TOKYO J. MATH.
VOL. 30, NO. 2, 2007

Monodromy on the Central Hyperbolic Component of Polynomials with Just Two Distinct Critical Points

Hikaru YAZAWA

Gakushuin University

(Communicated by T. Kawasaki)

Abstract. For any given integers $d_0, d_1 \geq 1$, let $\mathcal{F}$ be the family of polynomial maps f such that f has a fixed point at the origin, and moreover has just two distinct critical points 1 and $c_f \neq 1$ of multiplicies d_0 and d_1, respectively. For the central hyperbolic component $\mathcal{H}$ of $\mathcal{F}$, a monodromy map on $\mathcal{H}$ is obtained by Branner-Hubbard deformations. We show that for any given λ with $0 < |\lambda| < 1$ and for any given integer $n \geq 1$, the monodromy map transitively acts on the family of all polynomial maps $f \in \mathcal{H}$ with $f'(0) = \lambda$ and $f^{\circ n}(c_f) = 1$.

1. Introduction

For a polynomial map of degree ≥ 2 on the complex plane, if the polynomial map has an attracting fixed point, then we obtain Branner-Hubbard deformations by wringing the standard complex structure on the attracting basin. In this paper, we consider Branner-Hubbard deformations of a polynomial map which has just two distinct critical points and an attracting fixed point at the origin with a non-zero multiplier.

For any given integers $d_0, d_1 \geq 1$, let $\mathcal{F}$ be the family of polynomial maps f such that f has a fixed point at the origin, and moreover has just two distinct critical points 1 and $c_f \neq 1$ of multiplicies d_0 and d_1, respectively. A *hyperbolic component* of $\mathcal{F}$ is defined to be a connected component of the family of all polynomial maps $f \in \mathcal{F}$ such that both critical points are contained in the attracting basins of f. In particular, the *central hyperbolic component* $\mathcal{H}$ of $\mathcal{F}$ is defined to be the family of all polynomial maps $f \in \mathcal{F}$ such that f has an attracting fixed point at the origin whose immediate basin contains both critical points. Moreover, for any complex number λ with $|\lambda| < 1$, let $\mathcal{H}_\lambda$ be the family of all polynomial maps $f \in \mathcal{H}$ with $f'(0) = \lambda$.

In the case of $0 < |\lambda| < 1$, the monodromy map on $\mathcal{H}_\lambda$ is obtained by making use of the technique of Branner-Hubbard deformations. For any polynomial map $f \in \mathcal{H}_\lambda$, we obtain the Branner-Hubbard deformation f_s which has an attracting fixed point with multiplier $\lambda|\lambda|^{s-1}$ at the origin, where $s = u + iv$ be a complex number with $v > 0$. In particular, for each

Received June 27, 2006; revised October 23, 2006

$s(t) = 1 + 2\pi i t/\log|\lambda|$ with $t \in \mathbf{R}$, we have $\lambda|\lambda|^{s(t)-1} = e^{2\pi i t}\lambda$, and hence the Branner-Hubbard deformation $f_{s(t)}$ is called the *turning deformation*. Using this Branner-Hubbard deformation, we define the *monodromy map* $\mathcal{M}_\lambda : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ by $\mathcal{M}_\lambda(f) = f_{s(1)}$.

In this paper, we study the action of $\mathcal{M}_\lambda$ on the family of all polynomial maps $f \in \mathcal{H}_\lambda$ with the critical orbit relation $f^{\circ n}(c_f) = 1$ for some integer $n \geq 1$. Our main result is as follows.

MAIN THEOREM. *In the case of* $0 < |\lambda| < 1$, *for any given integer* $n \geq 1$, *the monodromy map* $\mathcal{M}_\lambda$ *transitively acts on the family of all polynomial maps* $f \in \mathcal{H}_\lambda$ *with* $f^{\circ n}(c_f) = 1$.

In §2–3, we prove this theorem. In §4, we give an immediate application of the pinching deformations of Haissinsky [4] to polynomial maps obtained by this theorem.

2. Preliminaries

We first state some of definitions and facts about the dynamics of polynomial maps. For proofs and further details, see Milnor [8].

Let f be a polynomial map of degree ≥ 2 having an attracting fixed point with multiplier λ ($|\lambda| < 1$) at the origin. We denote by $\mathcal{A}(f, 0)$ the *basin of attraction*, i.e.,

$$\mathcal{A}(f, 0) = \{z \in \mathbf{C} \mid f^{\circ k}(z) \to 0 \ (k \to \infty)\}.$$

A connected component of $\mathcal{A}(f, 0)$ is called the *immediate attracting basin* if the connected component contains 0.

In the case of $\lambda = 0$, the dynamics of f on a neighborhood of 0 is understood by the Böttcher theorem. The polynomial map f takes the form $f(z) = a_d z^d + a_{d+1} z^{d+1} \cdots$, with $d \geq 2$ and $a_d \neq 0$. The Böttcher theorem asserts that there exists a conformal isomorphism φ defined on a neighborhood of 0 satisfying $\varphi(0) = 0$ and $\varphi \circ f(z) = (\varphi(z))^d$ for all z in the neighborhood. This φ need not be extended throughout the attracting basin. However, the function $|\varphi(z)|$ is extended throughout the attracting basin.

In the case of $0 < |\lambda| < 1$, the dynamics of f on a neighborhood of 0 is understood by the Kœnigs theorem. There exists a holomorphic map φ defined on $\mathcal{A}(f, 0)$ satisfying $\varphi(0) = 0$ and $\varphi \circ f(z) = \lambda\varphi(z)$ for all $z \in \mathcal{A}(f, 0)$. Moreover, φ is unique up to multiplication by a non-zero constant. For a small radius $r > 0$, this holomorphic map φ has a holomorphic inverse map $\psi : \mathbf{D}_r \to \mathcal{A}(f, 0)$ with $\psi(0) = 0$. In particular, there exists the largest radius R such that ψ is well defined on $\mathbf{D}_R$. Note that ψ extends homeomorphically over $\partial\mathbf{D}_R$, and the image $\psi(\partial\mathbf{D}_R)$ contains a critical point of f.

DEFINITION 1. For f as above, a critical point of f is said to be *first in* $\mathcal{A}(f, 0)$ if the critical point is contained in $\psi(\partial\mathbf{D}_R)$.

For any polynomial map $f \in \mathcal{H}_\lambda$ with $0 < |\lambda| < 1$, let φ_f be the holomorphic map satisfying $\varphi_f(0) = 0$, $\varphi_f(1) = 1$, and $\varphi_f \circ f(z) = \lambda\varphi_f(z)$ for all $z \in \mathcal{A}(f, 0)$. Note that

$\mathcal{A}(f,0)$ contains both critical points 1 and c_f. Hence for a radius r with $0 < r < 1$, the inverse map ψ_f of φ_f with $\psi_f(0) = 0$ can be defined on $\mathbf{D}_r$. In particular, if f satisfies $f^{\circ n}(c_f) = 1$ for some integer $n \geq 1$, then the critical point 1 is first in $\mathcal{A}(f,0)$, and hence ψ_f is well defined on $\mathbf{D}$.

For the rest of this section, we will prove the following.

PROPOSITION 2. *For any given integer $n \geq 1$, there exists a polynomial map $f \in \mathcal{H}$ which satisfies the inequality $0 < |f'(0)| < 1$ and the critical orbit relation $f^{\circ n}(c_f) = 1$.*

To prove Proposition 2, we consider the parameter space of polynomial maps $f_{a,b}$ with two critical points $a, b \in \mathbf{C}$.

Let $f_{a,b}(z)$ be the unique polynomial such that $f_{a,b}(0) = 0$ and $f'_{a,b}(z) = d(z-a)^{d_0}(z-b)^{d_1}$. Since $f_{0,0}(z) = z^{d_0+d_1+1}$, there exists a small $\varepsilon > 0$ such that for $f_{a,b}$ with $|a|, |b| < \varepsilon$, the attracting basin $\mathcal{A}(f_{a,b}, 0)$ contains both critical points a and b. For such a polynomial $f_{a,b}$, the polynomial $g_{a,b}(z) = f_{a,b}(az)/a$ belongs to $\mathcal{H}$ if $a \neq 0$ and if $a \neq b$. To find a polynomial $f_{a,b}$ with $f^{\circ n}_{a,b}(b) = a$ for some integer $n \geq 1$, we introduce algebraic curves of parameters (a, b) related to critical orbit relations.

For any positive integer n, let $F_n(a,b) = f^{\circ n}_{a,b}(b) - a$, and let $\mathcal{S}_n$ be the set of all zeros of F_n. Note that $\mathcal{S}_n$ has no isolated point. Since $(0,0) \in \mathcal{S}_n$, there exists a parameter (a, b) such that the polynomial $g_{a,b}(z) = f_{a,b}(az)/a$ belongs to $\mathcal{H}$.

LEMMA 3. *For any given integer $n \geq 1$ and for any complex number $b \neq 0$, if $(0,b) \in \mathcal{S}_n$, then the immediate attracting basin of $\mathcal{A}(f_{0,b}, 0)$ does not contain the critical point b.*

PROOF. We use the Böttcher theorem for $f_{0,b}$. Suppose that $b \in \mathbf{C} \setminus \{0\}$. Then there exists a holomorphic map φ_b on a neighborhood of the origin satisfying $\varphi_b(0) = 0$ and $\varphi_b \circ f_{0,b}(z) = (\varphi_b(z))^{d_0}$ for all z in the neighborhood. Following Milnor [8], we extend the function $|\varphi_b|$ to a continuous function G_b which is well defined and satisfies $G_b(f_{0,b}(z)) = (G_b(z))^{d_0}$ for all $z \in \mathcal{A}(f_{0,b}, 0)$.

In the case of $(0,b) \in \mathcal{S}_n \setminus \{(0,0)\}$, since $f^{\circ n}_{0,b}(b) = 0$, it follows from the equality above that $G_b(b) = 0$. Hence for any real number r with $0 < r < 1$, any connected component of the set $\{z \in \mathcal{A}(f_{0,b}, 0) \mid G_b(z) = r\}$ is a Jordan curve. In particular, φ_b extends continuously on such Jordan curves surrounding the origin. Hence φ_b is well defined on the immediate basin of $\mathcal{A}(f_{0,b}, 0)$, or equivalently the inverse map ψ_b of φ_b is well defined on $\mathbf{D}$.

Thus $f_{0,b}$ satisfies $\varphi_b \circ f_{0,b} \circ \psi_b(w) = w^{d_0}$ for all $w \in \mathbf{D}$, and hence the immediate basin does not contain the critical point b. □

Now we consider the section $\mathcal{S}_n \cap (\{0\} \times \mathbf{C})$.

LEMMA 4. $(0,0)$ *is an isolated point of* $\mathcal{S}_n \cap (\{0\} \times \mathbf{C})$.

PROOF. There exists a small $\varepsilon > 0$ such that if $|z| < \varepsilon$, then $|f_{0,b}(z)| < |z|$. Suppose that $(0,b) \in \mathcal{S}_n \cap (\{0\} \times \mathbf{C})$ and $|b| < \varepsilon$. Then by the inequality above, the immediate attracting basin of 0 for $f_{0,b}$ contains the critical point b. Thus it follows from Lemma 3 that $b = 0$. □

Proposition 2 is proved immediately by Lemma 4 as follows.

PROOF OF PROPOSITION 2. For a radius $r > 0$, define $I_r^2 = \{(a, b) \in \mathbf{C}^2 \,|\, |a| < r,\ |b| < r\}$. It follows from Lemma 4 that for a small radius $r > 0$, there exists a parameter $(a, b) \in (\mathcal{S}_n \cap I_r^2) \setminus \{(0, 0)\}$ satisfying $0 < |f'_{a,b}(0)| < 1$ and $|f_{a,b}(z)| < |z|$ for all z with $|z| \leq r$. Then each inverse image of the circles $|z| = r$ under $f_{a,b}^{\circ k}$ for all integer $k \geq 1$ is connected, and hence the attracting basin is also connected. Thus it follows that $g_{a,b} \in \mathcal{H}$ and $g_{a,b}^{\circ n}(c_g) = 1$. □

3. Proof of Main Theorem

In this section, for a polynomial map $f \in \mathcal{H}$ with $f'(0) = 0$, we define the Branner-Hubbard deformations of obtained by wringing the standard complex structure on $\mathcal{A}(f, 0)$, and prove Main Theorem. For details of Branner-Hubbard deformations, see, for example, [2], [3], [5], and [9].

Suppose that f is a polynomial map in $\mathcal{H}$ with the critical orbit relation $f^{\circ n}(c_f) = 1$ for some integer $n \geq 1$. Note that $f'(0) \neq 0$, and f has the first critical point 1.

For any complex number $s = u + iv$ with $u > 0$, we define the Branner-Hubbard deformation f_s as follows. Consider the complex structure obtained by the pull-back of the standard complex structure σ_0 on $\mathbf{C}$. Let $l_s(z) = z\,|z|^{s-1}$, and let σ_s be the f-invariant almost complex structure such that

$$\sigma_s = \begin{cases} \sigma_0 & \text{on } \mathbf{C} \setminus \mathcal{A}(f, 0)\,, \\ (l_s \circ \varphi_f)^* \sigma_0 & \text{on } \psi_f(\mathbf{D})\,. \end{cases}$$

Using the Measurable Riemann Mapping Theorem, we have a unique quasiconformal map h_s satisfying $h_s(0) = 0$, $h_s(1) = 1$ and $h_s^* \sigma_0 = \sigma_s$. We define $f_s = h_s \circ f \circ h_s^{-1}$. Then it follows from [9] that $\varphi_{f_s} = l_s \circ f \circ h_s^{-1}$ and $f_s'(0) = l_s(f'(0))$. Since l_s maps $\mathbf{D}$ onto itself, we have the following:

PROPOSITION 5. *For any integer $n \geq 1$ and for any complex number λ with $0 < |\lambda| < 1$, there exist polynomial maps $f \in \mathcal{H}$ with $f'(0) = \lambda$ and $f^{\circ n}(c_f) = 1$.*

Without loss of generality, we may suppose that $\lambda = f'(0) \in (0, 1)$. Then it follows from $\lambda \in (0, 1)$ that ψ_f can be extended to the univalent function

$$\psi_{f,0} : \mathbf{C} \setminus [1, \infty) \to \mathcal{A}(f, 0)\,.$$

For example, consider the case of $d = 3$ and $n = 1$. Since $\varphi_f(c_f) = \lambda^{-1}$ holds,

$$c_f = \lim_{y \downarrow 0} \psi_{f,0}(\lambda^{-1} + iy) \text{ or } c_f = \lim_{y \uparrow 0} \psi_{f,0}(\lambda^{-1} + iy)$$

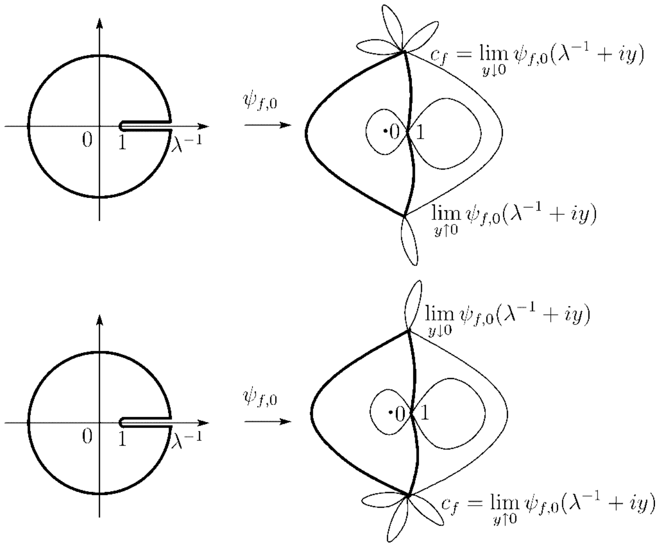

FIGURE 1. The case of $d = 3$ and $n = 1$.

holds as in Figure 1.

The quasiconformal map $h_{s(m)}$ is a Dehn twist of a fundamental annulus in $\psi_f(\mathbf{D})$. Consider the inverse image of $\hat{U}_{f,m} = l_{s(m)}^{-1}(\mathbf{C} \setminus [1, \infty))$ under φ_f. The inverse map ψ_f of φ_f is extended to the univalent map $\psi_{f,m}$ from $\hat{U}_{f,m}$ into $\mathcal{A}(f, 0)$. Let $U_{f,m} = \psi_{f,m}(\hat{U}_{f,m})$. Note that $h_{s(m)}(U_{f,m}) = \psi_{f_{s(m)},0}(\mathbf{C} \setminus [1, \infty))$. Thus, if $c_f = \lim_{y\downarrow 0} \psi_{f,0}(\lambda^{-1} + iy)$ holds, then $c_{f_{s(1)}} = h_{s(1)}(c_f) = \lim_{y\uparrow 0} \psi_{f_{s(1)},0}(\lambda^{-1} + iy)$ holds as in Figure 2.

Again, suppose that $d \geq 3$ and $n \geq 1$ are any given integers. To prove Main Theorem, considering the behavior of $h_{s(m)}$ on $\mathcal{A}(f, 0)$ for each integer $m \geq 1$, we describe a dynamical location of $c_{f_{s(m)}}$ on $\mathcal{A}(f_{s(m)}, 0)$.

LEMMA 6. *For f as above, suppose that $c_f \neq \lim_{y\downarrow 0} \psi_{f,0}(\lambda^{-1} + iy)$. Then there exists an integer $m \geq 1$ such that $c_{f_{s(m)}} = \lim_{y\downarrow 0} \psi_{f_{s(m)},0}(\lambda^{-n} + iy)$.*

PROOF. For any real number $r > 0$, let $D_f(0; r)$ be the connected component of 0 in $\{z \in \mathcal{A}(f, 0) \mid |\varphi_f(z)| < r\}$. We denote by $O_f^-(1)$ the set of all points z such that $f^{\circ k}(z) = 1$ for some integer $k \geq 1$. Since $f^{\circ n}(c_f) = 1$, we obtain $\partial D_f(0; \lambda^{-k}) \cap O_f^-(1) \neq \emptyset$ for all $k \geq 1$. For a small $\varepsilon > 0$, consider the ring domain $D_f(0; \lambda + \varepsilon) \setminus D_f(0; \lambda - \varepsilon)$. Since the preimage of this ring domain under $f^{\circ n}$ contains c_f, the preimage is simply connected. Hence, the preimage of the circle $|z| = \lambda^{-n}$ under φ_f is just one connected curve. Thus $\partial D_f(0; \lambda^{-n})$ contains c_f as in Figure 3.

To prove this lemma, we study the process of the Dehn twist on $D_f(0; \lambda^{-n})$.

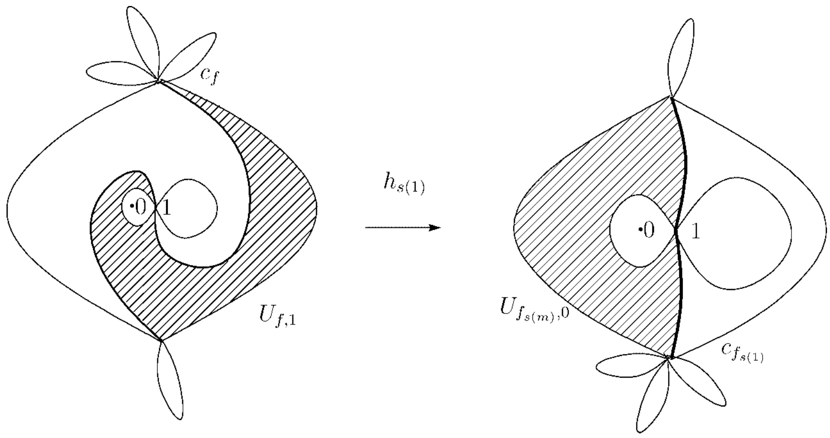

FIGURE 2. The case of $d = 3$ and $n = 1$.

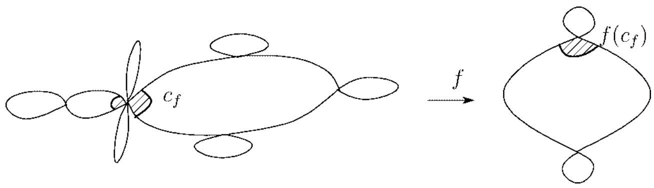

FIGURE 3. This figure shows the curve $|\varphi_f(z)| = \lambda^{-n}$ containing c_f.

First we label each point of $\partial D_f(0; \lambda^{-k}) \cap O_f^-(1)$ for all $k \geq 1$. Recall that f has a critical point 1 with multiplicity d_0, in other words f takes the form

$$f(w) = f(1) + a_{d_0}(w-1)^{d_0} + \cdots ,$$

where $d_0 \geq 2$ and $a_{d_0} \neq 0$. Since f restricted to $D_f(0; \lambda^{-j}) \setminus \{1\}$ is d_0-to-one for any given integer $j \geq 1$, the boundary $\partial D_f(0; \lambda^{-j})$ contains just d_0^j points of $O_f^-(1)$. Now we define

$$z(j, 0) = \lim_{y \downarrow 0} \psi_{f,0}(\lambda^{-j} + iy) ,$$

and moreover, we denote by $z(j, k)$ the k-th point of $O_f^-(1) \cap \partial D_f(0; \lambda^{-j})$ from $z(j, 0)$ along the curve $\partial D_f(0; \lambda^{-k})$ counterclockwise. For example, if $d = 3$ and $n = 1$, then $O_f^-(1) \cap \partial D_f(0; \lambda^{-1})$ consists of just two points $z(1, 0)$ and $z(1, 1)$ as in Figure 5.

Next, for any angle $\theta \in (0, 1)$, consider the curve

$$C_m(\theta) = \{\psi_{f,m} \circ l_m^{-1}(re^{2\pi i\theta}) \mid r > 0\} ,$$

which passes through a subarc of $\partial D_f(0; \lambda^{-j}) \cap U_{f,m}$. To describe such subarcs, we introduce the order relation $\prec_j$ in $\partial D_f(0; \lambda^{-j}) \cap O_f^-(1)$ as follows. For the positive orientation of

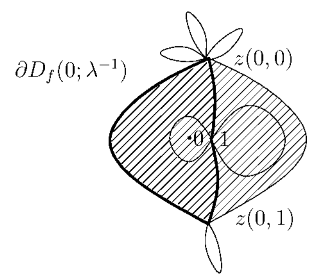

FIGURE 4. The case of $d = 3$ and $n = 2$.

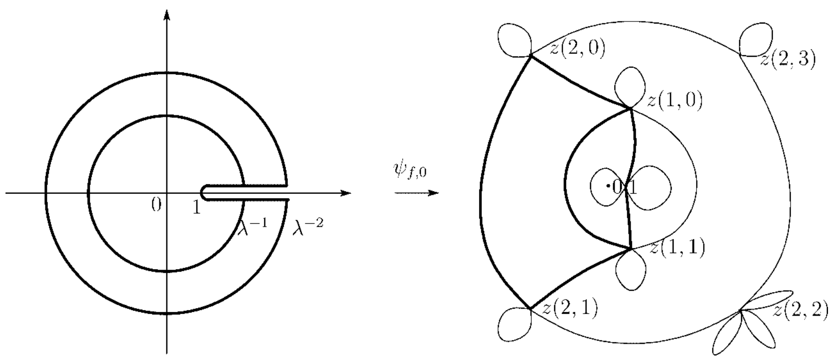

FIGURE 5. The case of $d = 3$ and $n = 2$.

$\partial D_f(0;\lambda^{-j})$ with respect to $D_f(0;\lambda^{-j})$, let $\gamma_j : [0,1) \to \partial D_f(0;\lambda^{-j})$ be an orientation preserving homeomorphism such that $\gamma_j(0) = z(j,0)$. For any points $z, w \in \partial D_f(0;\lambda^{-j})$ with $z \neq w$, we say that $z \prec_j w$ if $\gamma_j^{-1}(z) < \gamma_j^{-1}(w)$. As in Figure 6, let $A(j,k)$ be the open subarc of $\partial D_f(0;\lambda^{-j})$ which consists of all $z \in \partial D_f(0;\lambda^{-j})$ satisfying

$$z(j, k \bmod d_0^j) \prec_j z \prec_j z(j, k+1 \bmod d_0^j)\,.$$

It follows from straightforward computation that $C_m(\theta)$ passes through

$$A(1,m)\,,$$
$$A(2, m + d_0 m) = A(2, m(d_0+1))\,,$$
$$A(3, m + d_0 m(d_0+1)) = A(3, m(d_0^2 + d_0 + 1))\,,$$
$$\ldots,$$
$$\text{and } A\left(n, m + d_0 m \sum_{k=0}^{n-2} d_0^k\right) = A\left(n, m \sum_{k=0}^{n-1} d_0^k\right).$$

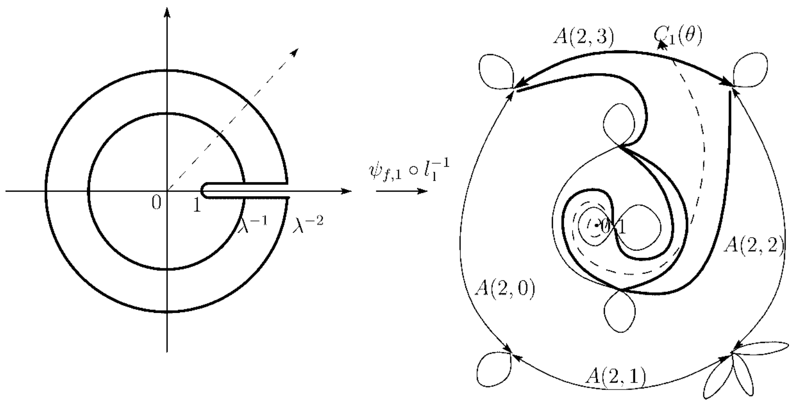

FIGURE 6. The case of $d = 3$ and $n = 2$.

Now suppose that $z(n, k_0) = c_f$, where k_0 is some integer with $1 \leq k_0 \leq d_0 - 1$. Then c_f and $z(n, k_0 + 1 \bmod d_0^n)$ are the end-points of $A(n, k_0)$. Let $M = (d_0 - 1)(-k_0 + d_0^n)$. Since $M \sum_{k=0}^{n-1} d_0^k \equiv k_0 \pmod{d_0^n}$, it follows that $C_M(\theta)$ passes through $A(n, k_0)$. Thus c_f is contained in the curve

$$\left\{ \lim_{\theta \to +0} \psi_{f,M} \circ l_M^{-1}(re^{2\pi i\theta}) \,\middle|\, r > 0 \right\},$$

and hence we have $c_{f_{s(M)}} = \lim_{y \downarrow 0} \psi_{f_{s(M)},0}(\lambda^{-n} + iy)$. □

PROOF OF MAIN THEOREM. For each λ with $0 < |\lambda| < 1$, consider the family of all polynomial maps $f \in \mathcal{H}_\lambda$ such that $f^{\circ n}(c_f) = 1$. Then by Proposition 2, we obtain $\mathcal{H}_\lambda \neq \emptyset$. In the case of $\lambda \in (0, 1)$, as in Lemma 6, we obtain $M_\lambda(f) \neq f$, i.e., $f_{s(1)} \neq f$. Moreover, it follows from Lemma 6 that the action of M_λ on $\mathcal{H}_\lambda$ is transitive. Now for any given complex number $\mu \in \mathbf{D}$, any polynomial map $g \in \mathcal{H}_\mu$ is quasiconformally conjugate to a polynomial map $f \in \mathcal{H}_\lambda$. Thus we obtain Main Theorem. □

4. Parabolic fixed points whose basin contains two distinct critical points

For a polynomial map $f_0 \in \mathcal{H}_\lambda$ with $\lambda \in (0, 1)$, we use the technique of the Haissinsky pinching deformations. The limit polynomial f_∞ of the Haissinsky pinching deformations of f_0 has a parabolic fixed point which attract two distinct critical points of f_∞. In this section, we classify such parabolic fixed points into four types, and moreover, for f_0 obtained by Lemma 6, we consider the type of the parabolic fixed point of the limit polynomial f_∞.

For a parabolic fixed point α, we define the *parabolic basin* to be the union of all Fatou component U such that the orbit of each point of U converges to α. Suppose that f has a parabolic fixed point α whose basin contains just two distinct critical points. A Fatou component of f is *critical* if the Fatou component contains at least one critical point. We denote by c_0 a critical point in the immediate parabolic basin, and denote by c_1 another critical point. Let U_k be the Fatou component which contains c_k, where $k = 0, 1$. Then there are four possibilities as follows.

CASE 1: The critical Fatou components are adjacent, i.e., $U_0 = U_1$.

CASE 2: The critical Fatou components are bitransitive. That is, the Fatou components satisfy that $U_0 \cap U_1 = \emptyset$, and moreover, there exist the smallest positive integers $p, q \geq 0$ satisfying $f^{\circ p}(U_0) = U_1$ and $f^{\circ q}(U_1) = U_0$.

CASE 3: The immediate parabolic basin captures U_1. That is, the immediate parabolic basin does not contain U_1, and hence contains $f^{\circ k}(U_1)$ for some integer $k \geq 1$.

CASE 4: There are two disjoint cycles in the immediate parabolic basin. That is, the immediate parabolic basin contains U_0 and U_1 which satisfy $f^{\circ n}(U_0) \cap f^{\circ m}(U_1) = \emptyset$ for any integers $n, m \geq 0$.

DEFINITION 7. For any parabolic fixed point whose basin contains just two distinct critical points, we will say that the type is *adjacent*, *bitransitive*, *capturing*, or *disjoint* if its critical Fatou components satisfy the Case 1, 2, 3, or 4, respectively.

Now suppose that the critical orbit relation $f_0^{\circ n}(c_{f_0}) = 1$ holds for an integer $n \geq 1$. Following [4], we define the pinching curves on $\mathcal{A}(f_0, 0)$ as follows.

First, we define straight lines. Let $p, q \geq 1$ be any integers with $0 < p/q < 1$. We define the lattice Λ by

$$\Lambda = \{-N \log \lambda + 2iM\pi \bigm| (N, M) \in \mathbf{N} \times \mathbf{Z}\},$$

and take the vector $\tau = q \log \lambda - 2ip\pi$. Moreover, let $L_y = \{t\tau + yi \bigm| t \in \mathbf{R}\}$, and let L be the union of all L_y with $L_y \cap \Lambda \neq \emptyset$.

Next, we define an f-invariant set of curves on $\mathcal{A}(f_0, 0)$. Let $\hat{V}_0$ be a connected component of the complement $\mathbf{C} \setminus \exp(L)$ whose boundary $\partial \hat{V}_0$ contains $\varphi_{f_0}(c_{f_0})$. The inverse map ψ_{f_0} of φ_{f_0} can be extended to the univalent map Ψ_{f_0} from $\hat{V}_0$ to $V_0 = \Psi_{f_0}(\hat{V}_0)$. Let $\hat{\gamma}_0$ be a curve such that each connected component of the inverse image under the exponential function is parallel to L_y, and let $\gamma_0 = \Psi_f(\hat{\gamma}_0)$. The union $S = \bigcup_{k=0}^{\infty} f^{-k}(\gamma_0)$ is called the *support* of the pinching.

It follows from [4] that there exists a sequence of quasiconformal deformations such that the limit function is a polynomial map with a parabolic fixed point at the origin as follows. There exists a sequence of quasiconformal maps $(h_t)_{t \geq 0}$ satisfying the following properties:

1. the limit function h_∞ is locally quasiconformal on $\mathbf{C} \setminus S$;
2. $h_t(0) = 0$, $h_t(1) = 1$ and $h_t(\infty) = \infty$;
3. h_t (resp. $f_t = h_t \circ f \circ h_t^{-1}$) converges uniformly on $\widehat{\mathbf{C}}$ to a locally quasiconformal map h_∞ (resp. a limit polynomial f_∞);
4. moreover, $h_\infty(f^{\circ k}(\overline{\gamma_0})) = 0$ for any integer $k \geq 1$.

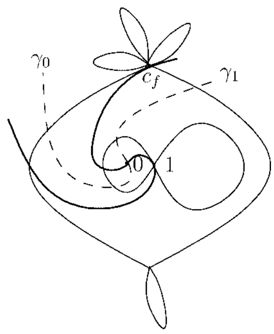

FIGURE 7. The pinching curves.

The *pinching deformation* is defined to be the sequence $(f_t = h_t \circ f_0 \circ h_t^{-1})_{t \geq 0}$. The limit function f_∞ is a polynomial map with a parabolic fixed point at the origin, and f_0 is semi-conjugate to f_∞ by h_∞, i.e., $h_\infty \circ f_0 = f_\infty \circ h_\infty$. The critical points of f_∞ is 1 and $h_\infty(c_{f_0})$. It follows from the equality $f_\infty^{\circ n}(h_\infty(c_{f_0})) = 1$ that the immediate parabolic basin of 0 for f_∞ contains 1.

Combining the pinching deformations and Main Theorem, we obtain the following.

COROLLARY 8. *For any given integer $n \geq 1$, there exists a polynomial map P in $\partial\mathcal{H}$ having a parabolic fixed point with the bitransitive type such that $P^{\circ n}(U_1) = U_0$ and $P^{\circ k}(U_1) \cap U_0 = \emptyset$ for all integer k with $0 \leq k < n$, where U_0 and U_1 are the critical Fatou components such that $1 \in U_0$ and $c_P \in U_1$.*

PROOF. By Main Theorem for any given integer $n \geq 1$, we obtain the polynomial map $f_0 \in \mathcal{H}_\lambda$ with $\lambda \in (0, 1)$ such that f satisfies $f_0^{\circ n}(c_{f_0}) = 1$ and $c_{f_0} = \lim_{y \downarrow 0} \psi_{f_0,0}(\lambda^{-1} + iy)$. Note that 1 is the first critical point of f_0 in $\mathcal{A}(f_0, 0)$.

We define a pinching curves separating two critical points. Let $p \geq 1$ and $q \geq n$ be any integers such that p/q is an irreducible fraction with $0 < p/q < 1$. Define γ_0, V_0 and S as above. Let $\hat{C}$ be any smooth open arc with end-points 0 and λ^{-n} satisfying $\hat{C} \cap [1, \infty) = \emptyset$ and $\hat{C} \subset \hat{V}_0$. Then the smooth curve $C = \psi_{f_0,0}(\hat{C})$ has end-points 0 and c_{f_0}, and hence we obtain $c_{f_0} \in \partial V_0$ as in Figure 7. Thus the cycle of connected components $\mathcal{A}(f_0, 0) \setminus S$ contains both critical points. Let $f_t = h_t \circ f_0 \circ h_t^{-1}$ be the Haissinsky pinching deformation of f_0 defined by the support S. Since the image of this cycle under h_∞ is contained in the immediate parabolic basin of 0 for f_∞, we obtain Corollary 8. □

ACKNOWLEDGEMENT. The author expresses his thanks to the referee for helpful comments for improving the paper.

References

[1] A. F. BEARDON, *Iteration of rational functions*, *Complex analytic dynamical systems*, Graduate Texts in Mathematics, 132, Springer-Verlag, New York, 1991.

[2] B. BRANNER and J. H. HUBBARD, The iteration of cubic polynomials. I. The global topology of parameter space, Acta Math. **160** (1988), no. 3-4, 143–206.

[3] B. BRANNER and J. H. HUBBARD, The iteration of cubic polynomials. II. Patterns and parapatterns, Acta Math. **169** (1992), no. 3–4, 229–325.

[4] P. HAISSINSKY, Pincement de polynômes, Comment. Math. Helv. **77** (2002), no. 1, 1–23.

[5] Y. KOMORI and SH. NAKANE, Landing property of stretching rays for real cubic polynomials, Conform. Geom. Dyn. **8** (2004), 87–114.

[6] O. LEHTO and K. VIRTANEN, *Quasiconformal mappings in the plane*, Second edition, Springer-Verlag, New York-Heidelberg, 1973.

[7] J. MILNOR, Remarks on iterated cubic maps, Experiment. Math. **1** (1992), no. 1, 5–24.

[8] J. MILNOR, *Dynamics in one complex variable*, Friedr. Vieweg & Sohn, Braunschweig, 2000.

[9] C. PETERSEN and TAN LEI, Branner-Hubbard motions and attracting dynamics, *Dynamics on the Riemann sphere*, *A Bodil Branner Festschrift* (ed. P. HJORTH and C. PETERSEN), EMS., 2006, 45–70.

[10] C. PETERSEN and TAN LEI, *The central hyperbolic component of cubic polynomials*, prepublication de l'Universite de Cergy-Pontoise.

[11] TAN. LEI, On pinching deformations of rational maps, Ann. Sci. École Norm. Sup. (4) **35** (2002), no. 3, 353–370.

Present Address:
DEPARTMENT OF MATHEMATICS,
GAKUSHUIN UNIVERSITY,
MEJIRO, TOSHIMA-KU, TOKYO, 171–8588 JAPAN.