^{kl_i+2}\,,
\end{aligned}$$

we have

$$\prod_{i=1}^{n} D_{k\lambda_i}(x,y,z) = \sum_{I_1,\dots,I_6} \prod_{i\in I_1}(x^{kl_i+2}y^{km_i+1}) \prod_{i\in I_2}(-z^{km_i+1}x^{kl_i+2}) \prod_{i\in I_3}(y^{kl_i+2}z^{km_i+1})$$
$$\prod_{i\in I_4}(-x^{km_i+1}y^{kl_i+2}) \prod_{i\in I_5}(z^{kl_i+2}x^{km_i+1}) \prod_{i\in I_6}(-y^{km_i+1}z^{kl_i+2})$$
$$= \sum_{I_1,\dots,I_6} (-1)^{|I_2|+|I_4|+|I_6|} x^{k(l_{I_1,I_2}+m_{I_4,I_5})+2|I_1|+2|I_2|+|I_4|+|I_5|}$$
$$y^{k(l_{I_3,I_4}+m_{I_6,I_1})+2|I_3|+2|I_4|+|I_6|+|I_1|}$$
$$z^{k(l_{I_5,I_6}+m_{I_2,I_3})+2|I_5|+2|I_6|+|I_2|+|I_3|}\,.$$

See Section 2.5 for the symbol l_{I_1,I_2} etc. Since the binomial theorem shows that

$$D_0(x,y,z)^{-n+2} = (x-y)^{-n+2}(x-z)^{-n+2}(y-z)^{-n+2}$$
$$= \sum_{j_1,j_2,j_3} (-1)^{-3n+6-j_1-j_2-j_3}$$
$$\binom{-n+2}{j_1}\binom{-n+2}{j_2}\binom{-n+2}{j_3} x^{j_1+j_2} y^{-n+2-j_1+j_3} z^{-2n+4-j_2-j_3}\,,$$

where $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$, we see that

$$F_{k\lambda}(x,y,z) = -\frac{1}{6} \sum_{\substack{I_1,\dots,I_6\\ j_1,j_2,j_3}} (-1)^{|I_1|+|I_3|+|I_5|+j_1+j_2+j_3} \binom{-n+2}{j_1}\binom{-n+2}{j_2}\binom{-n+2}{j_3}$$
$$x^{k(l_{I_1,I_2}+m_{I_4,I_5})+2|I_1|+2|I_2|+|I_4|+|I_5|+j_1+j_2}$$
$$y^{k(l_{I_3,I_4}+m_{I_6,I_1})+2|I_3|+2|I_4|+|I_6|+|I_1|-j_1+j_3-n+2}$$
$$z^{k(l_{I_5,I_6}+m_{I_2,I_3})+2|I_5|+2|I_6|+|I_2|+|I_3|-j_2-j_3-2n+4}\,.$$

Note that $|I_1|+\cdots+|I_6| = n$. By considering the term of $x^{\frac{k(L+M)+6}{3}} \cdot y^{\frac{k(L+M)+6}{3}} z^{\frac{k(L+M)+6}{3}}$, we obtain (4.2). □

Under the assumption (A1), Lemma 2.12 shows

$$l_{I_1,I_2}+m_{I_4,I_5} \neq \frac{L+M}{3}\,, \quad l_{I_3,I_4}+m_{I_6,I_1} \neq \frac{L+M}{3}\,, \quad l_{I_5,I_6}+m_{I_2,I_3} \neq \frac{L+M}{3}$$

for any 6-partition $(I_1,\dots,I_6)$. Obviously, we have $l_{I_1,I_2}+l_{I_3,I_4}+l_{I_5,I_6} = L$ and $m_{I_4,I_5} + m_{I_6,I_1} + m_{I_2,I_3} = M$.

LEMMA 4.3. *Let us fix a sufficiently large $k \in \mathbf{Z}_{>0}$. There exist $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$ such that* (4.2) *hold if and only if $(I_1,\dots,I_6)$ satisfies*

$$l_{I_1,I_2}+m_{I_4,I_5} < \frac{L+M}{3}\,, \quad l_{I_5,I_6}+m_{I_2,I_3} > \frac{L+M}{3}\,.$$

In such a case, j_1, j_3 *are determined by* $(I_1, \dots I_6)$ *and* j_2*, while the range of* j_2 *is given as follows.*

(1) *If* $\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5} < l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}$, *i.e.* $l_{I_3,I_4} + m_{I_6,I_1} < \frac{L+M}{3}$,

$$0 \le j_2 \le k\left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right) - 2|I_1| - 2|I_2| - |I_4| - |I_5| + 2\,.$$

(2) *If* $\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5} > l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}$, *i.e.* $l_{I_3,I_4} + m_{I_6,I_1} > \frac{L+M}{3}$,

$$0 \le j_2 \le k\left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) + 2|I_5| + 2|I_6| + |I_2| + |I_3| - 2n + 2\,.$$

PROOF. It follows from (4.2) that

$$\begin{cases} j_1 = k\left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right) - j_2 - 2|I_1| - 2|I_2| - |I_4| - |I_5| + 2\,, \\ j_3 = k\left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) - j_2 + 2|I_5| + 2|I_6| + |I_2| + |I_3| - 2n + 2\,. \end{cases}$$

Hence j_1 and j_3 are determined by $(I_1, \dots I_6)$ and j_2. The conditions $j_1 \ge 0$ and $j_3 \ge 0$ imply that

$$\begin{cases} j_2 \le k\left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right) - 2|I_1| - 2|I_2| - |I_4| - |I_5| + 2\,, \\ j_2 \le k\left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) + 2|I_5| + 2|I_6| + |I_2| + |I_3| - 2n + 2\,. \end{cases}$$

There exists $j_2 \in \mathbf{Z}_{\ge 0}$ satisfying the above for $k \gg 0$, if and only if

$$\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5} > 0\,, \quad l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3} > 0\,.$$

$\square$

Now, for a 6-partition $(I_1, \dots, I_6)$ which satisfies the condition in Lemma 4.3, define

$$\begin{aligned} C_{k\lambda}(I_1, \dots, I_6) :&= \sum_{j_2} (-1)^{j_1+j_2+j_3} \binom{-n+2}{j_1}\binom{-n+2}{j_2}\binom{-n+2}{j_3} \\ &= \sum_{j_2} \binom{j_1+n-3}{n-3}\binom{j_2+n-3}{n-3}\binom{j_3+n-3}{n-3} \\ &= \sum_{j_2} \binom{j_2+n-3}{n-3} \\ &\qquad \binom{k\left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right) - j_2 - 2|I_1| - 2|I_2| - |I_4| - |I_5| + n - 1}{n-3} \\ &\qquad \binom{k\left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right) - j_2 + 2|I_5| + 2|I_6| + |I_2| + |I_3| - n - 1}{n-3}, \end{aligned}$$

where the range of j_2 is as in Lemma 4.3.

By applying Lemma 3.1 to $C_{k\lambda}(I_1, \dots, I_6)$, we obtain the following.

LEMMA 4.4. (1) *If* $l_{I_3,I_4} + m_{I_6,I_1} < \frac{L+M}{3}$,

$$\lim_{k\to\infty} \frac{1}{k^{3n-8}} \cdot C_{k\lambda}(I_1, \dots, I_6) = \frac{1}{(3n-8)!} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \left(\frac{L+M}{3} - l_{I_3,I_4} - m_{I_6,I_1}\right)^c \left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right)^{3n-8-c}.$$

(2) *If* $l_{I_3,I_4} + m_{I_6,I_1} > \frac{L+M}{3}$,

$$\lim_{k\to\infty} \frac{1}{k^{3n-8}} \cdot C_{k\lambda}(I_1, \dots, I_6) = \frac{1}{(3n-8)!} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \left(l_{I_3,I_4} + m_{I_6,I_1} - \frac{L+M}{3}\right)^c \left(l_{I_5,I_6} + m_{I_2,I_3} - \frac{L+M}{3}\right)^{3n-8-c}.$$

Combining all the results above, we obtain the following explicit formula for

$$\mathcal{V}(\lambda_1, \dots, \lambda_n) = \lim_{k\to\infty} \frac{1}{k^{3n-8}} \dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}.$$

THEOREM 4.5. *Let $n \geq 3$ be an integer and let $\lambda_i = (l_i - m_i)\omega_1 + m_i\omega_2 \in \Lambda_+$ with $l_i > m_i > 0$ $(i = 1, \dots, n)$ satisfy* (A1) *and* (A2). *For $\lambda = (\lambda_1, \dots, \lambda_n)$, denote by $\mathcal{I}_\lambda$ the set of all* 6*-partitions $(I_1, \dots, I_6)$ such that*

$$l_{I_1,I_2} + m_{I_4,I_5} < \frac{1}{3}(L+M), \quad l_{I_3,I_4} + m_{I_6,I_1} < \frac{1}{3}(L+M)$$

and denote by $\mathcal{J}_\lambda$ the set of all $(I_1, \dots, I_6)$ such that

$$l_{I_3,I_4} + m_{I_6,I_1} > \frac{1}{3}(L+M), \quad l_{I_5,I_6} + m_{I_2,I_3} > \frac{1}{3}(L+M).$$

Define the functions A_λ on $\mathcal{I}_\lambda$ and B_λ on $\mathcal{J}_\lambda$ as follows.

$$A_\lambda(I_1, \dots, I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3n-8)!} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \left(\frac{L+M}{3} - l_{I_3,I_4} - m_{I_6,I_1}\right)^c \left(\frac{L+M}{3} - l_{I_1,I_2} - m_{I_4,I_5}\right)^{3n-8-c},$$

$$B_\lambda(I_1,\dots,I_6) := \frac{-(-1)^{|I_1|+|I_3|+|I_5|}}{6(3n-8)!}\sum_{c=0}^{n-3}\binom{3n-8}{c}\binom{2n-6-c}{n-3}$$
$$\left(l_{I_3,I_4}+m_{I_6,I_1}-\frac{L+M}{3}\right)^c\left(l_{I_5,I_6}+m_{I_2,I_3}-\frac{L+M}{3}\right)^{3n-8-c}.$$

Then we have

$$\mathcal{V}(\lambda_1,\dots,\lambda_n)=\sum_{\mathcal{I}_\lambda}A_\lambda(I_1,\dots,I_6)+\sum_{\mathcal{J}_\lambda}B_\lambda(I_1,\dots,I_6)\,. \tag{4.6}$$

REMARK 4.7. It follows from Remark 3.3 that even if $\lambda_1,\dots,\lambda_n$ do not satisfy (A1), $\mathcal{V}(\lambda_1,\dots,\lambda_n)$ is given by the same formula (4.6), by replacing the definitions of $\mathcal{I}_\lambda$ and $\mathcal{J}_\lambda$ with

$$\mathcal{I}_\lambda=\left\{(I_1,\dots,I_6)\;\middle|\; l_{I_1,I_2}+m_{I_4,I_5}\le\frac{L+M}{3},\ l_{I_3,I_4}+m_{I_6,I_1}\le\frac{L+M}{3}\right\},$$
$$\mathcal{J}_\lambda=\left\{(I_1,\dots,I_6)\;\middle|\; l_{I_3,I_4}+m_{I_6,I_1}\le\frac{L+M}{3},\ l_{I_5,I_6}+m_{I_2,I_3}\le\frac{L+M}{3}\right\}.$$

REMARK 4.8. According to Remark 2.9(2), we obtain in the same manner the corresponding formula for

$$\frac{1}{k^{d'}}\cdot\dim_{\mathbf{C}}(V_{k\lambda_1}\otimes\cdots V_{k\lambda_n})^T\,,$$

where $d'=3n-2$.

Now, Remark 2.4 and Proposition 2.5 tell us that the following hold.

COROLLARY 4.9. *Let $n\ge 3$ be an integer and let $\lambda_i=(l_i-m_i)\omega_1+m_i\omega_2\in\mathfrak{t}^*_+$ with $l_i,m_i\in\mathbf{Q}$ and $l_i>m_i>0$ $(i=1,\dots,n)$. Suppose that $\mathcal{M}(\lambda_1,\dots,\lambda_n)$ satisfies the assumptions* (a0) *and* (a1) *in Section* 2.3. *Then* $\mathrm{vol}(\mathcal{M}(\lambda_1,\dots,\lambda_n))$ *is given by the right hand side of* (4.6).

REMARK 4.10. Although our method has been quite combinatorial, the data appearing in the formula above have geometric meanings, of course. It might be interesting to consider the meaning of each term in (4.6), in connection with the residue formula in [13] and with the results in [20].

4.2. Example 1. As a typical example, let us consider the case $\lambda_i=m_i(\omega_1+\omega_2)$, where $m_i\in 3\mathbf{Z}_{>0}$. Since $l_i=2m_i$ and $L=2M$, (A1) means that

$$2m_{I_1,I_2}+m_{I_4,I_5}\ne M\,,\quad 2m_{I_3,I_4}+m_{I_6,I_1}\ne M\,,\quad 2m_{I_5,I_6}+m_{I_2,I_3}\ne M\,,$$

namely,

$$m_{I_1,I_2}\ne m_{I_3,I_6}\,,\quad m_{I_3,I_4}\ne m_{I_2,I_5}\,,\quad m_{I_5,I_6}\ne m_{I_1,I_4}$$

for any 6-partition $(I_1, \dots, I_6)$ of $\{1, \dots, n\}$. Thus we assume that $m_I \neq m_J$, for any disjoint subsets $I, J \subset \{1, \dots, n\}$. Then we have

$$\mathcal{I}_\lambda = \{(I_1, \dots, I_6) \mid m_{I_1,I_2} < m_{I_3,I_6}\,, \quad m_{I_3,I_4} < m_{I_2,I_5}\}\,,$$
$$\mathcal{J}_\lambda = \{(I_1, \dots, I_6) \mid m_{I_3,I_4} > m_{I_2,I_5}\,, \quad m_{I_5,I_6} > m_{I_1,I_4}\}$$

and

$$\begin{aligned}
&-6(3n-8)! \cdot \mathcal{V}(\lambda_1, \dots, \lambda_n)\\
&= \sum_{\mathcal{I}_\lambda} (-1)^{|I_1|+|I_3|+|I_5|} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \left(m_{I_2,I_5} - m_{I_3,I_4}\right)^c \left(m_{I_3,I_6} - m_{I_1,I_2}\right)^{3n-8-c}\\
&+ \sum_{\mathcal{J}_\lambda} (-1)^{|I_1|+|I_3|+|I_5|} \sum_{c=0}^{n-3} \binom{3n-8}{c} \binom{2n-6-c}{n-3} \left(m_{I_3,I_4} - m_{I_2,I_5}\right)^c \left(m_{I_5,I_6} - m_{I_1,I_4}\right)^{3n-8-c}\,.
\end{aligned}$$

Note that (A1) is not satisfied when $m_1 = \cdots = m_n$. However, the same formula holds by Remark 4.7, which would be written as a sum over $(|I_1|, \dots, |I_6|) \in (\mathbf{Z}_{\geq 0})^6$ (see an analogous example in Section 5.2).

REMARK 4.11. As we will investigate in [24], there is another formula for $\mathcal{V}(\lambda_1, \dots, \lambda_n)$, which is quite different from the one given in Theorem 4.5. For example, let $\lambda_i = m_i(\omega_1 + \omega_2)$ with $m_i \in 3\mathbf{Z}_{>0}$ be as above. Then the following holds.

$$\mathcal{V}(\lambda_1, \dots, \lambda_n) = \frac{2^6}{\pi^2} \cdot \left(\frac{2M}{\pi}\right)^{3n-8} \sum_{c,d \in \mathbf{Z}_{\geq 0}} \frac{\prod_{i=1}^n \left(\sin\frac{\pi m_i(c+1)}{2M} \sin\frac{\pi m_i(d+1)}{2M} \sin\frac{\pi m_i(c+d+2)}{2M}\right)}{((c+1)(d+1)(c+d+2))^{n-2}}\,.$$

This corresponds to the so-called Witten's volume formula in 2-dimensional gauge theory (see, e.g. [29], [19], [22]). In fact, it is shown in [12] that $\mathcal{M}(\lambda_1, \dots, \lambda_n)$ is identified with the moduli space of flat G-connections over the n-punctured sphere, with fixed conjugacy classes for the holonomies around the punctures. We refer to [15], [27] for the case $G = SU(2)$.

5. Quotient of product of projective planes

5.1. Main Theorem 2. In this section, we study Case 2. Suppose that $n \geq 5$ is an integer and $\lambda_i = l_i\omega_1 \in \Lambda_+$ $(i = 1, \dots, n)$ satisfy (A1) and (A2). Let us consider $\dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}$, where $k \in \mathbf{Z}_{>0}$, and

$$\mathcal{V}(\lambda_1, \dots, \lambda_n) = \lim_{k\to\infty} \frac{1}{k^{2n-8}} \dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}\,.$$

LEMMA 5.1. *Let $\lambda_i = l_i\omega_1 \in \Lambda_+$ $(i = 1, \dots, n)$ be as above and let $L = l_1 + \cdots + l_n$. Then we have*

$$\dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}$$

$$= -\frac{1}{6} \sum_{\substack{I_1, I_2, I_3 \\ j_1, j_2, j_3}} (-1)^{|I_2|+j_1+j_2+j_3} \binom{|I_1| - n + 2}{j_1} \binom{|I_2| - n + 2}{j_2} \binom{|I_3| - n + 2}{j_3},$$

where the sum is taken over all 3*-partitions* (I_1, I_2, I_3) *of* $\{1, \dots, n\}$ *and all* $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$ *such that*

$$\begin{cases} k(l_{I_1} - \frac{L}{3}) + 2|I_1| + j_2 + j_3 - 2 = 0\,, \\ k(l_{I_2} - \frac{L}{3}) + 2|I_2| + |I_3| + j_1 - j_3 - n = 0\,, \\ k(l_{I_3} - \frac{L}{3}) + 2|I_3| + |I_1| + |I_2| - j_1 - j_2 - 2n + 2 = 0\,. \end{cases} \tag{5.2}$$

Fix a sufficiently large $k \in \mathbf{Z}_{>0}$*. Then there exist* $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$ *as in* (5.2) *if and only if* (I_1, I_2, I_3) *satisfies*

$$l_{I_1} < \frac{L}{3}\,, \quad l_{I_3} > \frac{L}{3}\,.$$

PROOF. As in the preceding section, we consider the coefficient of $x^{\frac{kL+6}{3}} \cdot y^{\frac{kL+6}{3}} \cdot z^{\frac{kL+6}{3}}$ in

$$F_{k\lambda}(x, y, z) = -\frac{1}{6} D_{k\lambda_1}(x, y, z) \cdots D_{k\lambda_n}(x, y, z) \cdot D_0(x, y, z)^{-n+2}\,.$$

Since

$$\begin{aligned} D_{k\lambda_i}(x, y, z) &= x^{kl_i+2}y - zx^{kl_i+2} + y^{kl_i+2}z - xy^{kl_i+2} + z^{kl_i+2}x - yz^{kl_i+2} \\ &= x^{kl_i+2}(y - z) - y^{kl_i+2}(x - z) + z^{kl_i+2}(x - y)\,, \end{aligned}$$

we have

$$\begin{aligned} \prod_{i=1}^{n} D_{k\lambda_i}(x, y, z) &= \sum_{I_1, I_2, I_3} \prod_{i \in I_1} (x^{kl_i+2}(y - z)) \prod_{i \in I_2} (-y^{kl_i+2}(x - z)) \prod_{i \in I_3} (z^{kl_i+2}(x - y)) \\ &= \sum_{I_1, I_2, I_3} (-1)^{|I_2|} x^{kl_{I_1}+2|I_1|} y^{kl_{I_2}+2|I_2|} z^{kl_{I_3}+2|I_3|} (x-y)^{|I_3|} (x-z)^{|I_2|} (y-z)^{|I_1|}\,. \end{aligned}$$

By expanding $D_0(x, y, z)^{-n+2} = (x - y)^{-n+2}(x - z)^{-n+2}(y - z)^{-n+2}$ to a power series on the domain $|x| < |y| < |z|$, we obtain

$$\begin{aligned} F_{k\lambda}(x, y, z) &= -\frac{1}{6} \sum_{I_1, I_2, I_3} (-1)^{|I_2|} x^{kl_{I_1}+2|I_1|} y^{kl_{I_2}+2|I_2|} z^{kl_{I_3}+2|I_3|} \\ &\qquad (x - y)^{|I_3|-n+2} (x - z)^{|I_2|-n+2} (y - z)^{|I_1|-n+2} \\ &= -\frac{1}{6} \sum_{\substack{I_1, I_2, I_3 \\ j_1, j_2, j_3}} (-1)^{|I_2|+j_1+j_2+j_3} \binom{|I_3| - n + 2}{j_3} \binom{|I_2| - n + 2}{j_2} \binom{|I_1| - n + 2}{j_1} \\ &\qquad x^{kl_{I_1}+2|I_1|+j_2+j_3} y^{kl_{I_2}+2|I_2|+|I_3|+j_1-j_3-n+2} z^{kl_{I_3}+2|I_3|+|I_1|+|I_2|-j_1-j_2-2n+4}\,. \end{aligned}$$

Now the lemma follows in the same way with the proofs of Lemma 4.1 and 4.3. □

For a 3-partition (I_1, I_2, I_3) of $\{1, \ldots, n\}$ with the condition in Lemma 5.1, define

$$C_{k\lambda}(I_1, I_2, I_3) := \sum_{j_1, j_2, j_3} (-1)^{j_1+j_2+j_3} \binom{|I_1| - n + 2}{j_1} \binom{|I_2| - n + 2}{j_2} \binom{|I_3| - n + 2}{j_3},$$

where the sum is taken over all $j_1, j_2, j_3 \in \mathbf{Z}_{\geq 0}$ satisfying (5.2). We divide the investigations into the following two cases.

(I) The case that $|I_1| \leq n - 3$, $|I_2| \leq n - 3$, and $|I_3| \leq n - 3$.

(II) Otherwise.

(I) If $|I_1| \leq n - 3$, $|I_2| \leq n - 3$, $|I_3| \leq n - 3$, it follows from (5.2) that

$$\begin{cases} j_1 = k\left(l_{I_3} - \frac{L}{3}\right) - j_2 + 2|I_3| + |I_1| + |I_2| - 2n + 2\,, \\ j_3 = k\left(\frac{L}{3} - l_{I_1}\right) - j_2 - 2|I_1| + 2\,. \end{cases}$$

The range of j_2 is given as follows.

(1) If $\dfrac{L}{3} - l_{I_1} < l_{I_3} - \dfrac{L}{3}$, *i.e.* $l_{I_2} < \dfrac{L}{3}$, then $0 \leq j_2 \leq k\left(\dfrac{L}{3} - l_{I_1}\right) - 2|I_1| + 2$.

(2) If $\dfrac{L}{3} - l_{I_1} > l_{I_3} - \dfrac{L}{3}$, *i.e.* $l_{I_2} > \dfrac{L}{3}$, then

$$0 \leq j_2 \leq k\left(l_{I_3} - \frac{L}{3}\right) + 2|I_3| + |I_1| + |I_2| - 2n + 2$$

From $\binom{|I|-n+2}{j} = (-1)^j \binom{n-|I|-3+j}{n-|I|-3}$, we observe that

$$\begin{aligned} C_{k\lambda}(I_1, I_2, I_3) &= \sum_{j_2} \binom{n - |I_2| - 3 + j_2}{n - |I_2| - 3} \binom{n - |I_3| - 3 + j_3}{n - |I_3| - 3} \binom{n - |I_1| - 3 + j_1}{n - |I_1| - 3} \\ &= \sum_{j_2} \binom{j_2 + n - |I_2| - 3}{n - |I_2| - 3} \\ &\qquad \binom{k\left(\frac{L}{3} - l_{I_1}\right) - j_2 - 2|I_1| - |I_3| + n - 1}{n - |I_3| - 3} \\ &\qquad \binom{k\left(l_{I_3} - \frac{L}{3}\right) - j_2 + 2|I_3| + |I_2| - n - 1}{n - |I_1| - 3}. \end{aligned}$$

According to Lemma 3.1, we conclude the following lemma.

LEMMA 5.3. *Let* $|I_1| \leq n - 3$, $|I_2| \leq n - 3$, *and* $|I_3| \leq n - 3$.

(1) *If* $l_{I_2} < \dfrac{L}{3}$,

$$\lim_{k \to \infty} \frac{1}{k^{2n-8}} \cdot C_{k\lambda}(I_1, I_2, I_3)$$

$$= \frac{1}{(2n-8)!} \sum_{c=0}^{n-|I_1|-3} \binom{2n-8}{c} \binom{n+|I_2|-6-c}{|I_1|+|I_2|-3} \left(\frac{L}{3} - l_{I_2}\right)^c \left(\frac{L}{3} - l_{I_1}\right)^{2n-8-c}.$$

(2) *If* $l_{I_2} > \dfrac{L}{3}$,

$$\lim_{k\to\infty} \frac{1}{k^{2n-8}} \cdot C_{k\lambda}(I_1, I_2, I_3)$$
$$= \frac{1}{(2n-8)!} \sum_{c=0}^{n-|I_3|-3} \binom{2n-8}{c} \binom{n+|I_2|-6-c}{|I_2|+|I_3|-3} \left(l_{I_2} - \frac{L}{3}\right)^c \left(l_{I_3} - \frac{L}{3}\right)^{2n-8-c}.$$

(II) When some of $|I_1|$, $|I_2|$, or $|I_3|$ are greater than $n-3$, then the sum

$$C_{k\lambda}(I_1, I_2, I_3) = \sum_{j_1, j_2, j_3} (-1)^{j_1+j_2+j_3} \binom{|I_1|-n+2}{j_1} \binom{|I_2|-n+2}{j_2} \binom{|I_3|-n+2}{j_3}$$

consists of a finite number of terms. Thus, we are not able to apply Lemma 3.1 directly. Nevertheless, we can verify the following.

PROPOSITION 5.4. *Also in the case where* $|I_1|, |I_2|, |I_3|$ *is greater than* $n-3$,

$$\lim_{k\to\infty} \frac{1}{k^{2n-8}} \cdot C_{k\lambda}(I_1, I_2, I_3)$$

is given by the formula in Lemma 5.3.

PROOF. For brevity, let us investigate only the case $|I_1| = 0$, $|I_2| = n-1$, and $|I_3| = 1$, which seems to be the most complicated one. In this case, we see

$$C_{k\lambda}(I_1, I_2, I_3) = \sum_{j_2=0}^{1} (-1)^{j_1+j_2+j_3} \binom{-n+2}{j_1} \binom{1}{j_2} \binom{-n+3}{j_3}$$
$$= \sum_{j_2=0}^{1} (-1)^{j_2} \binom{n-3+j_1}{n-3} \binom{1}{j_2} \binom{n-4+j_3}{n-4}.$$

It follows from (5.2) that $j_1 = k\left(l_{I_3} - \dfrac{L}{3}\right) - j_2 - n + 3$, and $j_3 = k \cdot \dfrac{L}{3} - j_2 + 2$. Note also that $l_{I_1} = 0$. Therefore, we obtain

$$C_{k\lambda}(I_1, I_2, I_3) = \binom{k(l_{I_3} - \frac{L}{3})}{n-3} \binom{k \cdot \frac{L}{3} + n - 2}{n-4} - \binom{k(l_{I_3} - \frac{L}{3}) - 1}{n-3} \binom{k \cdot \frac{L}{3} + n - 3}{n-4}$$
$$= \binom{k(l_{I_3} - \frac{L}{3})}{n-3} \binom{k \cdot \frac{L}{3} + n - 2}{n-4}$$
$$\frac{(n-3)(k \cdot \frac{L}{3} + n - 2) + (n-4)k(l_{I_3} - \frac{L}{3}) - (n-3)(n-4)}{k(l_{I_3} - \frac{L}{3})(k \cdot \frac{L}{3} + n - 2)},$$

which implies that

$$\lim_{k\to\infty} \frac{1}{k^{2n-8}} \cdot C_{k\lambda}(I_1, I_2, I_3) = \frac{(l_{I_3} - \frac{L}{3})^{n-4}}{(n-4)!} \frac{(\frac{L}{3})^{n-4}}{(n-4)!} + \frac{(l_{I_3} - \frac{L}{3})^{n-3}}{(n-3)!} \frac{(\frac{L}{3})^{n-5}}{(n-5)!}.$$

On the other hand, Lemma 5.5 below shows that the right hand sides of (1) and (2) in Lemma 5.3

$$\frac{1}{(2n-8)!} \sum_{c=0}^{n-3} \binom{2n-8}{c} \binom{2n-7-c}{n-4} \left(\frac{L}{3} - l_{I_2}\right)^c \left(\frac{L}{3}\right)^{2n-8-c}$$

$$\frac{1}{(2n-8)!} \sum_{c=0}^{n-4} \binom{2n-8}{c} \binom{2n-7-c}{n-3} \left(l_{I_2} - \frac{L}{3}\right)^c \left(l_{I_3} - \frac{L}{3}\right)^{2n-8-c}$$

are both equal to $\frac{(l_{I_3}-\frac{L}{3})^{n-4}}{(n-4)!} \frac{(\frac{L}{3})^{n-4}}{(n-4)!} + \frac{(l_{I_3}-\frac{L}{3})^{n-3}}{(n-3)!} \frac{(\frac{L}{3})^{n-5}}{(n-5)!}$. □

LEMMA 5.5. *For $p, q \in \mathbf{Z}_{>0}$, we have*

$$\frac{1}{(p+q)!} \sum_{c=0}^{p+1} \binom{p+q}{c} \binom{p+q+1-c}{q} x^c y^{p+q-c} = \frac{(x+y)^p y^q}{p!q!} + \frac{(x+y)^{p+1} y^{q-1}}{(p+1)!(q-1)!}.$$

PROOF.

$$\begin{aligned} \text{LHS} &= \frac{1}{(p+1)!q!} \sum_{c=0}^{p+1} (p+q+1-c) \binom{p+1}{c} x^c y^{p+q-c} \\ &= \frac{1}{(p+1)!q!} \frac{d}{dy}((x+y)^{p+1} y^q) = \frac{(x+y)^p y^q}{p!q!} + \frac{(x+y)^{p+1} y^{q-1}}{(p+1)!(q-1)!}. \end{aligned}$$

□

Thus, our explicit formula for

$$\mathcal{V}(\lambda_1, \dots, \lambda_n) = \lim_{k\to\infty} \frac{1}{k^{2n-8}} \dim_{\mathbf{C}} \left(V_{k\lambda_1} \otimes \cdots \otimes V_{k\lambda_n}\right)^{SU(3)}$$

is given as follows.

THEOREM 5.6. *Let $n \geq 5$ be an integer and suppose that $\lambda_i = l_i \omega_1 \in \Lambda_+$ ($i = 1, \dots, n$) satisfy* (A1) *and* (A2). *For $\lambda = (\lambda_1, \dots, \lambda_n)$, denote by $\mathcal{I}_\lambda$ the set of all* 3*-partitions (I_1, I_2, I_3) of $\{1, \dots, n\}$ such that*

$$l_{I_1} < \frac{L}{3}, \quad l_{I_2} < \frac{L}{3},$$

and denote by $\mathcal{J}_\lambda$ the set of all (I_1, I_2, I_3) such that

$$l_{I_2} > \frac{L}{3}, \quad l_{I_3} > \frac{L}{3}.$$

Define the function A_λ on $\mathcal{I}_\lambda$ and B_λ on $\mathcal{J}_\lambda$ as follows.

$$A_\lambda(I_1, I_2, I_3) := \frac{-(-1)^{|I_2|}}{6(2n-8)!} \sum_{c=0}^{n-|I_1|-3} \binom{2n-8}{c} \binom{n+|I_2|-6-c}{|I_1|+|I_2|-3} \left(\frac{L}{3} - l_{I_2}\right)^c \left(\frac{L}{3} - l_{I_1}\right)^{2n-8-c},$$

$$B_\lambda(I_1, I_2, I_3) := \frac{-(-1)^{|I_2|}}{6(2n-8)!} \sum_{c=0}^{n-|I_3|-3} \binom{2n-8}{c} \binom{n+|I_2|-6-c}{|I_2|+|I_3|-3} \left(l_{I_2} - \frac{L}{3}\right)^c \left(l_{I_3} - \frac{L}{3}\right)^{2n-8-c}.$$

Then we have

$$\mathcal{V}(\lambda_1, \ldots, \lambda_n) = \sum_{\mathcal{I}_\lambda} A_\lambda(I_1, I_2, I_3) + \sum_{\mathcal{J}_\lambda} B_\lambda(I_1, I_2, I_3). \tag{5.7}$$

REMARK 5.8. As in the preceding section, it follows from Remark 3.3 that even if $\lambda_1, \ldots, \lambda_n$ do not satisfy (A1), $\mathcal{V}(\lambda_1, \ldots, \lambda_n)$ is given by the same formula (5.7) by replacing the definitions of $\mathcal{I}_\lambda$ and $\mathcal{J}_\lambda$ with

$$\mathcal{I}_\lambda = \left\{(I_1, I_2, I_3) \,\middle|\, l_{I_1} \le \frac{L}{3},\, l_{I_2} \le \frac{L}{3}\right\}$$

$$\mathcal{J}_\lambda = \left\{(I_1, I_2, I_3) \,\middle|\, l_{I_2} \ge \frac{L}{3},\, l_{I_3} \ge \frac{L}{3}\right\}.$$

By the same reason with Corollary 4.9, we observe the following.

COROLLARY 5.9. *Let $n \ge 5$ be an integer and let $\lambda_i = l_i\omega_1 \in \mathfrak{t}_+^*$ with $l_i \in \mathbf{Q}_{>0}$ $(i = 1, \ldots, n)$. Suppose that $\mathcal{M}(\lambda_1, \ldots, \lambda_n)$ satisfies* (a0) *and* (a1), *then* $\mathrm{vol}(\mathcal{M}(\lambda_1, \ldots, \lambda_n))$ *is given by the right hand side of* (5.7).

REMARK 5.10. In Case 2, it is known that $\mathcal{M}(\lambda)$ is identified with the symplectic quotient for the Grassmann manifold $Gr(3, n)$ of 3-dimensional subspaces in $\mathbf{C}^n$, by an action of the n-dimensional torus $U(1)^n$ (see, e.g. [8], [5]). It seems to be interesting to investigate the relation between this fact and our result.

5.2. Example 2. As a typical example, let us consider the case $\lambda_1 = \cdots = \lambda_n = l\omega_1$, where $l \in 3\mathbf{Z}_{>0}$. Since $L = ln$, (A1) means that

$$|I_1| \ne \frac{n}{3}, \quad |I_2| \ne \frac{n}{3}, \quad |I_3| \ne \frac{n}{3}$$

for any 3-partition (I_1, I_2, I_3) of $\{1, \ldots, n\}$. Thus we assume that $n \not\equiv 0 \pmod 3$.

By setting $|I_1| = i_1$, $|I_2| = i_2$, $|I_3| = i_3$, we can express $\mathcal{V}(\lambda_1, \ldots, \lambda_n)$ as a sum over $(i_1, i_2, i_3) \in (\mathbf{Z}_{\ge 0})^3$ as follows.

$$\frac{-6(2n-8)!}{l^{2n-8}} \cdot \mathcal{V}(\lambda_1, \ldots, \lambda_n)$$

$$= \sum_{\substack{i_1 < \frac{n}{3}, i_2 < \frac{n}{3} \\ i_1 + i_2 + i_3 = n}} \sum_{c=0}^{n-i_1-3} \frac{(-1)^{i_2} n!}{i_1! i_2! i_3!} \binom{2n-8}{c} \binom{n+i_2-6-c}{i_1+i_2-3} \left(\frac{n}{3} - i_2\right)^c \left(\frac{n}{3} - i_1\right)^{2n-8-c}$$

$$+\sum_{\substack{i_2>\frac{n}{3},i_3>\frac{n}{3}\\ i_1+i_2+i_3=n}}\sum_{c=0}^{n-i_3-3}\frac{(-1)^{i_2}n!}{i_1!i_2!i_3!}\binom{2n-8}{c}\binom{n+i_2-6-c}{i_2+i_3-3}\left(i_2-\frac{n}{3}\right)^c\left(i_3-\frac{n}{3}\right)^{2n-8-c}.$$

This formula is obtained also in [20] by a completely different method.

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