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Strongly n-trivial Links are Boundary Links

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Abstract. A link is said to be *strongly n-trivial* if there exists a diagram such that one can choose $n+1$ crossing points with the property that changing crossings on any $0 < m \leq n+1$ points of these $n+1$ points yields a trivial link. It is shown that for a positive integer n the components of a strongly n-trivial link admit mutually disjoint Seifert surfaces.

1. Introduction

Let n be a non-negative integer. A knot (or link) L in S^3 is said to be *strongly n-trivial* if there exists a diagram of L such that one can choose $n+1$ crossing points with the property that changing crossings on any $0 < m \leq n+1$ points of these $n+1$ points yields a trivial knot (or link). The collection of such crossing changes is called a *strong n-trivializer* for L. The link illustrated in Figure 1 is a non-trivial strongly 1-trivial link, where the strong 1-trivializer is indicated by dotted circles. H. Howards and J. Luecke [9] gave a construction of non-trivial strongly n-trivial knots for any given positive integer n via "finger moves" [9, Section 6]. Strongly n-trivial links can be constructed as in Figure 2. Given a strongly n-trivial link, it is not so easy to detect the triviality of the link. Conversely, N. Askitas and E. Kalfagianni [1] showed that any strongly n-trivial knot is obtained from the unknot by "finger moves" on a Brunnian Suzuki graph.

It is well-known that any Vassiliev invariant of order $\leq n$ vanishes for strongly n-trivial knots. Further, Askitas and Kalfagianni [1] showed that if a knot K is strongly n-trivial for $n \geq 2$, then $\nabla_K(z) = 1$ [1, Theorem 1.2]. For a strongly 1-trivial knot K, it can be seen that $a_2(K) = 0$ or ± 1, where $a_2(K)$ denotes the second coefficient of the Conway polynomial $\nabla_K(z)$, according to whether the trivializers are "unlinked" or "linked". This has been observed by T. Stanford (cf. [11]). Note that every strongly n-trivial knot has unknotting number one. It is well-known that every Alexander-Conway polynomial of knots is realized as that of unknotting number one knots (cf. [12], [13], [4], [19]).

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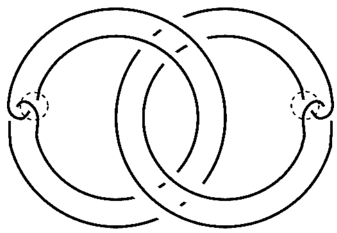

FIGURE 1. A strongly 1-trivial link.

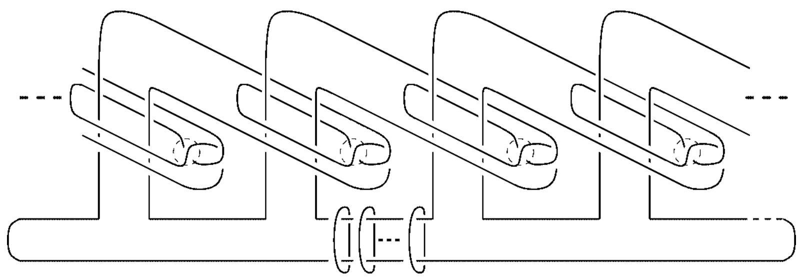

FIGURE 2

The strong n-triviality also gives some geometric restrictions. Several authors study minimal genus Seifert surfaces for strongly n-trivial knots by using techniques in 3-dimensional topology. Howard and Luecke [9] showed that if K is a non-trivial strongly n-trivial knot, then n is less than $6g(K) - 3$, where $g(K)$ denotes the genus of K. In particular, the trivial knot is the only knot that is strongly n-trivial for all n ([9, Corollary 1.3]). In [17] and [18], I. Torisu study strongly n-trivial 2-bridge knots and links.

THEOREM 1.1 ([17, Theorem 1.1]). *A 2-bridge knot K is strongly n-trivial for $n \geq 1$ if and only if K is the trivial knot or the trefoil knot or the figure-eight knot.*

THEOREM 1.2 ([18]). *There is no non-trivial strongly n-trivial 2-bridge link for $n \geq 1$.*

In this note, we give some other geometric restriction to strongly n-trivial links. A link is called a *boundary link* if the components bound mutually disjoint Seifert surfaces. Note that a trivial link is a boundary link. It is well-known that if L is a boundary link, then $\Delta_L(t) = 0$. We also remark that the exterior $E(L)$ of a boundary link L contains a closed incompressible surface which is not ∂-parallel (Proposition 4.3). Then, we show the following as a main result of this note.

THEOREM 1.3. *Every strongly n-trivial link is a boundary link for $n \geq 1$.*

As applications of Theorem 1.3 we have the following:

COROLLARY 1.4. *For a strongly n-trivial link L, $\Delta_L(t) = 0$ for $n \geq 1$.*

COROLLARY 1.5. *The exterior of a non-splittable strongly n-trivial link contains a closed essential surface of positive genus for $n \geq 1$.*

COROLLARY 1.6. *For a non-trivial strongly n-trivial link L with μ components, the rank of $\pi_1(S^3 - L)$ is greater than or equal to $\mu + 1$ for $n \geq 1$.*

Recall that for a 2-bridge knot/link $L = S(a, b)$ in the Schubert form, $\Delta_L(-1) = a$ since the double branched covering space is the Lens space of type (a, b). Recall also that the exterior of a 2-bridge link does not contain closed essential surfaces (cf. [7], [3]) and the rank of the fundamental group is two. Now we can reprove Theorem 1.2 by using any one of these facts together with Theorem 1.3.

2. On Dehn surgery creating reducible 3-manifolds

See [8] and [10] for basic terminology in 3-dimensional topology, and [14] for surgery, Seifert surfaces, boundary links. Throughout this paper, $N(\cdot)$ denotes the regular neighborhood and $E(\cdot)$ denotes the exterior, namely the complement of the interior of $N(\cdot)$. Unless stated otherwise, a link has at least two components. We consider Seifert surfaces for links L as properly embedded surfaces S in the link exteriors $E(L)$ such that for each component T of $\partial E(L)$, $T \cap S$ is a single circle intersecting a meridian exactly in one point.

We use the following results on non-trivial surgeries along a knot in reducible or ∂-reducible 3-manifolds that yield reducible or ∂-reducible 3-manifolds. In [16], M. Scharlemann studied surgeries producing reducible 3-manifolds. The following is a special case of his result.

THEOREM 2.1 (cf. [16, Theorem 6.1]). *Let M be a ∂-reducible 3-manifold. Let K be a knot in M such that the exterior $E(K)$ is irreducible and ∂-irreducible. If a non-trivial surgery along K yields a reducible 3-manifold, then K is a cable knot, and the surgery slope is that of the cabling annulus.*

In particular, on Theorem 2.1 if a non-trivial surgery along K yields a reducible 3-manifold M', then M' is a Lens space summand. In [5], D. Gabai showed that if a non-trivial surgery along a knot K in a solid torus yields a solid torus, then K is 0- or 1-bridge braid [5, Theorem 1.1 (1)]. The set of 1-bridge braids was classified in [6, Section 2]. In [2], J. Berge classified such knots and such non-trivial surgeries. See also [6, Section 3]. Then, we deduce the following lemma from [5, Theorem 1.1 (1)].

LEMMA 2.2. *Let K be a null-homologous knot in a solid torus V. If a non-trivial surgery along K yields a solid torus, then K bounds a disk in V.*

We have the following lemma by using these results.

LEMMA 2.3. *Let L_0 be a trivial link in S^3. Let ℓ be a trivial knot in S^3 which is disjoint from L_0. Suppose that ℓ is null-homologous in $E(L_0)$. If the result of twisting L_0 along ℓ is a trivial link, then ℓ bounds a disk in $E(L_0)$.*

PROOF. Note that the exterior of a trivial link is reducible and ∂-reducible. Suppose that $E(L_0 \cup \ell)$ is irreducible and ∂-irreducible. Then, by Theorem 2.1, ℓ is a non-trivial cable about some knot in $E(L_0)$ and the surgery slope is given by the cabling annulus. This implies that the result of the surgery on ℓ is a Lens space summand, a contradiction since $H_1(E(L))$ is torsion free. Suppose that $E(L_0 \cup \ell)$ is ∂-reducible. Then some component of $L_0 \cup \ell$ bounds a disk whose interior is disjoint from $L_0 \cup \ell$. If ℓ bounds a disk, we are done. Otherwise, some component of L_0 bounds a disk D, and $E(L_0 \cup \ell)$ is reducible since $\partial N(D)$ is a separating sphere in $E(L_0)$ not bounding 3-balls. Now we may assume that $E(L_0 \cup \ell)$ is reducible. Then there is a sphere S in $E(L_0 \cup \ell)$ which separates S^3 into two 3-balls B_1, B_2 such that $B_i \cap (L_0 \cup \ell)$ is non-empty for $i = 1, 2$, $\ell \subset B_1$, and $E(B_1 \cap (\ell \cup L_0))$ is irreducible (cf. [8], [10]). The following lemma is easy to see.

LEMMA 2.4. *If a link in a 3-ball is trivial in S^3, then the components of the link bound mutually disjoint disks in the 3-ball.*

Thus, if $B_1 \cap (L_0 \cup \ell) = \ell$, then ℓ bounds a disk in B_1 by Lemma 2.4 since ℓ is a trivial knot in S^3. Suppose that B_1 contains exactly one component k_1 of L_0. Then each of k_1 and the result of twisting k_1 along ℓ is a trivial knot. Hence, by Lemma 2.2, ℓ bounds a disk in $E(k_1)$, and in $B_1 - k_1$ by Lemma 2.4 and the conclusion follows. Suppose that B_1 contains more than one components of L_0. Then the conclusion follows by Theorem 2.1 for the link $L_0' = B_1 \cap L_0$ in S^3 and $\ell' = B_1 \cap \ell$, since each of L_0' and the result of twisting L_0' along ℓ' is trivial by Lemma 2.4. That is, ℓ' bounds a disk in B_1, a conclusion for L_0 and ℓ as required. This completes the proof of Lemma 2.3. □

3. Proof of Theorem 1.3

For convenience, we introduce (n, m)-triviality for links and prove the following:

THEOREM 3.1. *For any positive integer n, each $(n+1, n)$-trivial link is a boundary link.*

For positive integers n and m such that $n \geq m$, we say that a link L is (n, m)-*trivial* if there is a diagram of L such that one can choose n crossing points with the property that for any integer n' with $n \geq n' \geq m$, changing crossings of any n' points of the n points yields a trivial link. Equivalently, for an (n, m)-trivial link L, there is a diagram of a trivial link L_0 which includes n crossing points with the property that changing crossings on the n points yields L and for any integer m' with $n - m \geq m' \geq 0$, chaining crossings on any m' points of the n points yields a trivial link. Observe the following by definition:

PROPOSITION 3.2. (a) *A link is strongly n-trivial if and only if it is $(n+1, 1)$-trivial.*

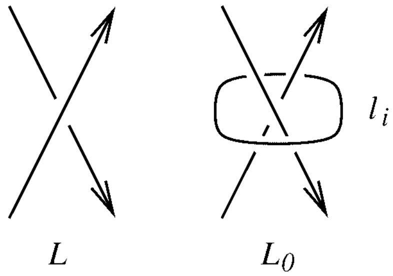

FIGURE 3. The -1-surgery along ℓ_i changes L_0 into L.

(b) *An (n, m)-trivial link is $(n, m+1)$-trivial for $n > m$.*

Then we show that $(n+1, n)$-trivial links admit surgery descriptions as follows:

LEMMA 3.3. *For an $(n+1, n)$-trivial μ-component link $L = K_1 \cup \cdots \cup K_\mu$, there is an $(n+\mu+1)$-component link $k_1 \cup \cdots \cup k_\mu \cup \ell_{\mu+1} \cup \cdots \cup \ell_{n+\mu+1}$ such that $L_0 = k_1 \cup \cdots \cup k_\mu$ is a trivial link, L is obtained from L_0 by suitable ± 1-surgeries along ℓ_i's, and each ℓ_i bounds a disk in $E(L_0)$.*

PROOF. Let L_0 denote the trivial link obtained from L by the crossing changes on the $n+1$ crossing points. We put a circle ℓ_i near each of the $n+1$ crossing points as in Figure 3 so that $\operatorname{lk}(\ell_i, L_0) = 0$. Then, we can recover L from L_0 by the corresponding ± 1-surgeries along all ℓ_i. Note that the link L_0^i obtained from L_0 by performing a suitable ± 1-surgery along a component ℓ_i is a trivial link for $i = \mu+1, \ldots, n+\mu+1$ since L is $(n+1, n)$-trivial. By the $(n+1, n)$-triviality of L and counting the linking numbers, we see that each of the $n+1$ crossings is made by the same component of L_0, a "self-crossing". Thus, each ℓ_i is null-homologous in $E(L_0)$ since $\operatorname{lk}(\ell_i, L_0) = 0$. Now by Lemma 2.3, we see that each ℓ_i bounds a disk in $E(L_0)$. This completes the proof. □

LEMMA 3.4. *Let $L_0 = k_1 \cup \cdots \cup k_\mu$ be a boundary link. Let $\ell_{\mu+1} \cup \cdots \cup \ell_{n+\mu+1}$ be a link in $E(L_0)$. Suppose that each ℓ_i bounds a disk in $E(L_0)$. Then $k_1, k_2, \ldots, k_\mu$ bound mutually disjoint Seifert surfaces disjoint from the ℓ_i's.*

PROOF. Let $S_1, S_2, \ldots, S_\mu$ be mutually disjoint Seifert surfaces for k_i's, and D_j the disk bounded by ℓ_j. We may assume that S_i and D_j are in general position. Then, $(S_1 \cup \cdots \cup S_\mu) \cap D_j$ consists of circles and proper arcs. If $(S_1 \cup \cdots \cup S_\mu) \cap D_j$ is empty or consists of circles for each j, we are done. Suppose $(S_1 \cup \cdots \cup S_\mu) \cap D_j$ has an arc component for some j, and let α be an outermost arc of D_j with respect to the arc components. Let β be the segment of ℓ_j corresponding to the outermost disk. Then, α is contained in some S_i and β is an arc such that the interior is disjoint from $S_1 \cup \cdots \cup S_\mu$ and, since α is bi-collared in D_j, the ends of β are on the same side of S_i. Attaching a tube to S_i along β, we obtain a new system of disjoint Seifert surfaces S_i' for the components of L_0. This procedure reduces the number of arc components of $(S_1 \cup \cdots \cup S_\mu) \cap D_j$ (and may produce circle components). Repeating

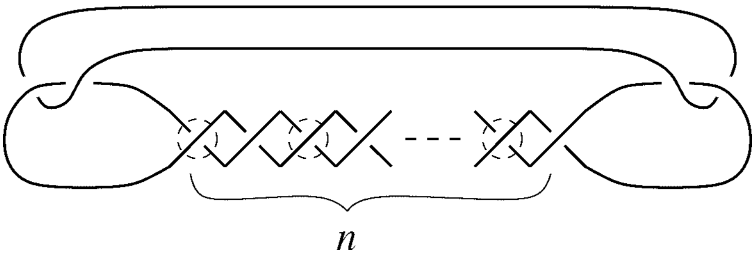

FIGURE 4. This link is not strongly n-trivial but (n, n)-trivial.

this procedure for $j = \mu + 1, \mu + 2, \ldots, n + \mu + 1$, we get a desired system of disjoint Seifert surfaces. This completes the proof. □

Now we are ready to prove Theorem 3.1.

PROOF OF THEOREM 3.1. Let L_0, ℓ_i, k_j be as in Lemma 3.3. Then L_0 is a boundary link since it is a trivial link. By Lemma 3.4, there are mutually disjoint Seifert surfaces $S_1, \ldots, S_\mu$ for k_i's disjoint from any of ℓ_i. Now S_j's become mutually disjoint Seifert surfaces S_i^*'s for K_i's after corresponding ± 1-surgeries along ℓ_i. This completes the proof of Theorem 3.1. □

PROOF OF THEOREM 1.3. This follows from Proposition 3.2 and Theorem 3.1. □

We remark that Theorem 3.1 (and the corollaries corresponding to Corollaries 1.4, 1.5, 1.6) are sharp in the sense that the link illustrated in Figure 4 is a 2-bridge (n, n)-trivial link but is not a boundary link.

The (n, m)-triviality gives a wider class than strongly n-trivial knots and links. Given a positive integer $n \geq 3$, let n_1, n_2 be positive integers such that $n - 1 = n_1 + n_2$. Then for a strongly n_1-trivial knot K_1 and a strongly n_2-trivial knot K_2, it is easy to see that the composition $K = K_1 \# K_2$, which is not an unknotting number one knot by Scharlemann [15], is an $(n + 1, n)$-trivial knot but is not strongly n-trivial. Notice that if L is unknotting number n, then L is (n, n)-trivial. There are a plenty of links that are boundary links but not strongly n-trivial for any $n \geq 1$, that is, the opposite direction of Theorem 1.3 is not true. However, it appears unknown whether any boundary link is $(n, n - 1)$-trivial for some n.

4. Proofs of Corollaries

It is well-known that $\Delta_L(t) = 0$ for any boundary link L and hence Corollary 1.4 follows. Corollary 1.5 follows from Theorem 1.3 and Lemma 4.3. Corollary 1.6 follows from Theorem 1.3 and Lemma 4.4.

LEMMA 4.1. *The components of a μ-component boundary link L admit mutually disjoint Seifert surfaces $S_1, \ldots, S_\mu$ such that each S_i is incompressible in $E(L)$.*

PROOF. We refer to $\sum g(S_i)$ as a complexity for the system $S_1, \dots, S_\mu$ of mutually disjoint Seifert surfaces for the components of L. Let S_i give the minimal complexity among disjoint Seifert surfaces. Then, we shall prove that, replacing S_i's if necessary by Seifert surfaces of the same complexity, each S_i is incompressible in $E(L)$. If not, there is an embedded disk D in $E(L)$ such that $D \cap S_i = \partial D$ and ∂D does not bound a disk in S_i. We may assume that D and the other S_j's are in general position and $D \cap (S_1 \cup \cdots \cup S_\mu)$ is non-empty and consists of circles including ∂D. Let D' be an innermost disk of D with respect to $D \cap (S_1 \cup \cdots \cup S_\mu)$, and S_k the component containing $\partial D'$. If $\partial D'$ does not bound a disk in S_k, then the compression of S_k along D' yields the ones with smaller complexity, a contradiction. If $\partial D'$ bounds a disk D'' in S_j, then replace S_j with $S'_j = (S_j - D'') \cup D'$ to obtain a new system with the same complexity. Repeating this process finitely many times, we reach a contradiction to the minimality of the complexity. This completes the proof. □

LEMMA 4.2. *Let M be a compact, orientable, irreducible 3-manifold with non-empty boundary ∂M. If ∂M is disconnected, then M contains a closed incompressible surface of positive genus.*

PROOF. We construct disjoint surfaces F_i in M successively as follows: Start with a component F_0 of ∂M and put $F_1 = \partial N(F_0, M) - F_0$. Put $W_1 = N(F_0, M)$. If F_i has a sphere component P, then by the irreducibility P bounds a 3-ball C on the side not containing W_i and regard $F_i - P$ as F_i and $W_i \cup C$ as W_i. At this stage F_i is not empty because F_i separates F_0 from $\partial M - F_0$. Then, each component of F_i has positive genus and $W'_i = W_1 \cup \cdots \cup W_i$ is a compression body in which F_i is incompressible. If F_i is incompressible in M, we stop. If F_i is compressible in M, then it is compressible in $\mathrm{cl}(M - W'_i)$ and we compress F_i to obtain a new surface F_{i+1} and write W_{i+1} the compression body between F_i and F_{i+1}. Since the complexity of F_{i+1} is fewer than that of F_i, F_n is incompressible for some n. Now it is elementary to show that each component of F_n is incompressible in M. □

LEMMA 4.3. *The exterior of a non-splittable boundary link contains a closed essential surface of positive genus.*

PROOF. A non-splittable boundary link L with μ-component ($\mu \geq 2$) admits mutually disjoint incompressible Seifert surfaces $S_1, S_2, \dots, S_\mu$ by Lemma 4.1. If $E(S_1 \cup S_2 \cup \cdots \cup S_\mu)$ has a reducing sphere R, then R becomes a splitting sphere of L. Hence $E(S_1 \cup S_2 \cup \cdots \cup S_\mu)$ is irreducible. By Lemma 4.2, $E(S_1 \cup S_2 \cup \cdots \cup S_\mu)$ contains a closed incompressible surface F of positive genus. By the incompressibility of S_i and the irreducibility of $E(S_1 \cup S_2 \cup \cdots \cup S_\mu)$ we see that F is incompressible in $E(L)$ by the standard innermost disk argument. If F is parallel to a component of $\partial E(L)$, then F bounds a solid torus V in S^3 whose core is a component k_i of L. However k_i bounds the Seifert surface S_i in V. This implies that k_i is null-homologous in V and cannot be a core, a contradiction. Hence F is a desired one. This completes the proof. □

LEMMA 4.4. *Let L be a non-trivial μ-component boundary link for $\mu > 1$. Then the rank of $\pi_1(S^3 - L)$ is greater than or equal to $\mu + 1$.*

PROOF. Let $S_1, \ldots, S_\mu$ be disjoint Seifert surfaces for the components of L. Let B_μ be the bouquet of μ loops. Then there is a continuous map $f : E(L) \to B_\mu$ such that $f(E(S_1 \cup \cdots \cup S_\mu))$ is the vertex of B_μ and $f(N(S_i))$ is the ith loop of B_μ. Notice that f is a surjection which induces an epimorphism $f_* : \pi_1(S^3 - L) \to F_\mu$, where F_μ is the free group of rank μ. Therefore the rank of $\pi_1(S^3 - L)$ is greater than or equal to μ. If $\pi_1(S^3 - L)$ is generated by exactly μ elements, then it can be shown that f_* is injective since F_μ is the free group of rank μ and hence $\pi_1(S^3 - L)$ is a free group. This implies that L is trivial, a contradiction. □

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