TOKYO J. MATH.
VOL. 30, NO. 1, 2007

Ergodic Measures of SRB Attractors

Jin HATOMOTO

Tokyo Metropolitan University

(Communicated by M. Okada)

Abstract. In the stochastic context, an invariant set is decomposed into the union of ergodic basins, and each of basin possesses the fractal structure determined by ergodic measures. This paper is to show that when a hyperbolic SRB measure is mixing, the set of measures with zero entropy and the set of measures with positive entropy but without SRB are both dense on the set of all invariant measures on the closure of the ergodic basin in the Pesin set, and moreover that in the set of invariant measures as above a measure with ergodicity and SRB exists uniquely.

1. Introduction

The set of invariant Borel probability measures of a compact metric space is compact convex with respect to the weak $*$-topology and its extreme points are ergodic measures. Every invariant measure is decomposed by the ergodic measures ([12]). This means that ergodic measures play an important role in the study of stochastic dynamics.

The set of points satisfying Birkhoff's ergodic theorem for any continuous function is called the *ergodic basin*. If the ergodic basin has the positive Lebesgue measure, then the measure is said to be a *Sinai-Ruelle-Bowen measure* (abbrev. SRB measure). Sinai, Ruelle and Bowen showed the existence of an SRB measure for a hyperbolic attractor ([6]). Our aim of this paper is to investigate a characteristic of the set of ergodic measures under the assumption of the existence of an SRB measure.

In the context of nonuniformly hyperbolic system, the theory of SRB measures has been developed by Pesin, Katok, Ledrappier, Young and several other mathematicians ([15], [20], [23]).

In [23] Pugh and Shub proved that a hyperbolic measure satisfying SRB condition is an SRB measure. Here let us say that an invariant measure μ satisfies *SRB condition* if μ has absolutely continuous conditional measures on unstable manifolds.

Firstly, we shall show that if a hyperbolic ergodic measure satisfies SRB condition, then its support is an SRB attractor (which is called an ergodic attractor in [23]) and that the attractor has similar properties to hyperbolic attractors (Theorem 3.2).

Received October 25, 2005
2000 *Mathematics Subject Classification*: Primary 37C50, 37D25.
Key Words and Phrases: SRB attractors, SRB condition, Shadowing property.

Secondly, under the same assumption, the set of ergodic measures will be divided into several classes according to their entropy and some of them satisfy the properties as stated in [27] (Theorem 4.1). A key point of the investigation is a theorem (Theorem 2.4), which is a version of nonuniformly shadowing property obtained by Katok ([12], Theorem S.4.14) and Pollicott ([22], Theorem 5.1).

Finally, we shall find several properties of measures satisfying SRB condition or absolute continuity (Theorem 5.4, Theorem 5.7).

Throughout this paper, let f be a C^2-diffeomorphism of an n-dimensional closed manifold M and μ be an f-invariant Borel probability measure on M. We denote by $d(\cdot,\cdot)$ and $\|\cdot\|$ the distance and the norm induced by the Riemannian metric $\langle\cdot,\cdot\rangle$ on M respectively. Let m denote the Lebesgue measure on M. The *support* of μ is the set of all $x \in M$ satisfying that $\mu(U) > 0$ for any neighborhood U of x, and is denoted by $\mathrm{Supp}(\mu)$. To simplify the notation we will often put $S = \mathrm{Supp}(\mu)$.

An f-invariant set A is called an *SRB attractor* or an *ergodic attractor* of μ if the set A has full μ-measure and there exists a set $W \subset M$ with positive Lebesgue measure such that (i) for $w \in W$, $\mathrm{dist}(f^n(w), A) = \inf\{d(f^n(w), y) : y \in A\} \to 0$ as $n \to \infty$, (ii)if we set

$$R(\mu) = \left\{ x \in M \;\middle|\; \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi d\mu \quad (\varphi : M \to \mathbf{R}\text{: continuous}) \right\}$$

(which is called an *ergodic basin* of μ), then $W \subset R(\mu)$ except for an m-null set. We remark that μ is an SRB measure if and only if there exists an SRB attractor A of μ(Remark 3.1(a)).

By [19], there exists a set $Y_\mu \subset S = \mathrm{Supp}(\mu)$ with full μ-measure such that every $x \in Y_\mu$ has a $D_x f$-invariant decomposition $\mathrm{T}_x M = \bigoplus_{i=1}^{t(x)} E_i(x)$ into subspaces $E_i(x)$ and real numbers $\chi_1(x) > \chi_2(x) > \cdots > \chi_{t(x)}(x)$ $(1 \le t(x) \le n)$ which satisfy the following properties:

$$(1)\quad \lim_{n\to\pm\infty} \frac{1}{n} \log \|D_x f^n(v)\| = \chi_i(x) \quad (v \in E_i(x)\backslash\{0\}),$$

$$(2)\quad \lim_{n\to\pm\infty} \frac{1}{n} \log \sin(\angle(D_x f^n(E_i(x)), D_x f^n(E_j(x)))) = 0 \quad (i \ne j,\ 1 \le i, j \le t).$$

Here $D_x f$ denotes the derivative of f at x and $\angle(A, B) = \min\left\{ \cos^{-1} \frac{|\langle u,v\rangle|}{\|u\|\|v\|} \mid u \in A,\ v \in B \right\}$ for subspaces $A, B \subset \mathbf{R}^n$. For $x \in Y_\mu$, $\dim E_i(x)$ is the dimension of $E_i(x)$, which means the multiplicity of $\chi_i(x)$. We call $\chi_i(x)$ $(1 \le i \le t(x) \le n)$ *Lyapunov exponents* of μ at $x \in Y_\mu$. If μ is ergodic, then $t(x)$, $\chi_i(x)$ and $\dim E_i(x)$ are constants $t(x) = t$, $\chi_i(x) = \chi_i$ and $\dim E_i(x) = \dim E_i$ for μ-a.e.x.

An invariant measure μ is called *hyperbolic* if all Lyapunov exponents of μ are different from 0.

Here after assume that μ is ergodic and hyperbolic, and put

$$\chi^s = \max\{\chi_i < 0 \,|1 \le i \le t\}, \quad \chi^u = \min\{\chi_i > 0 \,|1 \le i \le t\},$$

$$E^s(x) = \bigoplus_{i:\chi_i<0} E_i(x), \quad E^u(x) = \bigoplus_{i:\chi_i>0} E_i(x).$$

Then these subspaces are $D_x f$-invariant and satisfy $T_x M = E^s(x) \oplus E^u(x)$. Fix $\varepsilon > 0$ sufficiently small and write Λ_l ($l > 0$) the set of all points x satisfying the following:

(i) for $v \in E^s(x)$ and $n > 0$

$$\|D_x f^n(v)\| \leq \exp(2\varepsilon l)\exp(n(\chi^s + \varepsilon))\|v\|,$$

(ii) for $v \in E^u(x)$ and $n > 0$

$$\|D_x f^{-n}(v)\| \leq \exp(2\varepsilon l)\exp(n(-\chi^u + \varepsilon))\|v\|,$$

(iii) for $n \in \mathbf{Z}$

$$\sin(\angle(D_x f^n(E^s(x)), D_x f^n(E^u(x)))) \geq \exp(-\varepsilon(l + |n|)).$$

Then

(3) Λ_l is a closed set,

(4) $f(\Lambda_l) \subset \Lambda_{l+1}$ for $l > 0$,

(5) the above subspaces $E^s(x)$ and $E^u(x)$ depend continuously on $x \in \Lambda_l$,

(6) $\Lambda = \bigcup_{l=1}^{\infty} \Lambda_l$ is f-invariant.

$\Lambda = \bigcup_{l=1}^{\infty} \Lambda_l$ is said to be a *Pesin set* with respect to μ (abbrev. w.r.t. μ).

It is well known (see [20]) that for every $x \in \Lambda$ there exist the *local stable* and *unstable manifolds* $W^s_{loc}(x)$ and $W^u_{loc}(x)$ such that

$$f(W^s_{loc}(x)) \subset W^s_{loc}(f(x)), \quad f^{-1}(W^u_{loc}(x)) \subset W^u_{loc}(f^{-1}(x))$$

and $E^\sigma(x) = T_x W^\sigma_{loc}(x)$ ($\sigma = s, u$). The *stable* and *unstable manifolds* $W^s(x)$ and $W^u(x)$ are defined by

$$W^s(x) = \bigcup_{n\geq 0} f^{-n}(W^s_{loc}(f^n(x))), \quad W^u(x) = \bigcup_{n\geq 0} f^n(W^u_{loc}(f^{-n}(x))).$$

Let $\mathcal{B}$ be the Borel σ algebra. For a measurable partition ξ of M denote by $\mathcal{B}_\xi$ the set of all Borel subsets which consist of the unions of the elements of ξ. A measurable partition ξ defines a family of measures $\{\mu^\xi_x\}$ (μ-a.e.x) such that for μ-a.e.x and $B \in \mathcal{B}$, $\mu^\xi_x(B)$ is a $\mathcal{B}_\xi$-measurable function of x and

$$\mu(E \cap B) = \int_E \mu^\xi_x(B)\,d\mu(x) \quad (E \in \mathcal{B}_\xi).$$

If there exists a sequence $\{\xi_i\}_{i\geq 1}$ of countable measurable partitions such that

$$\xi_1 \leq \xi_2 \leq \cdots \leq \vee_{i\geq 1}\xi_i = \xi,$$

then $\mu^\xi_x(\xi(x)) = 1$ where $\xi(x)$ denotes the element of ξ containing x. The family of measures $\{\mu^\xi_x\}$ (μ-a.e.x) is said to be *the canonical system* of *conditional measures* of μ w.r.t. ξ.

We assume that a measurable partition ξ^u of M is *subordinate to the W^u-foliation*, that is, ξ^u satisfies that (7) $\xi^u(x) \subset W^u(x)$ and (8) $\xi^u(x)$ contains an open set in $W^u(x)$ for μ-a.e.x. Let $\{\mu_x^u\}$ (μ-a.e.x) denote a canonical system of conditional measures of μ w.r.t. ξ^u and m_x^u denote the Lebesgue measure on $W^u(x)$. If μ_x^u is absolutely continuous w.r.t. m_x^u for μ-a.e.x ($\mu_x^u \ll m_x^u$), then we say that μ *satisfies SRB condition* (for f).

It is well known that μ satisfies SRB condition for f if and only if the measure theoretic entropy $h_\mu(f)$ has

$$h_\mu(f) = \sum_{i:\chi_i>0} \chi_i \dim E_i$$

([13] Theorem 4.8, [14] Theorem 1.2, [15] Theorem A). It follows from the proof of Theorem A in [15] that if μ satisfies SRB condition for f then

$$\mu_x^u \sim m_x^u|_{\xi^u(x)} \quad (\mu\text{-a.e.}x). \tag{1.1}$$

Here the notation $\mu_x^u \sim m_x^u|_{\xi^u(x)}$ indicates that both relations $\mu_x^u \ll m_x^u|_{\xi^u(x)}$ and $\mu_x^u \gg m_x^u|_{\xi^u(x)}$ hold. If a hyperbolic measure μ satisfies SRB condition for f, then its support is an SRB attractor (Remark 3.1(b)).

2. Nonuniformly Shadowing Property

Let μ be a hyperbolic ergodic measure and Λ be a Pesin set w.r.t. μ. For $h \in (0,1]$ we denote by $B^s(h) \subset \mathbf{R}^{\dim E^s}$ ($B^u(h) \subset \mathbf{R}^{\dim E^u}$) be an s-closed ball (u-closed ball) centered at 0 of radius h w.r.t. the Euclidean norm. For $\gamma \in (0,1)$, $\delta \geq 0$ and $h \in (0,1]$ we define

$$\mathcal{U}_0^{\gamma,\delta} = \{\mathrm{graph}(\varphi) \mid \varphi : B^u(h) \to B^s(h) \text{ is a } C^1 \text{ map satisfying } |D\varphi| \leq \gamma, |\varphi(0)| \leq \delta\},$$

$$\mathcal{S}_0^{\gamma,\delta} = \{\mathrm{graph}(\varphi) \mid \varphi : B^s(h) \to B^u(h) \text{ is a } C^1 \text{ map satisfying } |D\varphi| \leq \gamma, |\varphi(0)| \leq \delta\}.$$

For $x \in \Lambda$ we introduce an inner product $\langle\cdot,\cdot\rangle_x'$ of T_xM such that $\langle\cdot,\cdot\rangle_x'$ depends continuously on $x \in \Lambda_l$ for $l \geq 1$, the angle between $E^s(x)$ and $E^u(x)$ in $\langle\cdot,\cdot\rangle_x'$ is $\pi/2$ and for $n \in \mathbf{N}$

$$\|D_x f^n(v)\|' \leq \exp(n(\chi^s + \varepsilon))\|v\|' \quad (v \in E^s(x)),$$

$$\|D_x f^{-n}(v)\|' \leq \exp(n(-\chi^u + \varepsilon))\|v\|' \quad (v \in E^u(x)),$$

where $\|\cdot\|'$ is the norm induced by $\langle\cdot,\cdot\rangle_x'$. By using a linear map $C_\varepsilon(x)$ satisfying $\langle C_\varepsilon(x)(v), C_\varepsilon(x)(w)\rangle_x' = \langle v, w\rangle$ for every $u, v \in \mathbf{R}^n$ ([12], p. 666 Theorem S.2.10), we define the map $\Phi_x : \mathbf{R}^n \to M$ by $\Phi_x = \exp_x \circ C_\varepsilon(x)$.

There exists a Borel measurable function $q : \Lambda \to \mathbf{R}$ such that $e^{-\varepsilon} < q(x)/q(f(x)) < e^\varepsilon$ ([12] p. 673 Theorem S.3.1 (1)) and $\Phi_x|B(q(x))$ is injective. Here $B(q(x)) = B^s(q(x)) \times B^u(q(x))$. We put $f_x = \Phi_{f(x)}^{-1} \circ f \circ \Phi_x$ and $f_x^{-1} = \Phi_{f^{-1}(x)}^{-1} \circ f^{-1} \circ \Phi_x$, and denote by $D_0 f_x$ and $D_0 f_x^{-1}$ the derivative of f_x and f_x^{-1} at 0 respectively. Since f is C^2-class, $f_x - D_0 f_x$ and

$f_x^{-1} - D_0 f_x^{-1}$ are the Lipschitz continuous maps from $B(q(x))$ to $\mathbf{R}^n$ with Lipschitz constant ε. For simplicity we replace $B(\alpha \cdot q(x))$ by $B(\alpha)$ for any $0 < \alpha \leq 1$.

We set

$$\mathcal{U}_x^{\gamma,\delta} = \{\Phi_x(V) \mid V \in \mathcal{U}_0^{\gamma,\delta}\}, \quad \mathcal{S}_x^{\gamma,\delta} = \{\Phi_x(V) \mid V \in \mathcal{U}_0^{\gamma,\delta}\}$$

for $x \in \Lambda$. For $\gamma > 0$ small, the local stable (unstable) manifold of $x \in \Lambda$ belongs to $\mathcal{S}_x^{\gamma,\delta}$ ($\mathcal{U}_x^{\gamma,\delta}$).

REMARK 2.1 ([11], p. 153). For $l \in \mathbf{N}$ there exists $r_l > 0$ such that for $x, y \in \Lambda$ with $y, f(x) \in \Lambda_l$ if $d(y, f(x)) < r_l$ then,

(a) for $V \in \mathcal{U}_x^{\gamma,\delta}$, $f(V) \cap \Phi_y(B(1)) \in \mathcal{U}_y^{\gamma,\delta}$,

(b) for $H \in \mathcal{S}_y^{\gamma,\delta}$, $f^{-1}(H) \cap \Phi_x(B(1)) \in \mathcal{S}_x^{\gamma,\delta}$.

For $v = (v_1, v_2) \in \mathbf{R}^n = \mathbf{R}^{\dim E^s} \bigoplus \mathbf{R}^{\dim E^u}$ define a norm $|||\cdot|||$ on $\mathbf{R}^n$ by

$$|||v||| = \max\{|v_1|, |v_2|\}.$$

Here $|\cdot|$ denotes the Euclidean norm on $\mathbf{R}^n$. Obviously

$$|||v||| \leq |v| \leq \sqrt{2}|||v|||.$$

LEMMA 2.2 ([11], p. 149). *For $V \in \mathcal{U}_0^{\gamma,\delta}$, $H \in \mathcal{S}_0^{\gamma,\delta}$ and $x \in \Lambda$,*

(a) $\exp(\chi^u - 2\varepsilon)|||y - z||| \leq |||f_x(y) - f_x(z)||| \quad (y, z \in V)$,

(b) $\exp(-\chi^s - 2\varepsilon)|||y - z||| \leq |||f_x^{-1}(y) - f_x^{-1}(z)||| \quad (y, z \in H)$.

LEMMA 2.3 ([12], p. 680). *For $l > 0$ there exists $r_l > 0$ such that for $x, y \in \Lambda_l$ with $d(x, y) < r_l$*

$$|\Phi_y^{-1} \circ \Phi_x(v) - v| < \varepsilon|v| \quad (v \in B(1)).$$

For $\alpha \in (0, 1)$ and $l > 0$ there exists $\beta_l > 0$ such that if $f(x), y \in \Lambda_l$ and $d(f(x), y) < \beta_l$, then $f(\Phi_x(B(\alpha)))$ ($f^{-1}(\Phi_y(B(\alpha)))$) intersects transversely $\Phi_y(B(\alpha))$ ($\Phi_x(B(\alpha))$) along the unstable (stable) direction as shown in Figure 1 (Figure 2) (see [22], Lemma 5.1).

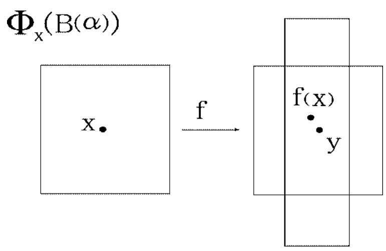

FIGURE 1. A transverse intersection (u-direction).

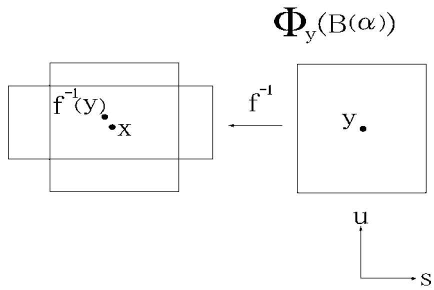

FIGURE 2. A transverse intersection (s-direction).

A sequence $\{x_i\}_{i\in\mathbf{Z}} \subset \Lambda$ is called a $\{\beta_l\}_{l\geq 1}$-*pseudo-orbit* if there exists a sequence $\{s_n\}_{n\geq 1} \subset \mathbf{N}$ such that for $n \in \mathbf{Z}$ $x_n \in \Lambda_{s_n}$, $|s_n - s_{n+1}| \leq 1$ and $d(f(x_n), x_{n+1}) \leq \beta_{s_n}$. Given $\alpha > 0$, a point $x \in M$ satisfying $d(f^i(x), x_i) < \alpha$ for $i \in \mathbf{Z}$ is called an α-*shadowing point* for a $\{\beta_l\}_{l\geq 1}$-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$.

THEOREM 2.4 (Uniqueness of Nonuniformly Shadowing Points). *For $\alpha \in (0,1)$ there exists a sequence $\{\beta_l\}_{l\geq 1}$ of positive real numbers such that for any $\{\beta_l\}_{l\geq 1}$-pseudo orbit there exists a unique shadowing point x with $f^i(x) \in \Phi_{x_i}(B(\alpha))$ for $i \in \mathbf{Z}$.*

Theorem 2.4 is a reformation of the nonuniformly shadowing lemma by Katok ([11], Main lemma) and Katok-Hasselblatt ([12], p. 690, Theorem S.4.14).

PROOF OF THEOREM 2.4. As the existence of shadowing points is shown as in [11] and [12], it remains only to show a uniqueness of the shadowing point.

A sequence $\{\beta_l\}_{l\geq 1}$ is chosen to satisfy the condition described as above. Assume that x and x' be two distinct shadowing points for a $\{\beta_l\}_{l\geq 1}$-pseudo orbit $\{x_i\}_{i=-\infty}^{\infty}$ such that

$$f^i(x),\ f^i(x') \in \Phi_{x_i}(B(\alpha)) \quad (i \in \mathbf{Z}).$$

Then we show that

$$\begin{aligned}|||\Phi_{x_0}^{-1}(x) - \Phi_{x_0}^{-1}(x')||| \leq &\left\{(1-\varepsilon)\exp(\chi^u - 2\varepsilon)\right\}^{-m} \\ &+ \left\{(1-\varepsilon)\exp(-\chi^s - 2\varepsilon)\right\}^{-m}\end{aligned} \tag{2.1}$$

for $m \in \mathbf{N}$. If we establish (2.1) and choose $\varepsilon > 0$ such that $(1-\varepsilon)\exp(\chi^u - 2\varepsilon) > 1$ and $(1-\varepsilon)\exp(-\chi^s - 2\varepsilon) > 1$, then (2.1) implies $x = x'$. This is a contradiction.

To prove (2.1), fix $m \in \mathbf{N}$ and take $H_m \in \mathcal{S}_{x_m}^{\gamma,\delta}$ with $f^m(x') \in H_m$. By Remark 2.1(2) we have

$$f^{m-1}(x') \in H_{m-1} \equiv f^{-1}(H_m) \cap \Phi_{x_{m-1}}(B(\alpha)) \in \mathcal{S}_{x_{m-1}}^{\gamma,\delta}$$

and again

$$f^{m-2}(x') \in H_{m-2} \equiv f^{-1}(H_{m-1}) \cap \Phi_{x_{m-2}}(B(\alpha)) \in \mathcal{S}_{x_{m-2}}^{\gamma,\delta}.$$

Repeating this inductively, we can find $H_j \in \mathcal{S}_{x_j}^{\gamma,\delta}$ $(j = -m, \dots, m)$ such that $f^j(x') \in H_j$ and $f(H_j) \subset H_{j+1}$.

Next take $V_{-m} \in \mathcal{S}_{x_{-m}}^{\gamma,\delta}$ with $f^{-m}(x) \in V_{-m}$. By Remark 2.1(1) we have

$$f^{-m+1}(x) \in V_{-m+1} \equiv f(V_0) \cap \Phi_{x_{-m+1}}(B(\alpha)) \in \mathcal{U}_{x_{-m+1}}^{\gamma,\delta}.$$

Using induction, we have that $V_j \in \mathcal{U}_{x_j}^{\gamma,\delta}$ $(j = -m, \dots, m)$ such that $f^j(x) \in V_j$ and $f^{-1}(V_j) \subset V_{j-1}$.

For $j = -m, \ldots, m$ put $\{z_j\} = H_j \cap V_j$. Then $f(z_j) = z_{j+1}$. Lemmas 2.2 and 2.3 ensure that

$$|||\Phi_{x_0}^{-1}(x) - \Phi_{x_0}^{-1}(z_0)||| \le \left\{(1-\varepsilon)\exp(\chi^u - 2\varepsilon)\right\}^{-m},$$
$$|||\Phi_{x_0}^{-1}(x') - \Phi_{x_0}^{-1}(z_0)||| \le \left\{(1-\varepsilon)\exp(-\chi^s - 2\varepsilon)\right\}^{-m},$$

from which (2.1) is concluded. □

For $n \in \mathbf{N}$ set $Y_n = \{0, \ldots, n-1\}$ and denote by $Y_n^{\mathbf{Z}}$ the infinite product topological space of Y_n. Then $Y_n^{\mathbf{Z}}$ is a compact metric space equipped with the metric d defined by

$$d(x, y) = \sum_{k=-\infty}^{\infty} \frac{|x_i - y_i|}{n^{|k|}} \; (x = (x_i)_{i\in\mathbf{Z}}, y = (y_i)_{i\in\mathbf{Z}} \in Y_n^{\mathbf{Z}}).$$

Define the *left shift map* $\sigma : Y_n^{\mathbf{Z}} \to Y_n^{\mathbf{Z}}$ by $\sigma((x_i)) = (x_{i+1})$ for $(x_i)_{i\in\mathbf{Z}} \in Y_n^{\mathbf{Z}}$. An f-invariant set $\Gamma \subset M$ is said to be a *topological horseshoe* if there exists $n > 0$ such that $\sigma : Y_n^{\mathbf{Z}} \to Y_n^{\mathbf{Z}}$ and $f : \Gamma \to \Gamma$ are topologically conjugate.

THEOREM 2.5. *Let μ be a hyperbolic ergodic measure and assume that $h_\mu(f) > 0$. Then for $\alpha > 0$ and $h \in (0, h_\mu(f))$ there exist $\Gamma \subset M$ and $n, q \in \mathbf{N}$ such that*

(a) *$f^n : \Gamma \to \Gamma$ and $\sigma : Y_q^{\mathbf{Z}} \to Y_q^{\mathbf{Z}}$ are topologically conjugate,*

(b) *if $\Gamma' = \bigcup_{i=0}^{n-1} f^i(\Gamma)$, then $|h(f|_{\Gamma'}) - h| \le \alpha$. Here $f|_{\Gamma'}$ is the restriction of f on Γ and $h(f|_{\Gamma'})$ is the topological entropy of $f|_{\Gamma'}$,*

(c) *if μ satisfies SRB condition for f, then $\Gamma \subset S = Supp(\mu)$.*

The estimation (b) of Theorem 2.5 is a some refined version of Theorem S.5.9 in [12], whose proof is similar to that in [12].

For $m \in \mathbf{N}$, $\rho > 0$ and $\delta \in (0, 1)$ let

$$B_m(x, \rho) = \{y \in M \mid d(f^i(x), f^i(y)) \le \rho, \; (0 \le i \le m-1)\}$$

and

$$N_f(m, \rho, \delta) = \inf\{k \mid \mu(\bigcup_{i=1}^{k} B_m(x_i, \rho)) > 1 - \delta, \; (x_1, \ldots x_k \in M)\}.$$

LEMMA 2.6 ([11] p. 143). *Let ν be an ergodic measure and assume that $h_\nu(f) < \infty$. Then for $\delta \in (0, 1)$*

$$h_\nu(f) = \lim_{\rho\to 0} \limsup_{n\to\infty} \frac{1}{n} \log N_f(n, \rho, \delta)$$
$$= \lim_{\rho\to 0} \liminf_{n\to\infty} \frac{1}{n} \log N_f(n, \rho, \delta).$$

PROOF OF THEOREM 2.5. For $\alpha > 0$ choose $\rho > 0$ such that

$$(h_\mu(f) - 2\rho)/(1 + \rho) > h_\mu(f) - \alpha/2, \quad \alpha > \rho.$$

We define

$$\tilde{\Lambda}_l = \mathrm{Supp}(\mu|_{\Lambda_l})\ (l \geq 1)\,, \quad \tilde{\Lambda} = \bigcup_{l \geq 1} \tilde{\Lambda}_l\,. \tag{2.2}$$

Obviously $\mu(\tilde{\Lambda}) = 1$ and $f(\tilde{\Lambda}) = \tilde{\Lambda}$. For $\delta \in (0, 1)$ there is $l \in \mathbf{N}$ such that

$$\mu(\tilde{\Lambda}_l) > 1 - \delta\,.$$

By lemma 2.6 we can find $0 < r < 1$ and $m_0 \in \mathbf{N}$ such that

$$\exp(m(h_\mu(f) - \rho)) \leq N_f(m, r, \delta) \leq \exp(m(h_\mu(f) + \rho)) \tag{2.3}$$

for $m \geq m_0$. Let ξ be a finite measurable partition of M such that $\xi \geq \{\tilde{\Lambda}_l, M \backslash \tilde{\Lambda}_l\}$ and $\mathrm{diam}(\xi) \leq \beta_l(r)$. Here $\beta_l(r)$ is the number in Theorem 2.4 and $\beta_l(r) < r$, and for two finite partitions $\alpha = \{A_i\}$, $\beta = \{B_i\}$, $\beta \geq \alpha$ means that for any $A \in \alpha$ there exists $B \in \beta$ such that $B \subset A$.

Let

$$\Lambda_l^m = \{x \in \tilde{\Lambda}_l \mid m \leq \exists q \leq [(1 + \rho)m] \text{ s.t } f^q(x) \in \xi(x)\}\,,$$

where $[a]$ denotes the Gauss symbol. By Birkhoff's ergodic theorem we have that $\mu(\tilde{\Lambda}_l \backslash \Lambda_l^m) \to 0$ $(m \to \infty)$. Thus there is a set $E_m \subset \Lambda_l^m$ such that

$$\sharp(E_m) = N_f(m, \beta_l(r), \delta)$$

and $B_m(x, \beta_l(r)/2) \cap B_m(y, \beta_l(r)/2) = \phi$ for $x, y \in E_m$ with $x \neq y$, where $\sharp(A)$ denotes the cardinality of a set A. For $m \geq 1$ sufficiently large it follows from (2.3) that

$$\sharp(E_m) \geq \exp(m(h_\mu(f) - \rho))\,, \tag{2.4}$$

$$\sharp(E_m) \leq \exp(m(h_\mu(f) + \rho)) \tag{2.5}$$

and

$$\frac{1}{m} \log \sharp(\xi) < \frac{\alpha}{2}\,. \tag{2.6}$$

For $m \leq q \leq [(1 + \rho)m]$ let

$$V_q = \{x \in E_m \mid f^q(x) \in \xi(x)\}\,.$$

Then $\{V_q\}_{q=m}^{[(1+\rho)m]}$ is a finite cover of E_m. Denote by V_{q_0} be a set with the maximal cardinality of $\{V_q\}_{q=m}^{[(1+\rho)m]}$. Obviously

$$\sharp(E_m) \leq (m\rho + 1)\sharp(V_{q_0})\,.$$

Since $e^x \geq x + 1$, by (2.4) we have

$$\sharp(V_{q_0}) \geq \exp(m(h_\mu(f) - 2\rho))\,. \tag{2.7}$$

Denote by $V_{q_0} \cap \xi(z_0)$ the set with the maximal cardinality of $\{V_{q_0} \cap \xi(x) | x \in \tilde{\Lambda}_l\}$ and put $b = \sharp(V_{q_0} \cap \xi(z_0))$ where $z_0 \in \tilde{\Lambda}_l$. Since $b\sharp(\xi) \geq \sharp(V_{q_0})$, by (2.7) we have

$$\log b \geq -\log \sharp(\xi) + m(h_\mu(f) - 2\rho)\,. \tag{2.8}$$

Write $V_{q_0} \cap \xi(z_0) = \{y_0, \ldots, y_{b-1}\}$ and $Y_b = \{0, 1, \ldots, b-1\}$. We construct $\{\beta_l(\delta)\}_{l\geq 1}$-pseudo orbit as follows. Firstly fix $a = (a_i)_{i\in\mathbf{Z}} \in Y_b^{\mathbf{Z}}$. Then y_{a_i} returns to $\xi(z_0)$ by the iteration of f^{q_0} such that $d(f^{q_0}(y_{a_i}), y_{a_{i+1}}) < \beta_l(\delta)/2$ for $i \in \mathbf{Z}$. Combining the finite orbit $y_{a_i}, \ldots, f^{q_0-1}(y_{a_i})$ with the finite orbit $y_{a_{i+1}}, \ldots, f^{q_0-1}(y_{a_{i+1}})$ for any $i \in \mathbf{Z}$, we obtain the following $\{\beta_l(\delta)\}_{l\geq 1}$-pseudo orbit.

$$\left\{ \cdots, \underbrace{y_{a_i}, \ldots, f^{q_0-1}(y_{a_i})}_{i-th}, \underbrace{y_{a_{i+1}}, \ldots, f^{q_0-1}(y_{a_{i+1}})}_{(i+1)-th}, \cdots \right\} \subset \Lambda_{l+q_0}\,.$$

Denote by $z(a)$ the above sequence (see Figure 3). Theorem 2.4 ensures the existence of a unique shadowing point $\bar{a}$ for $z(a)$, and so define $\varphi(a) = \bar{a}$. Put $\Gamma = \varphi(Y_b^{\mathbf{Z}})$. Using (2.1) in the proof of Theorem 2.4, we have that the map $\varphi : Y_b^{\mathbf{Z}} \to \Gamma$ is injective and continuous. Therefore $\sigma : Y_b^{\mathbf{Z}} \to Y_b^{\mathbf{Z}}$ and $f^{q_0}|_\Gamma : \Gamma \to \Gamma$ are topologically conjugate.

Put $\Gamma' = \bigcup_{i=0}^{q_0-1} f^i(\Gamma)$. Then Γ' is f-invariant and

$$\begin{aligned} h(f|_{\Gamma'}) &= \frac{1}{q_0} h(f^{q_0}|_\Gamma) \\ &= \frac{1}{q_0} \log b\,. \end{aligned}$$

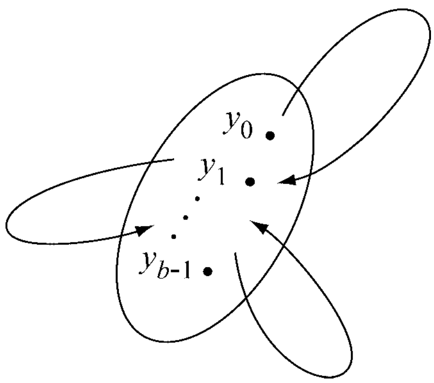

FIGURE 3. A pseudo orbit of $z(a)$.

By (2.8) we have

$$h(f|_{\Gamma'}) \geq -\frac{1}{q_0}\log \sharp(\xi) + \frac{m}{q_0}(h_\mu(f) - 2\rho)\,.$$

Since $m \leq q_0 \leq [(1+\rho)m]$, by (2.6)

$$\begin{aligned} h(f|_{\Gamma'}) &\geq -\frac{1}{m}\log \sharp(\xi) + \frac{1}{1+\rho}(h_\mu(f) - 2\rho) \\ &> h_\mu(f) - \alpha\,. \end{aligned}$$

The choice of $\alpha > \rho$ ensures that

$$b \leq \exp(m(h_\mu(f) + \alpha))$$

because of $b \leq \sharp(V_{q_0}) \leq \sharp(E_m)$ and (2.5). Therefore we have

$$\begin{aligned} h(f|_{\Gamma'}) &= \frac{1}{q_0}\log b \\ &\leq \frac{m}{q_0}(h_\mu(f) + \alpha) \\ &\leq h_\mu(f) + \alpha\,. \end{aligned}$$

This completes the proof of (a) and (b) in the case of $h = h_\mu(f)$.

To obtain (c), it suffices to show that every unstable manifold of $y \in V_{q_0} \cap \xi(z_0)$ is contained in $S = \mathrm{Supp}(\mu)$ (see [12], p. 690 Theorem S.4.14).

Since $V_{q_0} \subset \tilde{\Lambda}_l = \mathrm{Supp}(\mu|_{\Lambda_l})$, we have $\mu(U(y) \cap \Lambda_l) > 0$ for $y \in V_{q_0} \cap \xi(z_0)$ and any open neighborhood $U(y)$ of y. By SRB condition of μ we have $W^u(x) \subset S$ for μ-a.e.x (Theorem 3.2(b) in the next section). Thus there exists a sequence $\{x_n\}_{n\geq 1} \subset S \cap \Lambda_l$ such that $x_n \to y$ $(n \to \infty)$ and $W^u(x_n) \subset S$. Therefore we have $W^u(y) \subset S = \mathrm{Supp}(\mu)$ since local unstable manifolds depend continuously on Λ_l. This concludes (c) for $h = h_\mu(f)$.

Finally we continue the proof for $0 < h < h_\mu(f)$. By the above construction we can find Γ' satisfying $|h(f|_{\Gamma'}) - h_\mu(f)| \leq \alpha$ for $0 < \alpha < h_\mu(f) - h$. Since $h(f|_{\Gamma'}) > h_\mu(f) - \alpha > h$, we can take an equilibrium state ν on Γ' with $h_\nu(f|_{\Gamma'}) = h$ (in the same way as Theorem 4.1, Part (II)). Applying the argument above to ν again, we obtain $\Gamma_1 \subset \Gamma'$ and $k, j \in \mathbf{N}$ such that $f^k : \Gamma_1 \to \Gamma_1$ and $\sigma : Y_j^{\mathbf{Z}} \to Y_j^{\mathbf{Z}}$ are topologically conjugate. Put $\Gamma_1' = \bigcup_{i=0}^{k-1} f^i(\Gamma_1)$. Then

$$|h(f|_{\Gamma_1'}) - h| = |h(f|_{\Gamma_1'}) - h_\nu(f)| \leq \alpha\,.$$

This completes (a) and (b) for $0 < h < h_\mu(f)$. If μ satisfies SRB condition, then it is checked that $\Gamma' \subset S$. And so $\Gamma_1 \subset S$. The theorem is concluded. □

3. Hyperbolic Attractors and SRB Attractors

An f-invariant closed set Γ is said to be an *attractor* if there exists a neighborhood V of Γ such that $f(\mathrm{Cl}(V)) \subset V$ and $\bigcap_{i\geq 1} f^i(V) = \Gamma$. Here $\mathrm{Cl}(A)$ denotes the closure of a set A. An attractor Γ is said to be a *hyperbolic attractor* if Γ is hyperbolic and $f|_\Gamma$ is topologically transitive. The map $f : \Gamma \to \Gamma$ is said to be *topologically transitive* if there exists a point $x \in \Gamma$ such that its orbit $\{f^n(x)\}_{n\in\mathbf{Z}}$ is dense in Γ. An f-invariant closed set Γ is said to be a *hyperbolic set* if there exist $C > 0$ and $0 < \lambda < 1$ such that for any $x \in \Gamma$ there exists a decomposition $\mathrm{T}_x M = E^s(x) \oplus E^u(x)$ into subspaces $E^s(x)$ and $E^u(x)$ such that the following properties hold: (i) $D_x f(E^\sigma(x)) = E^\sigma(f(x))$ for $\sigma = s, u$, and (ii)

$$\|D_x f^n(v)\| \leq C\lambda^n \|v\| \quad (v \in E^s(x)),$$
$$\|D_x f^{-n}(v)\| \leq C\lambda^n \|v\| \quad (v \in E^u(x))$$

for $n \in \mathbf{N}$.

Denote by $P(f, \varphi)$ the *topological pressure* of f w.r.t. $\varphi \in C(M, \mathbf{R})$ (see [21] for the definition), where $C(M, \mathbf{R})$ denotes the set of real valued continuous functions on M. An f-invariant probability measure ν on M is said to be an *equilibrium state* for φ if $P(f, \varphi) = h_\nu(f) + \int \varphi d\nu$ holds. Denote by $P_\Gamma(f, \varphi^u)$ the topological pressure of f w.r.t. $\varphi^u|_\Gamma$. Here $\varphi^u(x) = -\log|\det(D_x f|_{E^u(x)})|$. If Γ is a hyperbolic attractor, then the following properties hold:

(9) $P_\Gamma(f, \varphi^u) = 0$.

(10) $W^u(x) \subset \Gamma$ for any $x \in \Gamma$.

(11) $f|_\Gamma$ is topologically transitive (by the definition of hyperbolic attractors).

(12) $m(W^s(\Gamma)) > 0$.

Here $W^s(\Gamma)$ denotes the union of all stable manifolds at the points in Γ. In [6] (p. 99 Theorem4.11) Bowen showed that both (9) and (12) are equivalent.

(13) $f|_\Gamma$ satisfies the uniformly shadowing property.

(14) The set of periodic points for $f|_\Gamma$, $P(f|_\Gamma)$, is dense in Γ.

(15) $f|_\Gamma$ is expansive.

(16) $f|_\Gamma$ has a unique equilibrium state for φ^u.

A sequence $\{x_i\}_{i\in\mathbf{Z}} \subset \Gamma$ is called a δ-*pseudo orbit* for f if $d(f(x_i), x_{i+1}) < \delta$ for $i \in \mathbf{Z}$. A point x is called an α-*shadowing point* for a δ-pseudo orbit $\{x_i\}_{i\in\mathbf{Z}}$ if $d(f^i(x), x_i) < \alpha$ for $i \in \mathbf{Z}$. We call that $f|_\Gamma$ satisfies the *uniformly shadowing property* if for any $\alpha > 0$ there exists $\delta > 0$ such that for any δ-pseudo orbit there exists at least one α-shadowing point in Γ.

Throughout this section, let μ be an ergodic measure and Λ be the Pesin set w.r.t. μ. For a Borel set R put

$$W^s(R) = \bigcup_{x\in R} W^s(x), \quad W^u(R) = \bigcup_{x\in R} W^u(x).$$

We say that μ satisfies the $condition\ (A)_s$ ($condition\ (A)_u$) if $m(W^s(R)) > 0$ ($m(W^u(R)) > 0$) for any Borel set R with $\mu(R) = 1$.

If μ is an ergodic hyperbolic measure satisfying SRB condition for f, then it follows from the proof of Theorem 3 in [23] that μ satisfies the condition $(A)_s$.

REMARK 3.1. The following statements hold:

(a) μ is an SRB measure if and only if there exists an SRB attractor of μ,

(b) if μ is a hyperbolic measure and satisfies SRB condition for f, then $S = \mathrm{Supp}(\mu)$ is an SRB attractor of μ.

PROOF. (a) is clear by the definition of SRB attractors. If μ satisfies SRB condition, then (b) follows from $(A)_s$ (see [23]). □

If μ is ergodic, then

$$P_{R(\mu)}(f, \varphi) = h_\mu(f) + \int \varphi d\mu \quad (\varphi \in C(S, \mathbf{R})) \tag{3.1}$$

([21]).

For $\psi \in L^1(\mu)$ define the norm on $L^1(\mu)$ by $\|\psi\|_1 = \int |\psi| d\mu$. Since $C(S, \mathbf{R})$ is dense in $L^1(\mu)$ w.r.t. $\|\cdot\|_1$, for $\psi \in L^1(\mu)$ there is $\{\varphi_i\}_{i\geq 1} \subset C(S, \mathbf{R})$ such that $\|\varphi_n - \psi\|_1 \to 0$ $(n \to \infty)$, and so define

$$\begin{aligned}\bar{P}_{R(\mu)}(f, \psi) &= \lim_{n\to\infty} P_{R(\mu)}(f, \varphi_n)\\ &= \lim_{n\to\infty}\left(h_\mu(f) + \int \varphi_n d\mu\right)\\ &= h_\mu(f) + \int \psi d\mu\,.\end{aligned}$$

THEOREM 3.2. *Let μ be an ergodic measure satisfying SRB condition for f and put $S = Supp(\mu)$. Then*

(a) $\bar{P}_{R(\mu)}(f, \varphi^u) = 0$,

(b) $W^u(x) \subset S$ *for μ-a.e.x*,

(c) *$f|_S$ is topologically transitive.*

Moreover, if μ is hyperbolic, then S is an SRB attractor of μ and

(d) *$m(W^s(R)) > 0$ for $R \subset S \cap \Lambda$ with $\mu(R) = 1$,*

(e) *$f|_S$ satisfies the nonuniformly shadowing property,*

(f) *the set of all hyperbolic periodic points of $f|_S$ is dense in S,*

(g) *in general $f|_S$ is not expansive.*

PROOF. (a) follows from the SRB condition of μ (see [15], theorem A). The ergodicity of μ implies (c).

To show (b), let ξ^u be a measurable partition which is subordinate to the W^u-foliation as in Sect.1, μ_x^u be the canonical system of conditional measure of μ w.r.t. ξ^u and m_x^u be the

Lebesgue measure on $W^u(x)$ (μ-a.e.x). Since

$$0 = \mu(S^c) = \int \mu_x^u(S^c) d\mu(x) ,$$

we have $\mu_x^u(S^c) = 0$ (μ-a.e.x). By (1.1)

$$m_x^u(\xi^u(x) \cap S^c) = 0 \quad (\mu\text{-a.e.}x \in M) . \tag{3.2}$$

For μ-a.e.x there exists an $r = r(x) > 0$ such that $U^u(x, r) \subset \xi^u(x)$ because ξ^u is subordinate to the W^u-foliation. Here

$$U^u(x, r) = \{y \in W^u(x) \mid d^u(y, x) < r(x)\}$$

and d^u denotes the Riemannian distance in $W^u(x)$.

If $U^u(x, r) \cap S^c \neq \phi$, then $m_x^u(U^u(x, r) \cap S^c) > 0$ since $U^u(x, r) \cap S^c$ is open in $W^u(x)$, and by (3.2)

$$\begin{aligned} 0 &= m_x^u(\xi^u(x) \cap S^c) \\ &\geq m_x^u(U^u(x, r) \cap S^c) \\ &> 0 . \end{aligned}$$

This is a contradiction. Therefore

$$U^u(x, r) \subset S \quad (\mu\text{-a.e.}x \in M) ,$$

and so $W^u(x) \subset S$ (see the proof of Proposition 3.1 in [14]).

(d) follows from the fact that μ satisfies $(A)_s$ (see [23]). By combining (b) and the proofs of Theorem S.4.14 in [12] and Main lemma in [11], we have (e) and (f). Since the sizes of the local stable and unstable manifolds of Λ are not constant, it does not ensure that $f|_S$ is expansive. □

4. Ergodic Measures of SRB Attractors

Let μ be a hyperbolic ergodic measure of M and Λ be a Pesin set w.r.t. μ. Denote by $\mathcal{M}(M)$ the set of Borel probability measures on M. Let $\{\varphi_i\}_{i\geq 1}$ be a sequence of continuous functions which is dense in $C(M, \mathbf{R})$. For $\lambda, \nu \in \mathcal{M}(M)$ define

$$D(\nu, \lambda) = \sum_{i=1}^{\infty} \frac{\left| \int \varphi_i d\nu - \int \varphi_i d\lambda \right|}{2^i \|\varphi_i\|_0} .$$

Here $\|\varphi\|_0 = \sup_{x\in M}\{|\varphi(x)|\}$ for $\varphi \in C(M, \mathbf{R})$. Then $\mathcal{M}(M)$ is compact. Let $\mathcal{M}_f(M)$ be the set of f-invariant measures in $\mathcal{M}(M)$ and put

$$\mathcal{M}_f(\tilde{\Lambda}) = \{\nu \in \mathcal{M}_f(M) \mid \nu(\tilde{\Lambda}) = 1\}$$

where $\tilde{\Lambda}$ is a set defined as in (2.2). The set of all ergodic probability measures of $S = \mathrm{Supp}(\mu)$, $\mathcal{E}(S)$, is decomposed into the union of subsets:

$$\mathcal{H}(S) = \{\nu \in \mathcal{E}(S) \mid \nu \text{ is hyperbolic } \},$$
$$\mathcal{N}(S) = \{\nu \in \mathcal{E}(S) \mid \nu \text{ is non hyperbolic } \}.$$

If $S = \Lambda$, then $\mathcal{N}(S) = \phi$. On the other hand, in the case of $S \neq \Lambda$, $\mathcal{N}(S)$ is not empty in general. Indeed, there exists a C^2-diffeomorphism of $\mathbf{T}^2$, called an 'almost Anosov', described in [10].

As shown in Table 1, we can decompose $\mathcal{H}(S)$ into the union of subsets where

(i) $\theta(S)$ is the set of measures with a point mass on the periodic orbits on S,

(ii) $\mathcal{Z}(S)$ is the set of measures with zero entropy and $\sharp(\text{support}) = \infty$.

(iii) $\mathcal{P}_1(S)$ is the set of measures with positive entropy, but not satisfy SRB condition for both f and f^{-1}.

(iv) $\mathcal{P}_2(S)$ is the set of measures satisfying SRB condition for f^{-1}, but not SRB condition for f,

(v) $\mathcal{P}_3(S)$ is the set of measures satisfying SRB condition for f, but not SRB condition for f^{-1},

(vi) $\mathcal{P}_4(S)$ is the set of measures satisfying SRB condition for both f and f^{-1}.

Note that $\mathcal{M}_f(\tilde{\Lambda}) \subset \mathrm{Cl}(\theta(S))$ ([9]). Since any element of $\mathcal{P}_3(S)$ or $\mathcal{P}_4(S)$ satisfies SRB condition, each element is an SRB measure (see Remark 3.1). Since there exist at most countable SRB measures in $\mathcal{E}(S)$, $\mathcal{P}_3(S)$ and $\mathcal{P}_4(S)$ are at most countable sets and so is $\mathcal{P}_2(S)$ by applying the same argument for f^{-1}.

In the case when $\mathcal{N}(S) \neq \phi$, denote by $\mathcal{P}_5(S)$ the subset of $\mathcal{N}(S)$ which consists of the elements satisfying SRB condition for f. Then $\nu \in \mathcal{P}_5(S)$ satisfies (a)–(c) in Theorem 3.2. Denote by $\mathcal{P}_6(S)$ the subset of $\mathcal{N}(S)$ which consists of the elements satisfying SRB condition for f and f^{-1}. By Theorem 3.2 we have that for $\nu \in \mathcal{P}_6(S)$

$$W^\sigma(x) \subset S \quad (\nu\text{-a.e.}x,\ \sigma = s, u).$$

f is said to be μ-*mixing* if $\lim_{n\to\infty} \mu(f^{-n}(A) \cap B) = \mu(A)\mu(B)$ for Borel sets A and B.

TABLE 1. A classification of $\mathcal{E}(S)$

			$\mathcal{H}(S)$	$\mathcal{N}(S)$
zero entropy	finite support		$\theta(S)$	
	infinite support		$\mathcal{Z}(S)$	
positive entropy	f : non SRB cond	f^{-1} : non SRB cond	$\mathcal{P}_1(S)$	
		f^{-1} : SRB cond	$\mathcal{P}_2(S)$	
	f : SRB cond	f^{-1} : non SRB cond	$\mathcal{P}_3(S)$	$\mathcal{P}_5(S) \supset \mathcal{P}_6(S)$
		f^{-1} : SRB cond	$\mathcal{P}_4(S)$	

THEOREM 4.1. *Let μ be a hyperbolic ergodic measure satisfying SRB condition for f. Assume that f is μ-mixing. Then the following holds:*

(a) *$\mathcal{Z}(S)$ is an uncoutable set and $\mathcal{M}_f(\tilde{\Lambda}) \subset Cl(\mathcal{Z}(S))$.*

(b) *$\mathcal{P}_1(S)$ is an uncoutable set and $\mathcal{M}_f(\tilde{\Lambda}) \subset Cl(\mathcal{P}_1(S))$.*

(c) *An ergodic measure satisfying SRB condition for f is unique in $\mathcal{M}_f(\tilde{\Lambda})$.*

The proof of Theorem 4.1 is decomposed into the following four parts.

PART (I). $\mathcal{Z}(S)$ is an uncoutable set.

PROOF. Using Theorem 2.5 for $h = h_\mu(f)$, we can find $k, q \in \mathbf{N}$ and $\Gamma \subset S$ such that $f^k|_\Gamma : \Gamma \to \Gamma$ and $\sigma : Y_q^{\mathbf{Z}} \to Y_q^{\mathbf{Z}}$ are topologically conjugate.

Now we construct a Sturmian shift (see [16], [18]) as follows: for an irrational number $\beta \in (0, 1/q)$ the family of sets

$$I_i = [i\beta, (i+1)\beta) \quad (i = 0, \dots q-2)\,, \quad I_{q-1} = [(q-1)\beta, 1)$$

is a partition of $S^1 = [0, 1)$ (mod 1). Define $T_\beta : S^1 \to S^1$ by

$$T_\beta(z) = z + \beta \pmod 1 \quad (z \in S^1)\,,$$

and $h_\beta^i : S^1 \to Y_2$ by

$$h_\beta^i(z) = \begin{cases} 1 & \text{if } T_\beta^i(z) \in I_0\,, \\ 0 & \text{if } T_\beta^i(z) \notin I_0 \end{cases}$$

for $i \in \mathbf{Z}$. Then the map $h_\beta : S^1 \to Y_2^{\mathbf{Z}}$ defined by

$$h_\beta(z) = (h_\beta^i(z))_{i \in \mathbf{Z}} \in Y_2^{\mathbf{Z}}$$

is injective such that $h_\beta \circ T_\beta = \sigma \circ h_\beta$. Let

$$C_0 = \{(x_i)_{i \in \mathbf{Z}} \in Y_2^{\mathbf{Z}} \mid x_0 = 1\} \quad \text{and} \quad C_1 = \{(x_i)_{i \in \mathbf{Z}} \in Y_2^{\mathbf{Z}} \mid x_0 = 0\}\,.$$

Then

$$h_\beta^{-1}(C_0) = [0, \beta) \quad \text{and} \quad h_\beta^{-1}(C_1) = [\beta, 1)$$

and we have that

$$\begin{aligned} h_\beta^{-1}\left(\bigcap_{j=-m}^{n} \sigma^{-j}(C_{k_j})\right) &= \bigcap_{j=-m}^{n} h_\beta^{-1} \circ \sigma^{-j}(C_{k_j}) \\ &= \bigcap_{j=-m}^{n} T_\beta^{-j} \circ h_\beta^{-1}(C_{k_j}) \\ &= \bigcap_{j=-m}^{n} T_\beta^{-j}(S^1 \setminus I_{k_j}) \end{aligned}$$

for $k_j \in \{0,1\}$ $(-m \le j \le n, m, n \in \mathbf{N})$. This implies that h_β is measurable.

We know that a Lebesgue measure λ on S^1 is a T_β-invariant ergodic probability measure. Define

$$\mu_\beta(E) = \lambda \circ h_\beta^{-1} \circ h(E) \quad (E \subset \Gamma)$$

where $h : \Gamma \to Y_q^{\mathbf{Z}}$ is the conjugacy. Then μ_β is f^k-invariant and $h_{\mu_\beta}(f^k) = 0$. Since

$$\beta = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_E(f^{ki}x) = \int 1_E d\mu_\beta \quad \mu_\beta\text{-a.e.}x$$

for $E = h^{-1}(C_0)$, we have that $\mu_\beta \ne \mu_\alpha$ if $\alpha \ne \beta$. $\square$

PART (II). For $\alpha \in (0, h_\mu(f))$ there exists $\nu \in \mathcal{P}_1(S)$ such that

$$h_\nu(f) = \alpha\,.$$

PROOF. For $\alpha \in (0, h_\mu(f))$ take $\rho > 0$ so small that $\alpha < h_\mu(f) - \rho$. By Theorem 2.5 there exist $k, q \in \mathbf{N}$ and $\Gamma \subset S$ such that $f^k|_\Gamma : \Gamma \to \Gamma$ and $\sigma : Y_q^{\mathbf{Z}} \to Y_q^{\mathbf{Z}}$ are topologically conjugate and $h(f|_{\Gamma_\rho}) > \alpha$ where $\Gamma_\rho = \bigcup_{i=0}^{k-1} f^{-i}(\Gamma)$.

Fix $\gamma > 0$ and define a continuous function φ_γ on $Y_q^{\mathbf{Z}}$ by

$$\varphi_\gamma(x) = \begin{cases} \gamma & \text{if } x_0 = 0\,, \\ 0 & \text{if } x_0 \ne 0\,. \end{cases}$$

Let ν_γ be the equilibrium state of $Y_q^{\mathbf{Z}}$ for φ_γ. Put $[i] = \{(x_i)_{i\in\mathbf{Z}} \in Y_q^{\mathbf{Z}} \mid x_0 = i\}$ for $0 \le i \le q-1$. Then

$$\nu_\gamma([0]) = \frac{e^\gamma}{e^\gamma + q - 1}\,, \quad \nu_\gamma([i]) = \frac{1}{e^\gamma + q - 1} \quad (1 \le i \le q-1)$$

(see [28]) and we have that

$$h_{\nu_\gamma}(\sigma) = \sum_{j=0}^{q-1} -\nu_\gamma([j]) \log \nu_\gamma([j])$$

and so $h_{\nu_\gamma}(\sigma) \to 0$ $(\gamma \to \infty)$. Then the entropy of an ergodic measure

$$\tilde{\nu}_\gamma = \frac{1}{k} \sum_{j=0}^{k-1} \nu_\gamma \circ h \circ f^{-i}$$

is given by

$$h_{\tilde{\nu}_\gamma}(f) = \frac{1}{k} h_{\nu_\gamma \circ h}(f^k) = \frac{1}{k} h_{\nu_\gamma}(\sigma)\,,$$

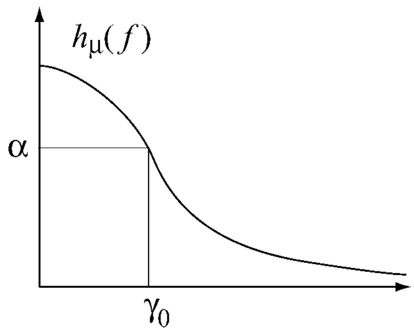

FIGURE 4. The graph of $h_{\tilde{\nu}_\gamma}(f)$.

where h is the conjugacy from Γ to $Y_q^{\mathbf{Z}}$. Thus in the case when $\gamma = 0$ we have that

$$h_{\tilde{\nu}_0}(f) = \frac{1}{k} h_{\nu_0}(\sigma) = \frac{1}{k} \log q > \alpha$$

because of $h_{\nu_0}(\sigma) = \log q$. Therefore we can find $\gamma_0 > 0$ such that $h_{\tilde{\nu}_{\gamma_0}}(f) = \alpha$ (see Figure 4). $\square$

It is well known that a Borel set

$$Q(f) = \left\{ x \in M \,\middle|\, \text{there exists } \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) \text{ for any } \varphi \in C(M, \mathbf{R}) \right\}.$$

satisfies $\nu(Q(f)) = 1$ for $\nu \in \mathcal{M}_f(M)$.

PART (III). For $\nu \in \mathcal{M}_f(\tilde{\Lambda})$ and $\rho > 0$ there exist $n \in \mathbf{N}$ and $\Gamma \subset S$ such that

(a) $f^n|_\Gamma : \Gamma \to \Gamma$ and $\sigma : Y_2^{\mathbf{Z}} \to Y_2^{\mathbf{Z}}$ are topologically conjugate, and

(b) $D(\nu, \lambda) < \rho$ for any f-invariant Borel probability measure λ of $\tilde{\Gamma}_\rho = \bigcup_{i=0}^{n-1} f^i(\Gamma)$.

PROOF. Choose $\nu \in \mathcal{M}_f(\tilde{\Lambda})$ and $\rho > 0$. Fix a dense subset $\{\varphi_i\}_{i\geq 1}$ of $C(M, \mathbf{R})$ and set $F = \{\varphi_j\}_{j=1}^{I_0}$ for $I_0 \in \mathbf{N}$. For convenience assume that $\psi \in F$ satisfies $\|\psi\|_0 = 1$. Since $\psi \in F$ is uniformly continuous, we can find $\delta_0 > 0$ such that $|\psi(x) - \psi(y)| < \bar{\rho}/4$ for $x, y \in M$ with $d(u, v) < \delta_0$ and $\psi \in F$. Put $\psi^*(x) = \lim_{n\to\infty}(1/n) \sum_{i=0}^{n-1} \psi(f^i(x))$ for $x \in Q(f)$ and $\psi \in C(M, \mathbf{R})$, and $\bar{\rho} = \rho/I_0$. Then for $x \in Q(f)$ we can take $N(x) \in \mathbf{N}$ such that

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \psi(f^i(x)) - \psi^*(x) \right| < \frac{\bar{\rho}}{4} \quad (n \geq N) \tag{4.1}$$

for $\psi \in F$. Moreover, Birkhoff's ergodic theorem ensures that

$$\int_{Q(f)} \psi^* d\nu = \int \psi d\nu\,.$$

Put $A = \sup\{|\psi^*(x)|\,|\, x \in Q(f),\ \psi \in F\}$ and define

$$Q_j(\psi) = \left\{ x \in Q(f) \,\middle|\, -A + \frac{j-1}{4}\bar{\rho} \le \psi^*(x) < -A + \frac{j}{4}\bar{\rho} \right\}$$

for $j = 1, \dots, [8A/\bar{\rho}]+1$ and $\psi \in F$. Since F is a finite set, we can define a finite measurable partition $\mathcal{Q} = \{Q_i\}_{i=1}^k$ of $Q(f)$ by

$$\mathcal{Q} \equiv \{Q_i\}_{i=1}^k \equiv \vee_{\psi \in F}\{Q_1(\psi), \dots, Q_{[8A/\bar{\rho}]+1}(\psi)\}$$

where $\alpha \vee \beta = \{A_i \cap B_j \mid A_i \in \alpha,\ B_j \in \beta\}$ for two finite partitions $\alpha = \{A_i\}$, $\beta = \{B_i\}$.

Here after notice that k be fixed. Without loss of generality, we may assume that $\nu(Q_i) > 0$ for $Q_i \in \mathcal{Q}$. Then we can find $l > 0$ such that $\nu(Q_j \cap \tilde{\Lambda}_l) > 0$ for $j = 1, \dots, k$. By the definition of $\tilde{\Lambda}_l$ there exists $z_j \in Q_j \cap \tilde{\Lambda}_l$ such that $\mu(U(z_j) \cap \Lambda_l \cap Q_j) > 0$, where $U(z_j)$ is a neighborhood of z_j with radius less than $\beta_l(\delta_0)$ in Theorem2.4 and so, by Poincare's recurrence theorem we can find $x_j \in U(z_j) \cap Q_j \cap \tilde{\Lambda}_l$ and $n_j > N(x_j)$ sufficiently large such that

$$d(x_j, f^{n_j}(x_j)) < \beta_l(\delta_0) \quad (j = 1, \dots, k)\,. \tag{4.2}$$

By the definition of $\mathcal{Q} = \{Q_i\}_{i=1}^k$,

$$\left| \int_{Q(f)} \psi^* d\nu - \sum_{j=1}^k \nu(Q_j)\psi^*(x_j) \right| < \frac{\bar{\rho}}{4}$$

and by (4.1)

$$\left| \int \psi d\nu - \sum_{j=1}^k \nu(Q_j) \frac{1}{n_j} \sum_{i=0}^{n_j-1} \psi(f^i(x_j)) \right| < \frac{\bar{\rho}}{2}\,. \tag{4.3}$$

For a fixed large integer s with $0 < 1/s < \bar{\rho}/2k$ we can find $\tilde{s}_1, \dots, \tilde{s}_k \in \mathbf{N}$ such that $\tilde{s}_j/s \le \nu(Q_j) \le (\tilde{s}_j + 1)/s$. Choose $s_1, \dots, s_k \in \mathbf{N}$ such that $s = \sum_{j=1}^k s_j$ and $s_j = \tilde{s}_j$ or $\tilde{s}_j + 1$. Then we have

$$\left| \nu(Q_j) - \frac{s_j}{s} \right| < \frac{\bar{\rho}}{2k}$$

and so, by (4.3)

$$\left| \int \psi d\nu - \frac{1}{s} \sum_{j=1}^k s_j \frac{1}{n_j} \sum_{i=0}^{n_j-1} \psi(f^i(x_j)) \right| < \bar{\rho}\,. \tag{4.4}$$

Take $y_0, y_1 \in \tilde{\Lambda}_l$ and $m_0, m_1 > 0$ such that

$$d(y_0, y_1) < \beta_l(\delta_0), \quad d(y_i, f^{m_i}(y_i)) < \beta_l(\delta_0) \quad (i = 0, 1). \tag{4.5}$$

Since $y_0, y_1 \in \tilde{\Lambda}_l$, we have $\mu(U(y_i) \cap \Lambda_l) > 0$ for any open neighborhood $U(y_i)$ of y_i. By the assumption of μ-mixing, there exist $M > 0$ and $w_0, w_1, \dots, w_k \in \tilde{\Lambda}_l$ such that

$$\begin{aligned}
&d(f^{m_i}(y_i), w_0) < \beta_l(\delta_0) \quad (i = 0, 1), \\
&d(f^M(w_0), x_1) < \beta_l(\delta_0), \\
&d(f^{n_j}(x_j), w_j) < \beta_l(\delta_0) \quad (1 \le j \le k), \\
&d(f^M(w_j), x_{j+1}) < \beta_l(\delta_0) \quad (1 \le j \le k-1), \\
&d(f^M(w_k), y_i) < \beta_l(\delta_0) \quad (i = 0, 1).
\end{aligned} \tag{4.6}$$

For simplicity put

$$n = \sum_{j=1}^{k} s_j n_j + (k+1)M + m_0 m_1 .$$

Since $n_1, \dots, n_k$ are large enough,

$$\frac{1}{n}\{(k+1)M + m_0 m_1\} < \frac{\bar{\rho}}{4} . \tag{4.7}$$

In the same way as stated in the proof of Theorem 2.5 we construct a $\{\beta_l(\delta_0)\}_{l \ge 1}$-pseudo orbit. To do so, fix $a = \{a_i\}_{i \in \mathbf{Z}} \in Y_2^{\mathbf{Z}}$ and set $b_i = 1 - a_i$ $(i \in \mathbf{Z})$. For $i \in \mathbf{Z}$ and the finite orbit $y_{a_i}, \dots, f^{m_{a_i}-1}(y_{a_i})$ we firstly construct a finite $\{\beta_l(\delta_0)\}$-pseudo orbit

$$\underbrace{\underbrace{y_{a_i}, \dots, f^{m_{a_i}-1}(y_{a_i})}, \dots, \underbrace{y_{a_i}, \dots, f^{m_{a_i}-1}(y_{a_i})}}_{m_{b_i}\text{-times}}$$

which passes m_{b_i}-times through the orbit $y_{a_i}, \dots, f^{m_{a_i}-1}(y_{a_i})$.

Next, for $1 \le j \le k$ we also construct a finite $\{\beta_l(\delta_0)\}$-pseudo orbit

$$\underbrace{\underbrace{x_j, \dots, f^{n_j-1}(x_j)}, \dots, \underbrace{x_j, \dots, f^{n_j-1}(x_j)}}_{s_j\text{-times}}$$

which passes s_j-times through the orbit $x_j, \dots, f^{n_j-1}(x_j)$. Under these pseudo orbits and the orbits $w_j, \dots, f^{M-1}(w_j)$ for $0 \le j \le k$, we construct a finite $\{\beta_l(\delta_0)\}$-pseudo orbit as follows:

$$z(a_i) = \Bigg(\underbrace{\underbrace{y_{a_i}, \dots f^{m_{a_i}-1}(y_{a_i})}, \dots, \underbrace{y_{a_i}, \dots f^{m_{a_i}-1}(y_{a_i})}}_{m_{b_i}\text{-times}}, w_0, \dots, f^{M-1}(w_0),$$

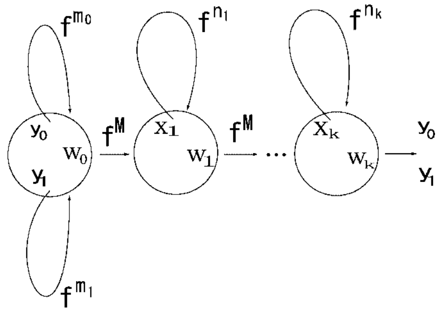

FIGURE 5. A pseudo orbit of $z(a)$.

$$\underbrace{\underbrace{x_1, \dots f^{n_1-1}(x_1)}, \dots, \underbrace{x_1, \dots f^{n_1-1}(x_1)}}_{s_1\text{-times}}, w_1, \dots, f^{M-1}(w_1),$$

$$\vdots$$

$$\underbrace{\underbrace{x_k, \dots f^{n_k-1}(x_k)}, \dots, \underbrace{x_k, \dots f^{n_k-1}(x_k)}}_{s_k\text{-times}}, w_k, \dots, f^{M-1}(w_k)\Big)$$

(see Figure 5).

It follows from (4.2), (4.5) and (4.6) that

$$z(a) = \{\dots, z(a_{-1}), z(a_0), z(a_1), \dots\}$$

is a $\{\beta_l(\delta_0)\}_{l\geq 1}$-pseudo orbit. By Theorem 2.4 there exists a unique shadowing point $\bar{a}$ for $z(a)$, and so define $\varphi : Y_2^{\mathbf{Z}} \to M$ by $\varphi(a) = \bar{a}$ and put $\Gamma_\rho = \varphi(Y_2^{\mathbf{Z}})$. By using (2.1) in the proof of Theorem 2.4, we have that φ is a continuous map such that $\varphi \circ \sigma = f^n \circ \varphi$. Using the same arguments in the proof of Theorem 2.5 (c), it follows that $\tilde{\Gamma}_\rho = \bigcup_{i=0}^{n-1} f^i(\Gamma_\rho) \subset S$.

Denote by $\mathcal{E}(\tilde{\Gamma}_\rho)$ the set of ergodic probability measures of $\tilde{\Gamma}_\rho$ and by $R(\lambda)$ the ergodic basin of λ. Fix $\lambda \in \mathcal{E}(\tilde{\Gamma}_\rho)$ and choose $z \in \Gamma_\rho \cap R(\lambda)$. Then there exists $p_0 \in \mathbf{N}$ such that

$$\left|\frac{1}{pn}\sum_{i=0}^{pn-1}\psi(f^i(z)) - \int \psi\, d\lambda\right| < \frac{\bar{\rho}}{4} \tag{4.8}$$

for $p \geq p_0$ and $\psi \in F$. Since $n_1, \dots, n_k$ and n are integers of (4.7), we have

$$\left|\frac{1}{pn}\sum_{i=0}^{pn-1}\psi(f^i(z)) - \left\{\frac{1}{s}\sum_{j=1}^{k} s_j \frac{1}{n_j}\sum_{i=0}^{n_j-1}\psi(f^i(x_j))\right\}\right| < \frac{3\bar{\rho}}{4}\,. \tag{4.9}$$

By (4.4), (4.8) and (4.9)

$$\left|\int \psi d\nu - \int \psi d\lambda\right| < 2\bar{\rho}\,.$$

$\square$

Let W_1 and W_2 be the unstable manifolds, and let $\pi : W_1 \to W_2$ be a holonomy map defined by sliding along the stable manifolds, i.e., for $x \in W_1$, $\pi(x) \in W_2 \cap W^s(x)$. Then it is known (see [3], [23]) that π is *absolutely continuous* i.e. $m_2^u \circ \pi \ll m_1^u$, where m_1^u and m_2^u be the Lebesgue measure on W_1 and W_2 respectively.

PART (IV). This part is to prove (c) of Theorem 4.1.

PROOF. Let ν be a hyperbolic ergodic probability measure supported on $\tilde{\Lambda}$ which satisfies SRB condition for f and put $\tilde{\Lambda}_l(\nu) = \mathrm{Supp}(\nu|_{\Lambda_l})$ $(l \geq 1)$ and $\tilde{\Lambda}(\nu) = \bigcup_{l\geq 1} \tilde{\Lambda}_l(\nu)$.

If $l \in \mathbf{N}$ is large enough, then we have $\nu(\tilde{\Lambda}_l \cap \tilde{\Lambda}_l(\nu)) \approx 1$ because of $\nu(\tilde{\Lambda} \cap \tilde{\Lambda}(\nu)) = 1$. Since $\nu(R(\nu)) = 1$, by the Borel Density Lemma there exists $x \in \tilde{\Lambda}_l(\nu)$ such that

$$\nu(B(x,r) \cap \tilde{\Lambda}_l \cap \tilde{\Lambda}_l(\nu) \cap R(\nu)) \approx \nu(B(x,r)) > 0$$

for $r > 0$ small enough. $\xi_\nu(z)$ denotes the connected component of $z \in B(x,r) \cap \tilde{\Lambda}_l(\nu)$ whose unstable manifold intersects $B(x,r)$ and put $\xi_\nu = \{\xi_\nu(z) | z \in B(x,r) \cap \tilde{\Lambda}_l(\nu)\}$. Let $\{\nu_z^u\}$ be the canonical system of conditional measures of ν w.r.t. ξ_ν. Then there exists $y \in B(x,r) \cap \tilde{\Lambda}_l(\nu)$ such that

$$\nu_y^u(\xi_\nu(y) \cap \tilde{\Lambda}_l \cap \tilde{\Lambda}_l(\nu) \cap R(\nu)) \approx \nu_y^u(\xi_\nu(y)) > 0\,.$$

Since ν satisfies SRB condition for f, it follows from (1.1) that $\nu_w^u \sim m_w^u|_{\xi_\nu(w)}$ for ν-a.e.$w \in B(x,r)$. Therefore

$$m_y^u(\xi_\nu(y) \cap \tilde{\Lambda}_l \cap \tilde{\Lambda}_l(\nu) \cap R(\nu)) \approx m_y^u(\xi_\nu(y)) > 0\,. \tag{4.10}$$

On the other hand, since $l > 0$ is large enough, we may assume that

$$\mu(B(x,r) \cap \tilde{\Lambda}_l \cap R(\mu)) \approx \mu(B(x,r)) > 0\,.$$

Let $\xi(y)$ be the connected component of $y \in \Lambda_l \cap B(x,r)$ whose unstable manifold intersects $B(x,r)$ and put $\xi = \{\xi(y) | y \in \Lambda_l \cap B(x,r)\}$. Let $\{\mu_z^u\}$ be the canonical system of conditional measures of μ w.r.t. ξ. Then there exists $y' \in B(x,r) \cap \tilde{\Lambda}_l \cap R(\mu)$ such that

$$\mu_{y'}^u(\xi(y') \cap \tilde{\Lambda}_l \cap R(\mu)) \approx \mu_{y'}^u(\xi(y')) > 0\,.$$

Since μ satisfies SRB condition for f, by (1.1) we have

$$m_{y'}^u(\xi(y') \cap \tilde{\Lambda}_l \cap R(\mu)) \approx m_{y'}^u(\xi(y')) > 0\,. \tag{4.11}$$

Since the holonomy map $\pi : \xi_\nu(y) \to \xi(y')$ is absolutely continuous, by (4.10) and (4.11) we can find $z \in \xi_\nu(y) \cap \tilde{\Lambda}_l \cap \tilde{\Lambda}_l(\nu) \cap R(\nu)$ such that

$$\pi(z) \in \xi(y') \cap \tilde{\Lambda}_l \cap R(\mu), \quad d(f^i(z), f^i(\pi(z))) \to 0 \quad (i \to \infty).$$

Therefore we have

$$\begin{aligned}\int \varphi d\nu &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(z)) \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(\pi(z))) \\ &= \int \varphi d\mu\end{aligned}$$

for $\varphi \in C(M, \mathbf{R})$, since $z \in R(\nu)$ and $\pi(z) \in R(\mu)$. This implies that $\mu = \nu$. □

5. SRB condition and absolute continuity of probability measures

We shall investigate the further properties of ergodic measures satisfying SRB condition.

REMARK 5.1. There exists a unique measure satisfying SRB condition for f on every hyperbolic attractor (see [6], Theorem 4.12).

Throughout this section, assume that μ is a hyperbolic ergodic measure and Λ is a Pesin set w.r.t. μ. Let ξ^u be a measurable partition of M which is subordinate to the W^u-foliation and $\{\mu_x^u\}$ (μ-a.e.x) be a canonical system of conditional measures of μ w.r.t. ξ^u. For $x \in \Lambda$ and $\sigma = s, u$ denote by $B^\sigma(x, \rho)$ the ball centered at x with radius ρ in $W^\sigma(x)$. Then there exists

$$\delta^\sigma = \lim_{\rho\to 0} \frac{\log \mu_x^\sigma(B^\sigma(x, \rho))}{\log \rho} \quad (\mu\text{-a.e.}x, \sigma = s, u),$$

and $\delta^\sigma \le \dim E^\sigma$ holds ([15]).

Let $F_0 \subset M$ be such that $\mu(F_0) = 1$, and μ_x^u and $\delta^u(x)$ exist for $x \in F_0$. Let $B(x, \rho)$ denote the ball centered at x with radius ρ.

LEMMA 5.2 ([29]). *Let ν be a finite Borel measure on M and $Z \subset M$ be $\nu(Z) > 0$. Assume that there exist $0 \le \underline{\delta} \le \bar{\delta}$ such that*

$$\underline{\delta} \le \liminf_{r\to 0} \frac{\log \nu(B(x, r))}{\log r} \le \limsup_{r\to 0} \frac{\log \nu(B(x, r))}{\log r} \le \bar{\delta}$$

for $x \in Z$. Then

$$\underline{\delta} \le HD(Z) \le \bar{\delta}.$$

Here $HD(Z)$ is the Hausdorff dimension of Z.

LEMMA 5.3 ([15]). *μ satisfies SRB condition for f if and only if $\delta^u = \dim E^u$.*

THEOREM 5.4. *μ satisfies SRB condition for f if and only if μ satisfies the following condition.*
(B) : for $N \subset M$ with $\mu(N) = 0$, $m_x^u(N \cap \xi^u(x)) = 0$ (μ-a.e.x) where m_x^u be the Lebesgue measure on $W^u(x)$.

PROOF. Assume that μ satisfies SRB condition for f and fix $N \subset M$ with $\mu(N) = 0$. Then we have

$$\begin{aligned} 0 = \mu(N) &= \int \mu_x^u(N) d\mu(x) \\ &= \int_{F_0} \mu_x^u(N) d\mu(x) . \end{aligned}$$

Thus there exists $F' \subset F_0$ with $\mu(F') = 1$ such that $\mu_x^u(N) = 0$ ($x \in F'$) and so $m_x^u(N \cap \xi^u(x)) = 0$ ($x \in F'$) by (1.1). Therefore μ satisfies the condition (B).

Conversely, assume that μ satisfies the condition (B). Since $\mu(F_0) = 1$, we have

$$\begin{aligned} 1 = \mu(F_0) &= \int \mu_x^u(F_0) d\mu(x) \\ &= \int_{F_0} \mu_x^u(F_0) d\mu(x) . \end{aligned}$$

Then there exists $F'' \subset F_0$ with $\mu(F'') = 1$ such that

$$\mu_x^u(F_0) = 1 \quad (x \in F'') . \tag{5.1}$$

Since μ satisfies the condition (B), we can find $F''' \subset M$ with $\mu(F''') = 1$ such that $m_x^u(F_0^c \cap \xi^u(x)) = 0$ ($x \in F'''$), from which

$$m_x^u(F_0 \cap \xi^u(x)) > 0 \quad (x \in F''') . \tag{5.2}$$

For $x \in F'' \cap F'''$, we have that by applying Lemma 5.2 to (5.1) and (5.2)

$$\delta^u = HD(\xi^u(x) \cap F_0) = \dim E^u \quad (x \in F'' \cap F''') .$$

Therefore μ satisfies SRB condition for f by Lemma 5.3. □

If

$$d_\mu(x) = \lim_{\rho \to 0} \frac{\log \mu(B(x, \rho))}{\log \rho} ,$$

then $d_\mu(x)$ is said to be the *pointwise dimension* of μ at x. We have (see [29]) that $d_\mu(x) = \inf\{HD(Z) \mid \mu(Z) = 1\}$ if $d_\mu(x)$ is constant for μ-a.e.x.

THEOREM 5.5 ([4], [15] p. 548). *$d_\mu(x) = \delta^s + \delta^u$ holds for μ-a.e.x.*

REMARK 5.6. If $\nu \in \mathcal{M}(M)$ and is absolutely continuous (w.r.t. m), then $d_\nu(x) = \dim M$ for ν-a.e.x.

THEOREM 5.7. *μ is an absolutely continuous measure (w.r.t. m) if and only if it satisfies SRB condition for f and the condition $(A)_u$ (which is defined in Sect.3).*

PROOF. Assume that μ satisfies both SRB condition for f and the condition $(A)_u$. By SRB condition of μ, we have $\mu_x^{f^i\xi^u} \sim m_x^u|_{f^i\xi^u(x)}$ (μ-a.e.x) for $i \in \mathbf{Z}$.

In order to show that μ is an absolutely continuous measure, it is enough to prove that if $\mu(E) > 0$ for $E \in \mathcal{B}$ with $E \subset F_0$ then $m(E) > 0$.

Since $\mu(E) > 0$, we can find $F_1 \subset E$ with $\mu(F_1) = \mu(E)$ such that

$$\mu_x^{f^i\xi^u}(E) > 0$$

for $x \in F_1$ and $i \in \mathbf{Z}$. By the regularity of μ, there exists a monotonically increasing sequence $\{C_p\}_{p\geq 1} \subset F_1 \cap Y_\mu$ of closed sets such that $\mu(\bigcup_{p\geq 1} C_p) = \mu(E)$. Since μ is ergodic, $\bigcup_{n\geq 0} f^n(\bigcup_{p\geq 1} C_p)$ has μ-measure 1 and so is

$$\begin{aligned} A &= \bigcup_{n\geq 0} f^n\left(\bigcup_{p\geq 1} C_p\right) \cap \Lambda \\ &= \bigcup_{n\geq 0}\bigcup_{p\geq 1}\bigcup_{l\geq 1}(f^n(C_p)\cap \Lambda_l)\,. \end{aligned}$$

Since μ satisfies the condition $(A)_u$, we have $m(W^u(A)) > 0$. Then there exist $l_1 \in \mathbf{N}$, $k_1 \in \mathbf{Z}$, $x_1 \in \Lambda_{l_1}$ and $r_{l_1} > 0$ such that $m(B^u) > 0$ where

$$B^u = \bigcup_{y\in B(x_1,r_{l_1})\cap f^{k_1}(C_p)\cap\Lambda_{l_1}} \eta^u(y)$$

and $\eta^u(y)$ is the connected component of $W^u(y) \cap B(x_1, r_{l_1})$ which contains y. Since $\eta^u = \{\eta^u(y) \mid y \in B(x_1, r_{l_1}) \cap f^{k_1}(C_p) \cap \Lambda_{l_1}\}$ is a measurable partition of B^u, we can take $F_2 \subset B^u$ with $m(F_2) = m(B^u)$ and a canonical system of conditional measures $\{m_w^{\eta^u} \mid w \in F_2\}$ of m w.r.t. η^u. For $w \in F_2$ we have that $m_w^u|_{\eta^u(w)} \sim m_w^{\eta^u}$ (see [1], p148), and so

$$m_w^{\eta^u}(f^{k_1}(E)) > 0 \quad (w \in F_2)\,. \tag{5.3}$$

Indeed, for $w \in F_2$ there exists $\bar{w} \in B(x_1, r_{l_1}) \cap f^{k_1}(C_p) \cap \Lambda_{l_1}$ such that $\eta^u(w) = \eta^u(\bar{w})$. Since $f^{-k_1}(\bar{w}) \in C_p \subset F_1 \cap Y_\mu$, we have $\mu_{f^{-k_1}(\bar{w})}^{f^{-k_1}\xi^u}(E) > 0$, and by SRB condition of μ, $m_{f^{-k_1}(\bar{w})}^u(E \cap f^{-k_1}\xi^u(f^{-k_1}(\bar{w}))) > 0$. Because of $m_w^u|_{\eta^u(w)} \sim m_w^{\eta^u}$, $m_{\bar{w}}^u(f^{k_1}(E) \cap \xi^u(\bar{w})) > 0$. Since $m_w^u = m_{\bar{w}}^u$, we have $m_w^u(f^{k_1}(E) \cap \xi^u(\bar{w})) > 0$ and then $m_w^{\eta^u}(f^{k_1}(E)) > 0$. Therefore (5.3) holds.

Using (5.3) we have

$$m(B^u \cap f^{k_1}(E)) = \int_{B^u} m_w^{\eta^u}(f^{k_1}(E))dm$$

$$= \int_{F_2} m_w^{\eta^u}(f^{k_1}(E))dm$$
$$> 0 .$$

Therefore $m(f^{k_1}(E)) > 0$ and so $m(E) > 0$. μ is absolutely continuous.

Conversely, assume that μ is absolutely continuous. It follows from Theorem 5.5 and Remark 5.6 that $n = \delta^u + \delta^s$. Since $\delta^\sigma \leq \dim E^\sigma (\sigma = s, u)$ and $n = \dim E^u + \dim E^s$, we have that $\delta^u = \dim E^u$ and $\delta^s = \dim E^s$. By Lemma 5.3 μ satisfies SRB condition for f. Moreover, applying Lemma 5.3 to f^{-1}, we have that μ satisfies SRB condition for f^{-1} and so μ satisfies the condition $(A)_u$ (see [23]). □

REMARK 5.8. A non-hyperbolic measure ν is not always an absolutely continuous measure even if ν satisfies SRB condition for f and f^{-1}.

References

[1] D. V. ANOSOV and YA. G. SINAI, Some smooth ergodic systems, Russ. Math. Surveys **22** No. 5 (1967), 107–172.

[2] N. AOKI, *The Series of Nonlinear Analysis*, I, II, III, IV, in Japanese, Kyoritu Publ., 2004.

[3] L. BARREIRA and Y. PESIN, Lectures on Lyapunov Exponents and Smooth Ergodic Theory, Appendix A by M. BRIN and Appendix B by D. DOLGOPYAT, H. HU and PESIN, Proc. Symposia Pure Math. AMS (2002).

[4] L. BARREIRA, Y. PESIN and J. SCHMELING, Dimension and product structure of hyperbolic measures, Ann. of Math. **149** (1999), 755–783.

[5] M. BENEDICKS and L. CARLESON, The dynamics of the Hénon map, Ann. of Math. **133** (1991), 73-169.

[6] R. BOWEN, *Equilibrium States and The Ergodic Theory of Anosov Diffeomorphisms*, Lecture Notes in Mathematics **470**, Springer-Verlag, 1975, 67–76.

[7] R. BOWEN, Some systems with unique equilibrium states, Math. syst. Theory, **8** (1975), 193–202.

[8] A. FATHI, M. R. HERMAN and J.-C. YOCCOZ, *A proof of Pesin's stable manifold theorem*, Springer Lect. Notes in Math. **1007**, Berlin, 1983.

[9] M. HIRAYAMA, Periodic probability measures are dense in the set of invariant measures, Discr and Conti Dynam. Syst. **9** (2003), 1185–1192.

[10] H. HU and L-S. YOUNG, Nonexistence of SBR measures for some diffeomorphism that are 'Almost Anosov', Ergod. Th. and Dynam. Sys. **15** (1995), 67–76.

[11] A. KATOK, Lyapunov Exponents entropy and periodic orbits for diffeomorphisms, I.H.E.S. Publ. Math. **51** (1980), 137–173.

[12] A. KATOK and B. HASSELBLATT, *Introduction to the Modern Theory of Dynamical Systems*, Cambrige Univ. Press, 1995.

[13] F. LEDRAPPIER, Proprietes ergodiques des measures de Sinai, Publ. Math. I.H.E.S. **59** (1984), 163–188.

[14] F. LEDRAPPIER and J. STRELCYN, A proof of the estimation from below in Pesin's entropy formula, Ergod. Th. and Dynam. Sys. **2** (1982), 203–219.

[15] F. LEDRAPPIER and L-S.YOUNG, The metric entropy of diffeomorphisms PartI, II, Ann. of Math. **122** (1985), 509–574.

[16] D. LIND and B. MARCUS, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995.

[17] J. MILNOR, Non-expansive Henon maps, Adv.in Math. **69** (1988), 109–114.

[18] M. MORSE and G. A. HEDLUND, Symbolic dynamics II. Sturmian trajectories, Amer. J. Math. **62** (1942), 1–42.

[19] Y. I. OSELEDEC, A multiplicative ergodic theorem, Lyapunov characteristic numbers for dynamical systems, Trudy Moskov Mat. Ostc. **19** (1968), 179–210.

[20] Y. B. PESIN, Characteristic Lyapunov exponents and smooth ergodic theory, Russ. Math. Surveys **32** (1977), 55–114.

[21] Y. B. PESIN, *Dimension Theory in Dynamical Systems*, Chicago Lectures in Math., The Univ. Chicago Press, 1997.

[22] M. POLLICOTT, *Lectures on ergodic theory and Pesin theory on compact manifolds*, LMS, 1993.

[23] C. PUGH and M. SHUB, Ergodic Attractors, Trans. A.M.S. **312** (1989), 1–54.

[24] V. A. ROHLIN, On the fundamental ideas of measure theory, Mat. Sbornik **25** (1949).

[25] D. RUELLE, An inequality for the entropy of differentiable maps, Bol. Soc. Bras. Math. **9** (1978), 83–87.

[26] M. SHUB, *Global Stability of Dynamical Systems*, Springer-Verlag, 1987.

[27] K. SIGMUND, Generic properties of invariant measures for Axiom A-diffeomorphisms, Invent. Math. **11** (1970), 99–109.

[28] P. WALTERS, *An introduction to ergodic theory*, Vol **79** of Graduate Texts in Mathmatics, New York-Berlin, Springer Verlag, 1982.

[29] L-S. YOUNG, Dimension entropy and Lyapunov exponents, Ergod. Th. and Dynam. Sys. **2** (1982), 109–124.

Present Address:
GRADUATE SCHOOL OF MATHEMATICS,
TOKYO METROPOLITAN UNIVERSITY,
MINAMI-OHSAWA, HACHIOJI-SHI, TOKYO, 192–0397 JAPAN.
e-mail: hatomoto-jin@c.metro-u.ac.jp